Acta Univ. Sapientiae, Informatica **12,** 1 (2020) 137–157

DOI: 10.2478/ausi-2020-0009

$\mathscr{P}$-energy of graphs

Prajakta Bharat JOSHI
Department of Mathematics,
CHRIST(Deemed to be University),
Bangalore, India.
email:
prajaktabharat.joshi@res.christuniversity.in

Mayamma JOSEPH
Department of Mathematics,
CHRIST(Deemed to be University),
Bangalore,
India.
email:
mayamma.joseph@christuniversity.in

Abstract. Given a graph $G = (V, E)$, with respect to a vertex partition $\mathscr{P}$ we associate a matrix called $\mathscr{P}$-matrix and define the $\mathscr{P}$-energy, $E_{\mathscr{P}}(G)$ as the sum of $\mathscr{P}$-eigenvalues of $\mathscr{P}$-matrix of G. Apart from studying some properties of $\mathscr{P}$-matrix, its eigenvalues and obtaining bounds of $\mathscr{P}$-energy, we explore the robust(shear) $\mathscr{P}$-energy which is the maximum(minimum) value of $\mathscr{P}$-energy for some families of graphs. Further, we derive explicit formulas for $E_{\mathscr{P}}(G)$ of few classes of graphs with different vertex partitions.

1 Introduction

In this paper, we are concerned with simple and undirected graph $G = (V, E)$ of order n and size m. For spectral and graph theoretic terminologies we refer Cvetković et al. and West respectively [4, 14].

If $A(G)$ is the adjacency matrix of a graph G, then its *energy* is the sum of the absolute values of all the eigenvalues of $A(G)$ [5]. In 1978, Gutman introduced this concept and thereafter, extensive studies on the same have

Computing Classification System 1998: G.2.2
Mathematics Subject Classification 2010: 05C50, 05C69
Key words and phrases: graph energy, vertex partitions, k-partition matrix, k-partition energy

been carried out by several researchers on its theoretical as well as practical aspects and several variations of graph energy can be found in the literature [2, 6, 7, 12].

An interesting variation of graph energy is *the* k*-partition energy* defined by Sampathkumar et al. [12]. They have introduced this concept using the idea of a matrix called L-*matrix*, $P_k(G)$ with respect to a vertex partition P_k that uniquely represents the given graph G. *The* k*-partition energy,* $E_{P_k}(G)$ *is sum of the absolute values of* k*-partition eigenvalues of* $P_k(G)$. For a given graph G, the value of $E_{P_k}(G)$ varies according to different vertex partitions. It can be observed that the properties of elements in the vertex partition is not taken into consideration while determining the value of $E_{P_k}(G)$. In the present study, we consider this aspect and introduce $\mathscr{P}$-energy as a variation of k-partition energy.

Let $\mathscr{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of the vertex set $V(G)$ of a graph $G = (V, E)$. Then the $\mathscr{P}$*-matrix of* G*,* $A_{\mathscr{P}}(G) = D(G) + P_k(G)$*, where* $D(G)$ *is the diagonal matrix with the* i^{th} *diagonal entry, the cardinality of the set* $V_r \in \mathscr{P}$ *containing the vertex* v_i. In other words, $A_{\mathscr{P}}(G) = (a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} |V_r| & \text{if } i = j \text{ and } v_i = v_j \in V_r, \text{ for } r = 1, 2, \ldots k \\ 2 & \text{if } v_iv_j \in E(G) \text{ with } v_i, v_j \in V_r, \\ 1 & \text{if } v_iv_j \in E(G) \text{ with } v_i \in V_r \text{ and } v_j \in V_s \text{ for } r \neq s, \\ -1 & \text{if } v_iv_j \notin E(G) \text{ with } v_i, v_j \in V_r, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{\mathscr{P}}(G)$ is denoted by $\phi_{\mathscr{P}}(G, \lambda)$ and *the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *of* $A_{\mathscr{P}}(G)$ *are called* $\mathscr{P}$*-eigenvalues.* If $g_1, g_2, \ldots, g_n$ are the multiplicities of $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ respectively, then the $\mathscr{P}$-spectrum of G is

$$\mathrm{Spec}_{\mathscr{P}}(G) = \{\lambda_1^{g_1}, \lambda_2^{g_2}, \ldots, \lambda_n^{g_n}\}$$

and accordingly *the* $\mathscr{P}$*-energy,* $E_{\mathscr{P}}(G)$ *is sum of the absolute values of* $\mathscr{P}$*-eigenvalues of* $A_{\mathscr{P}}(G)$.

For a given vertex partition $\mathscr{P}$ of $V(G)$, the diagonal entries of $A_{\mathscr{P}}(G)$ are positive numbers, whereas the diagonal entries of $P_k(G)$ are zeros and remaining entries of these matrices are same which belongs to the set $\{2, 1, 0, -1\}$. Since the absolute values of the eigenvalues of any matrix are directly proportional to the maximum value of the absolute values of entries in the given matrix, one immediate observation is that if the cardinality of every member of the given vertex partition $\mathscr{P}$ of a graph G is greater than 1, then $E_{\mathscr{P}}(G) \geq E_{P_k}(G)$.

Another interesting observation about $\mathscr{P}$-energy is that, as the order of vertex partition $\mathscr{P}$ of a given graph G increases, value of $E_{\mathscr{P}}(G)$ decreases. Hence, $\mathscr{P}$-energy of a graph G is maximum when the vertex partition $\mathscr{P} = \{V(G)\}$ and minimum when $\mathscr{P} = \{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}$. We call the maximum (minimum) value of $\mathscr{P}$-energy as the *robust(shear) $\mathscr{P}$-energy*, $E_{\mathscr{P}_r}(G)(E_{\mathscr{P}_s}(G))$, similar to the concepts of robust domination energy and shear domination energy of a graph introduced by Acharya et al. [1].

Example 1 *For a null graph H of order n, it can easily be verified that $E_{\mathscr{P}_r}(H) = n^2$ and $E_{\mathscr{P}_s}(H) = n$.*

Now, we state in the following remark some of the basic results from linear algebra which are required for the present study:

Remark 2 [11] *If A is a real or complex matrix of order $n \times n$ with the characteristic polynomial $\phi(G, \lambda)$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then*

(i) A principal sub-matrix of order $r \times r$ of A is a sub-matrix consisting of the same set of r rows and r columns and a principal minor of order $r \times r$ of A is the determinant of a principal sub-matrix of order $r \times r$.

