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Optimization of the Seventh-order Polynomial Interpolation 1P Kernel in the Time Domain

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Abstract.
This paper presents the optimization of the convolutional, seventh-order polynomial, one-parameter, interpolation kernel. In the first part of the paper, the seventh-order kernel is defined, and, after that, the process of the kernel optimization is described. The optimization criterion was the minimization of the interpolation error e. The optimization involved the selection of the optimal value of the kernel parameter α, and it was carried out in the time domain. In the second part of this paper, the experiment, which was realized with the aim of determining the precision of interpolation of the third-order, fifth-order, and the seventh-order interpolation kernels, is described. After that a comparative analysis of the interpolation precision is described. As a measure of the interpolation precision, the mean square error (MSE) was used. The results of the experiment are presented graphically and tabularly. Finally, using a comparative analysis, the precision of interpolation with the kernel, whose parameters were optimized in the time domain, in relation to the kernel, whose parameters were optimized in the spectral do-main, was analyzed. Based on the comparative analysis, a recommendation for the optimal parameter for the seventh-order kernel is given.

Key words and phrases: Convolution, interpolation, polynomial kernel, Taylor series.

1 Introduction

In many areas of digital signal processing (DSP), it is necessary to estimate the value of the discrete signal between two, time or spatial, neighboring samples. Estimation of the value of the discrete signal is performed using a numerical method, which is known as interpolation [1], [2]. Some of the typical cases of DSP, where the application of interpolation is necessary, are: a) image processing (spatial transformations, such as resampling, image dimension change, rotation, geometric deformation [3]–[7] b) speech processing (estimation of the fundamental frequency, emotional and health condition of the speaker [8], ...); c) processing of musical signals (extraction and transcription of solo and bass lines, recognition of chords and their transcription [9], evaluation of the parameters of the played tone, such as intonation, vibrato rate, vibrato extend [10], ...), etc. In the scientific literature, a large number of algorithms for interpolation (Lagrangian, Newtonian, Gaussian, Stirling, Bessel, Chebyshev, ...) are described [11]. The construction of the interpolation function using the described algorithms requires the use of a large number of samples. This increases the order of the interpolation function, which results in a long calculation time, and, because of this, their application in real-time is limited.

One of the methods of interpolation, which is suitable for implementation in DSP, is the so-called convolutional interpolation. The principle of convolutional interpolation is based on the realization of convolution between the discrete signal and the convolutional kernel r. The characteristics of the convolutional interpolation are directly dependent on the characteristics of the interpolation kernel. The ideal interpolation of the band limited signal can be realized with the ideal interpolation kernel, which is of the form $\sin(x)/x$ and denoted as *sinc*. Kernel *sinc* is defined in the interval $(-\infty, +\infty)$. Its spectral characteristic is a rectangular, i.e. box function, which: a) is flat in the pass-band and equal to one, b) is flat in the stop-band and equal to zero, and c) with an ideal slope in the transition band [12]. The interpolation kernel r, with the properties defined in this way, cannot be practically realized. The solution, which is self-evident, is to truncate the length of the kernel *sinc* to the final length L by applying the rectangular window function. This process is known as windovization. Truncate *sinc* kernel is, in the scientific literature, denoted by *sincw*. However, the shortening of the kernel leads to the degradation of the characteristic of the kernel *sinc*, which has: a) a ripple in the pass-band and stop-band, and b) a finite slope in the transition band. Therefore, convolutional interpolation with truncate kernel *sincw*, leads to a decrease of the interpolation precision. Because of all, this, in the last thirty years, intensive work has been done on the construction of the interpolation kernel r, of finite length L, which will be: a) good approximation of the *sinc* kernel in the time-space domain and spectral domain, and b) numerically simple, that is, it should be created from a relatively simple mathematical function, in order to reduce the interpolation execution time.