(ii) If $a_0, a_1, a_2, \dots, a_n$ are the coefficients of $\phi(G, \lambda)$, then $(-1)^r a_r$ is the sum of principal minors of order $r \times r$.

(iii) If the r^{th} symmetric function $S_r(A)$ is the sum of the product of the eigenvalues of A taken r at a time, then it is the sum of $r \times r$ principal minors of A.

(iv) Trace of A is the sum of diagonal entries of A and it can also be represented as $\mathrm{tr}(A) = S_1(A) = -a_1$.

(v) $\prod_{i=1}^{n} \lambda_i = |A|$.

Theorem 3 [10] *If λ is an eigenvalue of the matrix $(a_{ij})_{n \times n}$, then*

$$|\lambda| \leq n \max_{i,j} |a_{ij}|.$$

Lemma 4 [4] *If $C = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ is a symmetric block matrix of order 2×2, then the spectrum of C is the union of the spectra of $A + B$ and $A - B$.*

Lemma 5 [4] *If M, N, P, Q are matrices where M is invertible and $S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$, then* $\det S = \det M \cdot \det[Q - PM^{-1}N]$.

2 Properties of $\mathscr{P}$-eigenvalues of $A_{\mathscr{P}}(G)$

Before proceeding further, we present a few observations about $A_{\mathscr{P}}(G)$ with respect to the structure of a graph G.

Observation 1 *Given a graph $G = (V, E)$, the following are true for its $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$.*

(i) If $d(v_i)$ is the degree of $v_i \in V(G)$, then $d(v_i)$ is the number of positive off-diagonal entries of the i^{th} row corresponding to the vertex v_i in $A_{\mathscr{P}}(G)$.

(ii) The elements of $A_{\mathscr{P}}(G)$, excluding its main diagonal entries has one-one correspondence with $P_k(G)$ with respect to the same vertex partition of a given graph G.

(iii) For the matrix $A_{\mathscr{P}}(G)$,

$$\mathrm{tr}(A_{\mathscr{P}}(G)) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{k} |V_i|^2. \tag{1}$$

(iv) If m_1 is the number of edges of a graph G whose end vertices share the same partition, m_2 is the number of edges of G whose end vertices are in different partitions and m_3 is the number of pairs of non-adjacent vertices of G within the same partition, then

$$\sum_{1 \le i < j \le n} (a_{ij})^2 = 4m_1 + m_2 + m_3. \tag{2}$$

The following result that characterizes $\mathscr{P}$-matrix of a graph is similar to that of the characterization of L- matrix of a labeled graph as given in [13].

Theorem 6 *A symmetric matrix $A = (a_{ij})_{n \times n}$ with positive diagonal entries and off-diagonal entries belonging to the set {2, 1, 0, -1} is the $\mathscr{P}$-matrix graph G of order n with the vertex partition $\mathscr{P} = \{V_1, V_2, \ldots, V_k\}$ if and only if*

(i) $a_{ij}, a_{jk} \in \{2, -1\} \implies a_{ik} \in \{2, -1\}$,

(ii) $a_{ij} \in \{2, -1\}$ *and* $a_{jk} \in \{0, 1\} \implies a_{ik} \in \{0, 1\}$ *and*

(iii) $v_i \in V_r \implies a_{ii} = |V_r|$.

In the next result, we obtain the exact values of the coefficients of λ^n, λ^{n-1} and λ^{n-2} in the characteristic polynomial $\phi_{\mathscr{P}}(G, \lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda_1 + a_n$ of $A_{\mathscr{P}}(G)$.

Proposition 7 *If* G *is a graph with vertex partition* $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ *and* $\phi_{\mathscr{P}}(G, \lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n$*, then*

(i) $a_0 = 1$

(ii) $a_1 = -\sum_{i=1}^{k} |V_i|^2$

(iii) $a_2 = \sum_{1 \le i < j \le k} |V_i||V_j| - (4m_1 + m_2 + m_3)$.

Proof.

(i) It holds directly, as the characteristic polynomial $\phi_{\mathscr{P}}(G, \lambda)$ is a monic polynomial.

(ii) From Remark 2(ii) and Equation (1), we get the result.

(iii) From Remark 2(ii),

$$(-1)^2 a_2 = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ij}a_{ji}.$$

Since $A_{\mathscr{P}}(G)$ is a symmetric matrix,

$$a_2 = \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} (a_{ij})^2 = \sum_{1 \le i < j \le k} |V_i||V_j| - \sum_{1 \le i < j \le n} (a_{ij})^2. \tag{3}$$

Therefore from Equations (2) and (3), we obtain the result.

□

The trace of a matrix is the sum of the eigenvalues of that matrix, therefore the sum of $\mathscr{P}$-eigenvalues of $A_{\mathscr{P}}(G)$ of a graph G is non-zero. In the next proposition, we obtain its value in terms of cardinality of elements in the vertex partition $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ of G.

Proposition 8 *If* G *is a graph with the vertex partition* $\mathscr{P}$ *and* $\lambda_1, \lambda_2, \dots, \lambda_n$ *are the* $\mathscr{P}$*-eigenvalues, then*

(i) $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{k} |V_i|^2$

(ii) $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3.$

Proof.

(i) From Remark 2(iv) and Equation (1), the result holds.

(ii) For a matrix A of order $n \times n$, $tr(A^2) = (tr(A))^2 - 2S_2(A)$, where $S_2(A)$ is the 2^{nd} symmetric function. This can be written as

$$\sum_{i=1}^{n} \lambda_i^2 = (\sum_{i=1}^{n} a_{ii})^2 - 2S_2(A).$$

By Remark 2(iii),

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= (\sum_{i=1}^{n} a_{ii})^2 - 2\sum_{i<j}(a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= (\sum_{i=1}^{n} a_{ii})^2 - 2\sum_{i<j} a_{ii}a_{jj} + 2\sum_{i<j}(a_{ij})^2 \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j} a_{ii}a_{jj} - 2\sum_{i<j} a_{ii}a_{jj} + 2\sum_{i<j}(a_{ij})^2 \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}(a_{ij})^2 \qquad (4) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{n} a_{ii}^2 &= |V_1| \cdot |V_1|^2 + |V_2| \cdot |V_2|^2 + \dots + |V_k| \cdot |V_k|^2 \\ &= |V_1|^3 + |V_2|^3 + \dots + |V_k|^3 \\ &= \sum_{i=1}^{k} |V_i|^3. \qquad (5) \end{aligned}$$

Therefore from Equations (2), (4) and (5),

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{k} |V_i|^3 + 2(4m_1 + m_2 + m_3). \quad (6)$$

□

The next proposition given without proof, follows from Cauchy-Schwartz inequality and Proposition 8 (ii). Note that, the symbols m_1, m_2, m_3 for graph G_1 are as given in the Observation 1(iv) and m_1', m_2', m_3' are the corresponding values for the graph G_2.

Proposition 9 *Let G_1 and G_2 be two graphs with respect to vertex partition $\mathscr{P} = \{V_1, V_2, \ldots, V_k\}$ and $\mathscr{P}' = \{V_1', V_2', \ldots, V_k'\}$ respectively. If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\{\lambda_1', \lambda_2', \ldots, \lambda_n'\}$ are the $\mathscr{P}$-eigenvalues of $\mathscr{P}$-matrix of G_1 and G_2 respectively, then*

$$\sum_{i=1}^{n} \lambda_i \lambda_i' \leq \sqrt{\left[\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3\right]\left[\sum_{i=1}^{k} |V_i'|^3 + 8m_1' + 2m_2' + 2m_3'\right]}.$$

3 Bounds for $\mathscr{P}$-energy

Now we present some bounds for the $\mathscr{P}$-energy of a graph G in terms of its order and the cardinality of elements in its vertex partition. One obvious bound when $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ is

$$\sum_{i=1}^{k} |V_i|^2 \leq E_{\mathscr{P}}(G) \leq n^3.$$

The lower bound follows from the inequality $\sum_{i=1}^{n} \lambda_i \leq \sum_{i=1}^{n} |\lambda_i|$ whereas the upper bound is a direct deduction from Theorem 3.

Theorem 10 *For any graph G with vertex partition $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$,*

$$E_{\mathscr{P}}(G) \leq \sqrt{n\left\{\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3\right\}}. \quad (7)$$

Proof. By Cauchy-Schwartz inequality,

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \big(\sum_{i=1}^{n} a_i^2\big)\big(\sum_{i=1}^{n} b_i^2\big). \tag{8}$$

Replace $a_i = 1$ and $b_i = |\lambda_i|$ in Equation (8),

$$\begin{aligned}\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 &\leq \big(\sum_{i=1}^{n} 1\big)\big(\sum_{i=1}^{n} |\lambda_i|^2\big)\\ &\leq n\sum_{i=1}^{n} \lambda_i^2.\end{aligned}$$

From Equation (6),

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 \leq n\Big\{\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3\Big\}.$$

Hence,

$$E_{\mathscr{P}}(G) \leq \sqrt{n\Big\{\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3\Big\}}.$$

□

Theorem 11 *Let* G *be a graph with vertex partition* $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ *and* $|A_{\mathscr{P}}(G)|$ *be the determinant of* $A_{\mathscr{P}}(G)$. *Then*

$$E_{\mathscr{P}}(G) \geq \sqrt{\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3 + n(n-1)|A_{\mathscr{P}}(G)|^{2/n}}. \tag{9}$$

Proof. By the definition of $\mathscr{P}$-energy,

$$\begin{aligned}\big[E_{\mathscr{P}}(G)\big]^2 = \big(\sum_{i=1}^{n} |\lambda_i|\big)^2 &= \big(\sum_{i=1}^{n} |\lambda_i|\big)\big(\sum_{j=1}^{n} |\lambda_j|\big)\\ &= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i||\lambda_j|. \end{aligned} \tag{10}$$

By using arithmetic and geometric mean inequality, Equation (10) can be written as follows

$$\left[E_{\mathscr{P}}(G)\right]^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\dfrac{1}{n(n-1)}}$$
$$\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\dfrac{1}{n(n-1)}}.$$

Therefore,

$$\left[E_{\mathscr{P}}(G)\right]^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|\right)^{\frac{2}{n}}.$$

Hence from Remark 2(v) and Equation (6),

$$E_{\mathscr{P}}(G) \geq \sqrt{\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3 + n(n-1)|A_{\mathscr{P}}(G)|^{2/n}}.$$

□

Remark 12 *Let* H *be a null graph of order* n*. Then the upper and lower bounds given by Equations* (7) *and* (9) *are sharp for the vertex partition* $\mathscr{P} = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ *of* H.

4 $\mathscr{P}$-energy of some graph families

In this section, we examine the $\mathscr{P}$-energy of some families of graphs for the trivial partitions $\mathscr{P} = \{V(G)\}$ and $\mathscr{P} = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ respectively. Recall that the extreme values of $\mathscr{P}$-energy is obtained with respect to these partitions and the largest value of $E_{\mathscr{P}}(G)$ denoted by $E_{\mathscr{P}_r}(G)$ is referred to as the robust $\mathscr{P}$-energy and the smallest denoted by $E_{\mathscr{P}_s}(G)$ is referred to as the shear $\mathscr{P}$-energy.

Theorem 13 *For the complete graph* K_n*,*

$$E_{\mathscr{P}_r}(K_n) = n^2 \text{ and } E_{\mathscr{P}_s}(K_n) = n.$$

Proof. Let K_n be a complete graph and $\mathscr{P} = \{V(G)\}$. The $\mathscr{P}$-matrix of K_n is

$$A_{\mathscr{P}}(K_n) = [(n-2)I + 2J]_{n\times n}$$

where J is the matrix of order $n \times n$ whose all entries are 1 and I is identity matrix of order $n \times n$. The characteristic polynomial is $\phi_{\mathscr{P}}(K_n, \lambda) = |\lambda I - A_{\mathscr{P}}(K_n)|$. Thus,

$$\phi_{\mathscr{P}}(K_n,\lambda) = \begin{vmatrix} \lambda-n & -2 & -2 & \dots & -2 \\ -2 & \lambda-n & -2 & \dots & -2 \\ -2 & -2 & \lambda-n & \dots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & -2 & \dots & \lambda-n \end{vmatrix}_{n\times n}$$
$$= [\lambda-(n-2)]^{(n-1)}[\lambda-(3n-2)].$$