In recent decades, low-order polynomials ($n \leq 7$) have been intensively used for the construction of the interpolation kernel [13]. Numerically the simplest is the polynomial zeroth-order kernel [14]. Interpolation is performed by rounding to the nearest-neighbor sample [15], [16] . In addition to the fact that the interpolation time is very short, the interpolation error is huge. A linear, polynomial first-order interpolation kernel is described in [17]. A cubic polynomial third-order interpolation kernel is described in [18]. Convolutional interpolation using the third-order kernel is more precise than interpolation using the polynomial zeroth-order and first-order interpolation kernels. The parameterization of the polynomial third-order kernel was proposed by Robert Keys in [19]. The parameterization was performed in such a way that one of the coefficients of the kernel was replaced by the parameter α. By changing the parameter α, it is possible to influence the precision of interpolation. Later, in the scientific literature, third-order polynomial interpolation kernel, in honor of the author who proposed it, the one-parameter Keys (1P Keys) kernel was named. In addition, in [19] the optimization of the alpha parameter α in the time domain is shown ($\alpha^t_{opt,3}$ = -1/2). The optimization was performed with the criterion that the Taylor expansions of both, the interpolated function and the interpolation kernel, are equal up to the second term. In [20] the optimization of the 1P Keys kernel in the spectral domain is shown. The optimization criterion was minimization of the ripple of the spectral characteristic in pass-band and stop-band. In this way, the optimal value of the kernel parameter ($\alpha^f_{opt,3}$ = -1/2) was determined. It can be seen that for the third-order kernel, the optimal parameter is the same for optimization in the time and spectral domains. With the idea of increasing the precision of the interpolation, third-order polynomial two-parameter 2P (α, β) [21] and three-parameter 3P (α, β, γ) [22] kernels were constructed. A fifth-order polynomial one-parameter interpolation kernel, whose length is $L = 6$, is described in [20]. Optimization of the fifth-order kernel in the spectral domain, the optimal kernel parameter ($\alpha^f_{opt,5}$ = 3/64), was calculated. In [23] optimization of the kernel in the time domain was performed ($\alpha^f_{opt,5}$ = 3/64). In [24] the parameterization of a two-parameter fifth-order interpolation kernel, length $L = 8$, is described. The spectral characteristics of this kernel are described in [25]. The optimization of this kernel is performed in the spectral domain ($\alpha^f_{opt,5}$ = 171 / 1408, $\beta^f_{opt,5}$ = 525 / 7744) [26]. A seventh-order polynomial 1P kernel is described in [20]. In addition, the optimization of the 1P kernel in the spectral domain ($\alpha^f_{opt,7}$ = -71/83232) is described.

In this paper, the results of optimization of the seventh-order polynomial one-parameter kernel [20] are presented. The optimization was realized in the time domain. As an optimization criterion, the minimization of the interpolation error e was applied. The first part presents the optimization algorithm. First, the interpolation function g is determined. After that, assuming that the function f, which is to be interpolated, has at least six continuous derivatives in the interval where the interpolation is performed, the development of the function f in Taylor series is performed. The Taylor series has been expanded to the sixth term. Then, the interpolation error $e = f - g$, was formed. Finally, the minimization of the interpolation error was realized, so that the interpolated function f and interpolation function g agree up to the sixth term in the Taylor series expansion. The minimization of the interpolation error was achieved by minimizing the 27 coefficients in the Taylor series expansion. In this way, a system of 27 equations, with one variable, was formed. In that case, it is not possible to find a unique solution, and, there-fore, the least squares method (LSM) was applied. As a result of applying LSM, the optimal kernel parameter α_{opt} was calculated. With the aim of verifying the correctness of the choice of the optimal kernel parameters, an experiment was carried out. First, the algorithm, according to which the experiment was realized, is described, and four test functions f_1, ..., f_4, which represent signals with complex time form, are created. After that, the test functions are interpolated using convolutional interpolation with one parameter: a) third-order kernel, which is optimized in the spectral and time domain [19], [20], b) fifth-order kernel, which is optimized in the spectral and time domain [20], [23], c) seventh-order kernel, which is optimized in the spectral domain [20], and d) seventh-order kernel, which is optimized in the time domain, and whose optimization is presented in this paper. Then, the interpolation errors e and mean square errors MSE, for the case of interpolating test functions, were calculated. Finally, a comparative analysis of the interpolation precision of the kernel that was optimized in this paper, using optimization in the time domain, with kernels whose optimizations were performed in the spectral domain, was performed. As a measure of interpolation precision MSE was used. The results of the experiment are presented using graphs and tables.