Therefore,

$$\mathsf{Spec}_{\mathscr{P}}(K_n) = \{(3n-2)^1, (n-2)^{(n-1)}\}$$

and

$$E_{\mathscr{P}}(K_n) = n^2 \text{ with respect to the vertex partition } \mathscr{P} = \{V(G)\}.$$

Hence, $E_{\mathscr{P}_r}(K_n) = n^2$. Now, let $\mathscr{P}$ be a vertex partition of K_n such that $\mathscr{P} = \{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}$. The $\mathscr{P}$-matrix of K_n is

$$A_{\mathscr{P}}(K_n) = J_{n\times n}$$

and

$$\phi_{\mathscr{P}}(K_n,\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \dots & -1 \\ -1 & \lambda-1 & -1 & \dots & -1 \\ -1 & -1 & \lambda-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ -1 & -1 & -1 & \dots & \lambda-1 \end{vmatrix}_{n\times n}$$
$$= \lambda^{(n-1)}(\lambda-n).$$

Therefore,

$$\mathsf{Spec}_{\mathscr{P}}(K_n) = \{n^1, 0^{(n-1)}\}$$

and

$$E_{\mathscr{P}}(K_n) = n \text{ with respect to the vertex partition } \mathscr{P} = \{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}.$$

Hence, $E_{\mathscr{P}_s}(K_n) = n$. □

Remark 14 *For a complete graph* K_n *and a null graph* H,

$$E_{\mathscr{P}_r}(K_n) = E_{\mathscr{P}_r}(H) \text{ and } E_{\mathscr{P}_s}(K_n) = E_{\mathscr{P}_s}(H).$$

Note that K_n *and* H *are non-cospectral equi-*$\mathscr{P}$*-energetic graphs, since the* $\mathscr{P}$*-eigenvalues of* $\mathscr{P}$*-matrices of both the graphs differ but the values of their robust and shear* $\mathscr{P}$*-energies coincide.*

The following result deals with the robust and shear $\mathscr{P}$-energy of a star. We omit its proof, since its proof technique is similar to that of Theorem 13.

Theorem 15 *If* $K_{1,n-1}$ *is a star of order* $n \geq 2$, *then*

$$E_{\mathscr{P}_r}(K_{1,n-1}) = 2n - 4 + \sqrt{n^2 + 12n - 12} \text{ and}$$
$$E_{\mathscr{P}_s}(K_{1,n-1}) = (n-2) + 2\sqrt{n-1}.$$

If we join the maximum degree vertex of two copies of $K_{1,r-1}$ of order $r(r \geq 2)$, then the resultant graph is called a double star $B_{r,r}$ of order $n = 2r$.

Theorem 16 *If* $B_{r,r}$ *is a double star of order* n, *then*

$$E_{\mathscr{P}_r}(B_{r,r}) = n^2 \quad \text{for } n \geq 2,$$
$$E_{\mathscr{P}_s}(B_{r,r}) = \begin{cases} n - 1 + \sqrt{2n-3} & \text{for } 2 \leq n < 8, \\ (n-4) + 2\sqrt{2n-3} & \text{for } n \geq 8. \end{cases}$$

Proof. Let $B_{r,r}$ be a double star of order n with respect to the vertex partition $\mathscr{P} = \{V(G)\}$. Then

$$A_{\mathscr{P}}(B_{r,r}) = \begin{pmatrix} n & 2 & 2 & \dots & 2 & 2 & -1 & -1 & \dots & -1 \\ 2 & n & -1 & \dots & -1 & -1 & -1 & -1 & \dots & -1 \\ 2 & -1 & n & \dots & -1 & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & -1 & -1 & \dots & n & -1 & -1 & -1 & \dots & -1 \\ 2 & -1 & -1 & \dots & -1 & n & 2 & 2 & \dots & 2 \\ -1 & -1 & -1 & \dots & -1 & 2 & n & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & 2 & -1 & n & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 & 2 & -1 & -1 & \dots & n \end{pmatrix}_{n \times n}.$$

Clearly, it is of the form $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$. To get $\mathrm{Spec}_{\mathscr{P}}(B_{r,r})$, we need to find $\mathrm{Spec}_{\mathscr{P}}(A+B)$ and $\mathrm{Spec}_{\mathscr{P}}(A-B)$ by solving its respective characteristic polynomials.

By applying a series of row and column operations on $\phi_{\mathscr{P}}(A+B,\lambda)$ and $\phi_{\mathscr{P}}(A-B,\lambda)$, and using Lemma 5,

$$\begin{aligned}\phi_{\mathscr{P}}(A+B,\lambda) =& [\lambda-(n+1)]^{(r-2)}\left[\lambda-\frac{(n+5)+\sqrt{n^2+12n-3}}{2}\right]\\ &\left[\lambda-\frac{(n+5)-\sqrt{n^2+12n-3}}{2}\right]\end{aligned} \tag{11}$$

and

$$\begin{aligned}\phi_{\mathscr{P}}(A-B,\lambda) =& [\lambda-(n+1)]^{(r-2)}\left[\lambda-\frac{(2n-1)+3\sqrt{2n-3}}{2}\right]\\ &\left[\lambda-\frac{(2n-1)-3\sqrt{2n-3}}{2}\right].\end{aligned} \tag{12}$$

Therefore from Equations (11) and (12), and by the Lemma 4,

$$\begin{aligned}Spec_{\mathscr{P}}(B_{r,r}) = \Bigg\{&\left[\frac{(n+5)+\sqrt{n^2+12n-3}}{2}\right]^1, \left[\frac{(2n-1)+3\sqrt{2n-3}}{2}\right]^1,\\ &(n+1)^{(n-4)}, \left[\frac{(2n-1)-3\sqrt{2n-3}}{2}\right]^1,\\ &\left[\frac{(n+5)-\sqrt{n^2+12n-3}}{2}\right]^1\Bigg\}.\end{aligned}$$

Hence, $E_{\mathscr{P}_r}(B_{r,r}) = n^2$, for $n \geq 2$. Now, we consider the vertex partition $\mathscr{P} = \{\{v_1\},\{v_2\},\ldots,\{v_n\}\}$ of $B_{r,r}$ and the corresponding $\mathscr{P}$-matrix of $B_{r,r}$ is

$$A_{\mathscr{P}}(B_{r,r}) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 0\\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0\\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0\\ 1 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1\\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 0 & \ldots & 0\\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 1 \end{pmatrix}_{n\times n}.$$