Further organization of this paper is as follows. In Section 2, the seventh-order 1P interpolation kernel is described. Section 3 describes the kernel optimization in the time domain. In Section 4, the experiment is described, the results presented and a comparative analysis performed. Section 5 is the Conclusion.

2 Seventh-order Polynomial One-parameter Kernel

The construction of the seventh-order polynomial one-parameter kernel shown in [20]. The 1P kernel is defined on the interval (-4, 4) and approximates the ideal *sinc* interpolation kernel. Outside of the interval (-4, 4) the interpolation kernel is zero. The 1P kernel is composed of piecewise seventh-order polynomials, which are defined on the subintervals (-4, -3), (-3, -2), (-2, -1), (-1,0), (0,1), (1, 2), (2, 3) and (3, 4). Therefore, the length of the kernel is $L = 8$. The kernel r is defined by:

$$r(s) = \begin{cases} a_{70}|s|^7 + \ldots + a_{10}|s| + a_{00}, & |s| \leq 1 \\ a_{71}|s|^7 + \ldots + a_{11}|s| + a_{01}, & 1 < |s| \leq 2 \\ a_{72}|s|^7 + \ldots + a_{12}|s| + a_{02}, & 2 < |s| \leq 3 \\ a_{73}|s|^7 + \ldots + a_{13}|s| + a_{03}, & 3 < |s| \leq 4 \\ 0, & \text{otherwise} \end{cases} . \tag{1}$$

The coefficients a_{ij}, where $0 \leq i \leq 7$ and $0 \leq j \leq 3$, are determined from the conditions: a) $r(0) = 1$ and $r(s) = 0$ for $|s| = 1, 2, 3$; and b) $r^{(l)}(s)$ must be continuous at $|s| = 0, 1, 2, 3, 4$ for $l = 0, 1, 2, 3, 4, 5$.

In order to satisfy the set conditions, based on the definition of the kernel, 31 equations with 32 unknown coefficients a_{ij} were formed. The system of equations formed in this way cannot be solved unambiguously. By parametrizing the kernel, that is, introducing the parameter α, and setting the parameter $a_{73} = \alpha$, the system of equations can be solved. The coefficient values were calculated: $a_{70} = 245\alpha + 821/1734$, $a_{60} = -621\alpha - 1148/867$, $a_{50} = 0$, $a_{40} = 760\alpha + 1960/867$, $a_{30} = 0$, $a_{20} = -384\alpha - 1393/578$, $a_{10} = 0$, $a_{00} = 1$. $a_{71} = 301\alpha + 1687/6936$, $a_{61} = -3309\alpha - 2492/867$, $a_{51} = 14952\alpha + 32683/2312$, $a_{41} = -35640\alpha - 128695/3468$, $a_{31} = 47880\alpha + 127575/2312$, $a_{21} = -36000\alpha - 13006/289$, $a_{11} = 14168\alpha + 120407/6936$, $a_{01} = -2352\alpha - 2233/1156$, $a_{72} = 57\alpha + 35/6936$, $a_{62} = -1083\alpha - 175/1734$, $a_{52} = 8736\alpha + 1995/2312$, $a_{42} = -38720\alpha - 4725/1156$, $a_{32} = 101640\alpha + 1575/136$, $a_{22} = -157632\alpha - 5670/289$, $a_{12} = 133336\alpha + 42525/2312$, $a_{02} = -47280\alpha - 8505/1156$, $a_{73} = 1\alpha$, $a_{63} = -27\alpha$, $a_{53} = 312\alpha$, $a_{43} = -2000\alpha$, $a_{33} = 7680\alpha$, $a_{23} = -17664\alpha$, $a_{13} = 22528\alpha$, $a_{03} = -12288\alpha$.

The kernel parameter α directly affects the time-spectral characteristics of the 1P kernel. Changing the value of the kernel parameter affects on the interpolation precision. By minimizing the interpolation error e, it is possible to determine the optimal value of the kernel parameter, and, in this way, optimize the interpolation kernel r. It is possible to optimize the interpolation kernel in: a) spectral and b) time domain. Optimization in the spectral domain implies the minimization of the difference between the amplitude spectral characteristics of the ideal kernel *sinc*, whose characteristic is the box function, H_{sinc}, and the analyzed interpolation 1P kernel *r*, whose spectral characteristic is *H*. The paper [20] describes the optimization of the 1P kernel in the spectral domain ($\alpha^{f}_{opt,7} = -71/83232$). The optimization criterion was the minimization of the ripple of the spectral characteristic *H*. In the further part of this paper, the interpolation kernel, optimized in the spectral domain, will be denoted by , and its spectral characteristic by r^{f}_{opt}, and its spectral characteristic by H^{f}_{opt}.

In the rest of this paper, the optimization of the polynomial seventh-order 1P kernel, which was performed in the time domain, is presented. The optimization criterion was the minimization of the interpolation error *e*.

3 Optimization of the 1P Kernel in the Time Domain

The interpolation function $g(x)$ is a special type of approximation function. Its fundamental property is that it is equal to the sampled data, that is, the values of the function $f(x)$ in the interpolation nodes. Then $g(x_k) = f(x_k)$, where $0 \le k \le N1$, and N is the total number of interpolation nodes, in the segment where the function is interpolated. Let us assume that x is a point, in which the interpolation of the function $f(x)$ should be performed. Let x be between two consecutive interpolation nodes, denoted as x_j and x_{j+1}. Let $s = (x - x_j)/h$, where h is the sampling increment. Then $(x - x_k)/h = (x - x_j + x_j - x_k)/h = s + j + k$. The interpolation, that is, the reconstructed function $g(x)$, is determined by convolutional interpolation [19], [23], [27] of the interpolation function $f(x)$ with the interpolation kernel r:

$$g\left(x\right) = \sum_{k} c_k r\left(\frac{x - x_k}{h}\right) = \sum_{k} c_k r\left(s + j - k\right), \tag{2}$$

where c_k is the value of the function $f(x)$ in the interpolation k-th node (k-th sample), and h is the sampling increment. By developing the sum from Equation 2, the reconstruction function can be written as:

$$\begin{aligned} g\left(x\right) = {} & c_{j-3} r\left(s+3\right) + c_{j-2} r\left(s+2\right) + c_{j-1} r\left(s+1\right) + c_j r\left(s\right) \\ & + c_{j+1} r\left(s-1\right) + c_{j+2} r\left(s-2\right) + c_{j+3} r\left(s-3\right) + c_{j+4} r\left(s-4\right), \end{aligned} \tag{3}$$

The value of kernel r, for the segment is $-4 \le s \le -3$, is:

$$r\left(s+3\right) = \alpha\, s^7 - 6\alpha s^6 + 15\alpha s^5 - 20\alpha s^4 + 15\alpha s^3 - 6\alpha s^2 + \alpha s, \tag{4}$$

Continuing this procedure, the kernel values in the other segments are determined:

$$
\begin{aligned}
r\,(s+2) = {} & \left(57\,\alpha + \frac{35}{6936}\right) s^7 + \left(-285\,\alpha - \frac{35}{1156}\right) s^6 + \left(528\,\alpha + \frac{175}{2312}\right) s^5 \\
& + \left(-380\,\alpha - \frac{175}{1734}\right) s^4 + \left(\frac{175}{2312} - 40\,\alpha\right) s^3 + \left(192\,\alpha - \frac{35}{1156}\right) s^2 \\
& + \left(\frac{35}{6936} - 72\,\alpha\right) s,
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
r\,(s+1) = {} & \left(301\,\alpha + \frac{1687}{6936}\right) s^7 + \left(-1202\,\alpha - \frac{2709}{2312}\right) s^6 + \left(1419\,\alpha + \frac{1155}{578}\right) s^5 \\
& + \left(20\,\alpha - \frac{35}{34}\right) s^4 + \left(-805\,\alpha - \frac{1505}{1734}\right) s^3 + \left(6\,\alpha + \frac{21}{17}\right) s^2 \\
& + \left(261\,\alpha - \frac{707}{1734}\right) s + \frac{725}{388} \cdot 10^{-15},
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
r\,(s) = {} & \left(245\,\alpha + \frac{821}{1734}\right) s^7 + \left(-621\,\alpha - \frac{1148}{867}\right) s^6 \\
& + \left(760\,\alpha + \frac{1960}{867}\right) s^4 + \left(-384\,\alpha - \frac{1393}{578}\right) s^2 + 1,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
r\,(s-1) = {} & \left(-245\,\alpha - \frac{821}{1734}\right) s^7 + \left(1094\,\alpha + \frac{203}{102}\right) s^6 + \left(-1419\,\alpha - \frac{1155}{578}\right) s^5 \\
& + \left(20\,\alpha - \frac{35}{34}\right) s^4 + \left(805\,\alpha + \frac{1505}{1734}\right) s^3 + \left(6\,\alpha + \frac{21}{17}\right) s^2 \\
& + \left(\frac{707}{1734} - 261\,\alpha\right) s,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
r\,(s-2) = {} & \left(-301\,\alpha - \frac{1687}{6936}\right) s^7 + \left(905\,\alpha + \frac{1841}{3468}\right) s^6 + \left(-528\,\alpha - \frac{175}{2312}\right) s^5 \\
& + \left(-380\,\alpha - \frac{175}{1734}\right) s^4 + \left(40\,\alpha - \frac{175}{2312}\right) s^3 + \left(192\,\alpha - \frac{35}{1156}\right) s^2 \\
& + \left(72\,\alpha - \frac{35}{6936}\right) s + \frac{822}{295} \cdot 10^{-14},
\end{aligned} \tag{9}
$$