It is of the form $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$. Thus,

$$\phi_{\mathscr{P}}(A+B,\lambda) = (\lambda-1)^{(r-2)}\left[\lambda - \left(\frac{3 \pm \sqrt{4r-3}}{2}\right)\right] \tag{13}$$

and

$$\phi_{\mathscr{P}}(A-B,\lambda) = (\lambda-1)^{(r-2)}\left[\lambda - \left(\frac{1 \pm \sqrt{4r-3}}{2}\right)\right]. \tag{14}$$

Therefore from Lemma 4 and Equations (13) and (14),

$$\begin{aligned} Spec_{\mathscr{P}}(B_{r,r}) = & \Bigg\{ \left(\frac{3+\sqrt{4r-3}}{2}\right)^{1}, \left(\frac{1+\sqrt{4r-3}}{2}\right)^{1}, \\ & \left(\frac{1-\sqrt{4r-3}}{2}\right)^{1}, \left(\frac{3-\sqrt{4r-3}}{2}\right)^{1}, 1^{(n-4)} \Bigg\}. \end{aligned}$$

Hence, $E_{\mathscr{P}_s}(B_{r,r}) = n-1+\sqrt{2n-3}$, for $2 \le n < 8$
and
$E_{\mathscr{P}_s}(B_{r,r}) = (n-4)+2\sqrt{2n-3}$, for $n \ge 8$. □

Theorem 17 *If* $K_{r,r}$ *is a complete bipartite graph of order* $n = 2r \ge 2$, *then*

$$E_{\mathscr{P}_r}(K_{r,r}) = n^2+2n-2 \text{ and } E_{\mathscr{P}_s}(K_{r,r}) = 2(n-1).$$

Proof. Let $K_{r,r}$ be a complete bipartite graph of order n and let $\mathscr{P} = \{V(G)\}$. The $\mathscr{P}$-matrix of $K_{r,r}$ is a 2×2 block matrix which can be represented as $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$.

$$A_{\mathscr{P}}(K_{r,r}) = \begin{pmatrix} [(n+1)I-J]_{r\times r} & [2J]_{r\times r} \\ [2J]_{r\times r} & [(n+1)I-J]_{r\times r} \end{pmatrix}.$$

Therefore, from Lemma 4 its $\mathscr{P}$-spectrum is given by

$$Spec_{\mathscr{P}}(K_{r,r}) = Spec_{\mathscr{P}}(A+B) \cup Spec_{\mathscr{P}}(A-B). \tag{15}$$

By applying successive row and column operations on $\phi_{\mathscr{P}}(A+B,\lambda)$ and $\phi_{\mathscr{P}}(A-B,\lambda)$, and simplifying using Lemma 5, we get

$$\phi_{\mathscr{P}}(A+B,\lambda) = [\lambda-(3r+1)][\lambda-(n+1)]^{(r-1)} \tag{16}$$

and

$$\phi_{\mathscr{P}}(A-B,\lambda) = [\lambda+(r-1)][\lambda-(2n+1)][\lambda-(n+1)]^{(r-2)}. \tag{17}$$

Thus, from Equations (15), (16) and (17)

$$\phi_{\mathscr{P}}(K_{r,r},\lambda) = [\lambda+(r-1)][\lambda-(3r+1)][\lambda-(2n+1)][\lambda-(n+1)]^{(n-3)}.$$

Therefore,

$$\mathrm{Spec}_{\mathscr{P}}(K_{r,r}) = \{(2n+1)^1, (3r+1)^1, (n+1)^{(n-3)}, [-(r-1)]^1\}$$

and

$$E_{\mathscr{P}_r}(K_{r,r}) = n^2+2n-2.$$

Now, if we consider $\mathscr{P} = \{\{v_1\},\{v_2\},\dots,\{v_n\}\}$ as the vertex partition of $K_{r,r}$, then the corresponding $\mathscr{P}$-energy will be shear $\mathscr{P}$-energy of G. So, consider $K_{r,r}$ with respect to $\mathscr{P} = \{\{v_1\},\{v_2\},\dots,\{v_n\}\}$ and the $\mathscr{P}$-matrix of $K_{r,r}$ is $A_{\mathscr{P}}(K_{r,r}) = \left(\begin{smallmatrix} I & J \\ J & I \end{smallmatrix}\right)$. Therefore, from Lemma 4

$$\mathrm{Spec}_{\mathscr{P}}(K_{r,r}) = \mathrm{Spec}_{\mathscr{P}}(I+J) \cup \mathrm{Spec}_{\mathscr{P}}(I-J). \tag{18}$$

By applying successive row and column operations on $\phi_{\mathscr{P}}(A+B,\lambda)$ and $\phi_{\mathscr{P}}(A-B,\lambda)$, and simplifying using Lemma 5, we get the corresponding $\mathscr{P}$-eigenvalues. Thus,

$$\mathrm{Spec}_{\mathscr{P}}(A+B) = \{1^{(r-1)}, (r+1)^1\} \tag{19}$$

and

$$\mathrm{Spec}_{\mathscr{P}}(A-B) = \{1^{(r-1)}, [-(r-1)]^1\}. \tag{20}$$

Therefore, from Equations (18), (19) and (20)

$$\mathrm{Spec}_{\mathscr{P}}(K_{r,r}) = \{(r+1)^1, 1^{(n-2)}, [-(r-1)]^1\}$$

and

$$E_{\mathscr{P}_s}(K_{r,r}) = 2(n-1).$$

□

Remark 18 *We observe that, the robust $\mathscr{P}$-energy of $K_{r,r}$, 2-partition energy of $K_{1,n-1}$ and color energy of $K_{1,n-1}$ with respect to minimum number of colors χ are same, that is $E_{\mathscr{P}_s}(K_{r,r}) = E_{P_2}(K_{1,n-1}) = E_{\chi}(K_{1,n-1})$.*

Now, we proceed to determine $E_{\mathscr{P}}(G)$ for some families of graphs with respect to non-trivial vertex partitions.

Theorem 19 *For the star* $K_{1,n-1}, n \geq 3$*, with vertex partition* $\mathscr{P} = \{\{v_1\}, \{v_2, v_3, \ldots, v_n\}\}$ *where* v_1 *is the central vertex and* $v_2, v_3, \ldots, v_n$ *are pendant vertices of* $K_{1,n-1}$*,*

$$E_{\mathscr{P}}(K_{1,n-1}) = n(n-2) + 2\sqrt{n-1}.$$

Proof. The $\mathscr{P}$-matrix of $K_{1,n-1}$ is

$$A_{\mathscr{P}}(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & n-1 & -1 & \ldots & -1 \\ 1 & -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \ldots & n-1 \end{pmatrix}_{n\times n}.$$

Thus, the characteristic polynomial of $A_{\mathscr{P}}(K_{1,n-1})$ is

$$\phi_{\mathscr{P}}(K_{1,n-1}, \lambda) = (\lambda - n)^{(n-2)}[\lambda - (1 \pm \sqrt{(n-1)})].$$

Hence,

$$\mathrm{Spec}_{\mathscr{P}}(K_{1,n-1}) = \{[1 + \sqrt{(n-1)}]^1, [1 - \sqrt{(n-1)}]^1, n^{(n-2)}\}$$

and

$$\begin{aligned} E_{\mathscr{P}}(K_{1,n-1}) &= n(n-2) + |1 + \sqrt{n-1}| + |1 - \sqrt{n-1}| \\ &= n(n-2) + 2\sqrt{n-1}, \text{ for } n \geq 3. \end{aligned}$$

□

Now, we derive $\mathscr{P}$-energy of a double star $E_{\mathscr{P}}(B_{s,s})$ for different partitions and for that we consider, $V(B_{s,s}) = \{u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_s\}$ such that u_1, v_1 are the maximum degree (central) vertices. Note that, the pendant vertices u_i's are attached to u_1 and the pendant vertices v_i's are attached to v_1, for $i = 2, 3, \ldots, s$.

Theorem 20 *If* $B_{s,s}$ *is a double star of order* $n \geq 6$ *with the vertex partition* $\mathscr{P} = \{\{u_1, v_1\}, \{u_2, u_3, \ldots, u_s, v_2, v_3, \ldots, v_s\}\}$ *where* u_1 *and* v_1 *are the central vertices, then*

$$E_{\mathscr{P}}(B_{s,s}) = \begin{cases} (n-4)(n-1) + \sqrt{n^2-3} + 5 & \textit{for } n = 6 \textit{ and } 8, \\ (n-4)(n-1) + \sqrt{n^2-3} + \frac{1}{2}(5 + \sqrt{2n+5}) & \textit{for } n = 10, \\ (n-4)(n-1) + \sqrt{n^2-3} + \sqrt{2n+5} & \textit{for } n \geq 12. \end{cases}$$

Proof. The $\mathscr{P}$-matrix of $B_{s,s}$ is a 2×2 block circulant matrix which can be represented as $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$. Therefore, by Lemma 4 its spectrum is given by

$$\mathrm{Spec}_{\mathscr{P}}(B_{s,s}) = \mathrm{Spec}_{\mathscr{P}}(A+B) \cup \mathrm{Spec}_{\mathscr{P}}(A-B). \tag{21}$$

By applying successive row and column operations on $\phi_{\mathscr{P}}(A+B,\lambda)$ and $\phi_{\mathscr{P}}(A-B,\lambda)$, we get

$$\mathrm{Spec}_{\mathscr{P}}(A+B) = \left\{(n-1)^{(s-2)}, \left[\frac{5+\sqrt{2n+5}}{2}\right]^1, \left[\frac{5-\sqrt{2n+5}}{2}\right]^1\right\} \tag{22}$$

and

$$\mathrm{Spec}_{\mathscr{P}}(A-B) = \left\{(n-1)^{(s-2)}, \left[\frac{(n-1)+\sqrt{n^2-3}}{2}\right]^1, \left[\frac{(n-1)-\sqrt{n^2-3}}{2}\right]^1\right\} \tag{23}$$

respectively. Therefore, from Equations (21), (22) and (23)

$$\begin{aligned}\mathrm{Spec}_{\mathscr{P}}(B_{s,s}) =& \left\{(n-1)^{(n-4)}, \left[\frac{(n-1)+\sqrt{n^2-3}}{2}\right]^1, \left[\frac{5+\sqrt{2n+5}}{2}\right]^1,\right. \\ & \left.\left[\frac{5-\sqrt{2n+5}}{2}\right]^1, \left[\frac{(n-1)-\sqrt{n^2-3}}{2}\right]^1\right\}\end{aligned}$$

and

$$\begin{aligned}E_{\mathscr{P}}(B_{s,s}) =& (n-4)(n-1) + \left|\frac{5+\sqrt{2n+5}}{2}\right| + \left|\frac{5-\sqrt{2n+5}}{2}\right| \\ & + \left|\frac{(n-1)+\sqrt{n^2-3}}{2}\right| + \left|\frac{(n-1)-\sqrt{n^2-3}}{2}\right|.\end{aligned}$$

Hence from this, the result follows. □

Next, we consider another partition $\mathscr{P}' = \{\{u_1, v_2, v_3, \ldots, v_s\}, \{v_1, u_2, u_3, \ldots, u_s\}\}$ of $V(B_{s,s})$ such that $\{u_1, v_2, v_3, \ldots, v_s\}$ and $\{v_1, u_2, u_3, \ldots, u_s\}$ are two independent sets where u_1, v_1 are the central vertices and $u_2, u_3, \ldots, u_s, v_2, v_3, \ldots, v_s$ are the pendent vertices of $B_{s,s}$.

Theorem 21 *Let $B_{s,s}$ be a double star of order $n \geq 6$ with the vertex partition $\mathscr{P}' = \{\{u_1, v_2, v_3, \ldots, v_s\}, \{v_1, u_2, u_3, \ldots, u_s\}\}$ where u_1, v_1 are the central vertices of $B_{s,s}$. Then*

$$E_{\mathscr{P}'}(B_{s,s}) = \frac{1}{2}\left[n^2 - 2n - 4 + \sqrt{n^2 + 20n - 28}\right].$$

Proof. The $\mathscr{P}$-matrix of $B_{s,s}$ is a 2×2 block circulant matrix which can be represented as $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$. Therefore, its spectrum is given by

$$\text{Spec}_{\mathscr{P}'}(B_{s,s}) = \text{Spec}_{\mathscr{P}'}(A+B) \cup \text{Spec}_{\mathscr{P}'}(A-B). \tag{24}$$

By applying successive row and column operations on $\phi_{\mathscr{P}}(A+B, \lambda)$ and $\phi_{\mathscr{P}}(A-B, \lambda)$, we get

$$\text{Spec}_{\mathscr{P}'}(A+B) = \{(s+1)^{(s-2)}, 2^1\} \tag{25}$$

and

$$\begin{aligned} \text{Spec}_{\mathscr{P}'}(A-B) = &\left\{ \left[\frac{(s+1)+\sqrt{s^2+10s-7}}{2}\right]^1, (s+1)^{(s-2)}, \right. \\ &\left. \left[\frac{(s+1)-\sqrt{s^2+10s-7}}{2}\right]^1 \right\}. \end{aligned} \tag{26}$$

Therefore, from Equations (24), (25) and (26)

$$\begin{aligned} \text{Spec}_{\mathscr{P}'}(B_{s,s}) = &\left\{ \left[\frac{(s+1)+\sqrt{s^2+10s-7}}{2}\right]^1, (s+1)^{(n-4)}, \right. \\ &\left. \left[\frac{(s+1)-\sqrt{s^2+10s-7}}{2}\right]^1, 2^1 \right\} \end{aligned}$$

and

$$\begin{aligned} E_{\mathscr{P}'}(B_{s,s}) = &2 + (s+1)(n-4) + \left|\frac{(s+1)+\sqrt{s^2+10s-7}}{2}\right| + \\ &\left|\frac{(s+1)-\sqrt{s^2+10s-7}}{2}\right|. \end{aligned}$$

On simplifying the above equation, we get the result. □

Another possibility of the vertex partition $\mathscr{P}$ having 2 elements for a $B_{s,s}$ is taking one copy of a star $K_{1,r-1}$ in each of the two elements of $\mathscr{P}$. The next result gives its corresponding $\mathscr{P}$-energy. We omit its proof as it is similar to the proofs of Theorems 20 and 21.

Theorem 22 *Let $B_{s,s}$ be a double star of order $n \geq 6$ with the vertex partition $\mathscr{P}'' = \{\{u_1, u_2, u_3, \ldots, u_s\}, \{v_1, v_2, v_3, \ldots, v_s\}\}$ where u_1, v_1 are the central vertices of $B_{s,s}$. Then*

$$E_{\mathscr{P}''}(B_{s,s}) = \frac{1}{2}\left[n^2 - 2n - 8 + \sqrt{n^2+20n-28} + \sqrt{n^2+28n-60}\right].$$

Remark 23 *From Theorems 20, 21 and 22, we observe that*

$$E_{\mathscr{P}} \geq E_{\mathscr{P}''} \geq E_{\mathscr{P}'}.$$

Theorem 24 *Let $K_{r,r}$ be a complete bipartite graph of order n with a vertex partition $\mathscr{P} = \{V_1, V_2\}$ such that V_1 and V_2 are two partite sets of $K_{r,r}$. Then*

$$E_{\mathscr{P}}(K_{r,r}) = \frac{1}{2}[n^2 + 2n - 4].$$

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_r\}$. The $\mathscr{P}$-matrix of $K_{r,r}$ is

$$A_{\mathscr{P}}(K_{r,r}) = \begin{pmatrix} [(r+1)I - J]_{r\times r} & J_{r\times r} \\ J_{r\times r} & [(r+1)I - J]_{r\times r} \end{pmatrix}_{n\times n}.$$

To get the $\mathscr{P}$-spectra of $K_{r\times r}$, by Lemma 4, it is sufficient to find $\mathscr{P}$-spectra of $[(r+1)I]_{r\times r}$ and $[(r+1)I - 2J]_{r\times r}$. Since $[(r+1)I]_{r\times r}$ is a diagonal matrix,

$$\mathrm{Spec}_{\mathscr{P}}((r+1)I) = \{(r+1)^r\}.$$

After applying a series of row and column operations on $\phi_{\mathscr{P}}([(r+1)I - 2J], \lambda)$ and using Lemma 5, we get the corresponding $\mathscr{P}$-eigenvalues as

$$\mathrm{Spec}_{\mathscr{P}}((r+1)I - 2J) = \{(r+1)^{(r-1)}, [-(r-1)]^1\}.$$

Therefore, from Lemma 4

$$\mathrm{Spec}_{\mathscr{P}}(K_{r,r}) = \{(r+1)^{(n-1)}, [-(r-1)]^1\}$$

and

$$E_{\mathscr{P}}(K_{r,r}) = \frac{1}{2}[n^2 + 2n - 4].$$

□

Theorem 25 *Let $K_{r,r}$ be a complete bipartite graph of order n with bipartite sets $\{u_1, u_2, \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_r\}$ and the vertex partition $\mathscr{P} = \{\{u_i, v_i\},$ for $1 \leq i \leq r\}$. Then*

$$E_{\mathscr{P}}(K_{r,r}) = 3n - 2.$$

Proof. The $\mathscr{P}$-matrix of $K_{r,r}$ is

$$A_{\mathscr{P}}(K_{r,r}) = \begin{pmatrix} 2I_{r\times r} & (J+I)_{r\times r} \\ (J+I)_{r\times r} & 2I_{r\times r} \end{pmatrix}_{n\times n}.$$

Thus, the characteristic polynomial of $A_{\mathscr{P}}(K_{r,r})$ is

$$\phi_{\mathscr{P}}(K_{r,r},\lambda) = [\lambda + (r-1)](\lambda - 1)^{(r-1)}(\lambda - 3)^{(r-1)}[\lambda - (r+3)].$$

Hence,

$$\mathsf{Spec}_{\mathscr{P}}(K_{r,r}) = \{(r+3)^1, 3^{(r-1)}, 1^{(r-1)}, [-(r-1)]^1\}$$

and

$$E_{\mathscr{P}}(K_{r,r}) = 3n - 2.$$

□

Now, consider a graph obtained by removing 1-factor F_1 from a complete bipartite graph $K_{r,r}$ and denote is by $K_{r,r} - F_1$ [3].

Theorem 26 *Let $K_{r,r} - F_1$ be a graph of order $n = 2r$, for $r \geq 3$ with a vertex partition $\mathscr{P} = \{V_1, V_2\}$ such that V_1 and V_2 are two partite sets of $K_{r,r} - F_1$. Then*

$$E_{\mathscr{P}}(K_{r,r} - F_1) = \frac{1}{2}[n^2 + 2n - 8].$$

Proof. The $\mathscr{P}$-matrix of $K_{r,r} - F_1$ for the given vertex partition is

$$A_{\mathscr{P}}(K_{r,r} - F_1) = \begin{pmatrix} [(r+1)I - J]_{r\times r} & (J - I)_{r\times r} \\ (J - I)_{r\times r} & [(r+1)I - J]_{r\times r} \end{pmatrix}_{n\times n}.$$

By Lemma 4 and 5,

$$\mathsf{Spec}_{\mathscr{P}}(K_{r,r} - F_1) = \{(r+2)^{(r-1)}, r^r, [-(r-2)]^1\}.$$

Therefore,

$$E_{\mathscr{P}}(K_{r,r} - F_1) = \frac{1}{2}[n^2 + 2n - 8].$$

□

In the next theorem, we consider the partition $\mathscr{P} = \{\{u_i, v_i\},\ \text{for } 1 \leq i \leq r\}$ and determine the corresponding $\mathscr{P}$-energy for $K_{r,r} - F_1$.

Theorem 27 *Let $K_{r,r} - F_1$ be a graph of order $n = 2r$, for $r \geq 3$ with a vertex partition $\mathscr{P} = \{\{u_i, v_i\},\ \text{for } 1 \leq i \leq r\}$. Then*

$$E_{\mathscr{P}}(K_{r,r} - F_1) = \begin{cases} 2n & \text{for } n = 6 \text{ and } 8, \\ 3n - 8 & \text{for } n > 8. \end{cases}$$

.

Proof. The $\mathscr{P}$-matrix of $K_{r,r} - F_1$ is

$$A_{\mathscr{P}}(K_{r,r} - F_1) = \begin{pmatrix} 2I_{r\times r} & (J-2I)_{r\times r} \\ (J-2I)_{r\times r} & 2I_{r\times r} \end{pmatrix}_{n\times n}.$$

Thus, the characteristic polynomial of $A_{\mathscr{P}}(K_{r,r} - F_1)$ is

$$\phi_{\mathscr{P}}(K_{r,r} - F_1, \lambda) = \lambda^{(r-1)}(\lambda-4)^{(r-1)}(\lambda-r)[\lambda+(r-4)].$$

Therefore,

$$\text{Spec}_{\mathscr{P}}(K_{r,r} - F_1) = \left\{ \frac{n}{2}^1, 4^{(\frac{n}{2}-1)}, 0^{(\frac{n}{2}-1)}, \left[-\left(\frac{n}{2}-4\right)\right]^1 \right\}.$$

Hence, $E_{\mathscr{P}}(K_{r,r} - F_1) = 2n$, for $n = 6, 8$ and $E_{\mathscr{P}}(K_{r,r} - F_1) = 3n - 8$, for $n > 8$. □

5 Conclusion

The significance of $\mathscr{P}$-energy stems from the importance of vertex partition problems in graph theory. As observed from the discussions, the value of $E_{\mathscr{P}}(G)$ depends on factors such as the number of elements in the partition, the nature of the vertex subsets in the partition and the specific properties that determines the partition. In this direction, there is also much scope for extension of the study of the concept of $\mathscr{P}$-energy as we can consider specific vertex partitions such as domatic partitions and equitable degree partitions and study the relation between the corresponding $\mathscr{P}$-energy and other graph parameters.

It is to be noted that there are various algorithms available for partitioning a graph (or a network) [8, 9]. Their applications are well known such as partitioning a network into clusters [15], community detection problem in social sciences etc. [8]. We have observed that, in [9], the authors have presented certain parameters for measuring some key aspects of the network like modularity, z-score etc using quantities such as number of partitions k, and the numbers m_1, m_2 which are mentioned in Observation 1(iv) in a similar context. So by using these algorithms and with the help of $\mathscr{P}$-energy, there is a possibility for developing a tool for a given network to find its specific properties such as spectral clustering or community structures.

References

[1] B. D. Acharya, S. Rao, P. Sumathi, V. Swaminathan, Energy of a set of vertices in a graph, *AKCE Int. J. Graphs. Combin.* **4,** 2 (2007) 144–152. ⇒139

[2] C. Adiga, E. Sampathkumar, M. A. Sriraj, A. S. Shrikanth, Color energy of a graph, *Proc. Jangjeon Math. Soc.* **16,** 3 (2013) 335–351. ⇒138

[3] D. Archdeacon, M. Debowsky, J. Dinitz, H. Gavlas, Cycle systems in the complete bipartite graph minus a one-factor, *Discrete Math.* **284,** 1-3 (2004) 37–43. ⇒155

[4] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980. ⇒137, 139

[5] I. Gutman, The energy of a graph *Ber. Math. Stat. Sekt. Forschungsz. Graz.* **103** (1978) 1–22. ⇒137

[6] I. Gutman, B. Furtula, The total π-electron energy saga, *Croat. Chem. Acta.* **90,** 3 (2017) 359–368. ⇒138

[7] I. Gutman, X. Li, J. Zhang, *Graph Energy*, Springer, New York, 2012. ⇒138

[8] M. Girvan, M. E. J. Newman, Community structure in social and biological networks, *PNAS USA* **99,** 12 (2002) 7821–7826. ⇒156

[9] R. Guimera, L. A. N. Amaral, Functional cartography of complex metabolic networks *nature*, **433,** 7028 (2005) 895–900. ⇒156

[10] M. Marcus, H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Courier Corporation, 1992. ⇒139

[11] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, Siam, 2000. ⇒139

[12] E. Sampathkumar, S. V. Roopa, K. A. Vidya, M. A. Sriraj, Partition energy of a graph, *Proc. Jangjeon Math. Soc.* **18,** 4 (2015) 473–493. ⇒138

[13] E. Sampathkumar, M. A. Sriraj, Vertex labeled/colored graphs, matrices and signed graphs, *J. Comb. Inf. Syst. Sci.* **38** (2013) 113–120. ⇒140

[14] D. B. West, *Introduction to Graph Theory*, Pearson, New Jersey, 2001. ⇒137

[15] M. Zhang, J. Deng, C. V. Fang, X. Zhang, L. J. Lu, Molecular network analysis and applications *Knowledge-Based Bioinformatics: John Wiley and Sons, Ltd.* (2010) 251–287. ⇒156

Received: April 23, 2020 • Revised: June 10, 2020