$$r\left(s-3\right)=\left(-57\,\alpha-\frac{35}{6936}\right)\,s^{7}+\left(114\,\alpha+\frac{35}{6936}\right)\,s^{6}-15\alpha s^{5}\\-20\,\alpha s^{4}-15\,\alpha s^{3}-6\alpha s^{2}-\alpha s, \tag{10}$$

$$r\left(s-4\right)=-\alpha s^{7}+\alpha\,s^{6}, \tag{11}$$

Substituting Equations 4 - 11 in Equation 2 the interpolation function is written in the form:

$$g\left(x\right)=A_{7}s^{7}+A_{6}s^{6}+A_{5}s^{5}+A_{4}s^{4}+A_{3}s^{3}+A_{2}s^{2}+A_{1}s+A_{0}, \tag{12}$$

where $A_7 = 821/1734c_j + 1687/6936c_{j-1} + 35/6936c_{j-2} - 821/1734c_{j+1} - 1687/6936c_{j+2} - 35/6936c_{j+3} + 245\alpha c_j + 301\alpha c_{j-1} + 57\alpha c_{j-2} + \alpha c_{j-3} - 245\alpha c_{j+1} - 301\alpha c_{j+2} - 57\alpha c_{j+3} - \alpha c_{j+4}$; $A_6 = 203/102c_{j+1} - 2709/2312c_{j-1} - 35/1156c_{j-2} - 1148/867c_j + 1841/3468c_{j+2} + 35/6936c_{j+3} - 621\alpha c_j - 1202\alpha c_{j-1} - 285\alpha c_{j-2} - 6\alpha c_{j-3} + 1094\alpha c_{j+1} + 905\alpha c_{j+2} + 114\alpha c_{j+3} + \alpha c_{j+4}$; $A_5 = 1155/578c_{j-1} + 175/2312c_{j-2} - 1155/578c_{j+1} - 175/2312c_{j+2} + 1419\alpha c_{j-1} + 528\alpha c_{j-2} + 15\alpha c_{j-3} - 1419\alpha c_{j+1} - 528\alpha c_{j+2} - 15\alpha c_{j+3}$; $A_4 = 1960/867c_j - 35/34c_{j-1} - 175/1734c_{j-2} - 35/34c_{j+1} - 175/1734c_{j+2} + 760\alpha c_j + 20c_{j-1} - 380\alpha c_{j-2} - 20\alpha c_{j-3} + 20\alpha c_{j+1} - 380\alpha c_{j+2} - 20\alpha c_{j+3}$; $A_3 = 175/2312c_{j-2} - 1505/1734c_{j-1} + 1505/1734c_{j+1} - 175/2312c_{j+2} - 805\alpha c_{j-1} - 40\alpha c_{j-2} + 15\alpha c_{j-3} + 805\alpha c_{j+1} + 40\alpha c_{j+2} - 15\alpha c_{j+3}$; $A_2 = 21/17c_{j-1} - 1393/578c_j - 35/1156c_{j-2} + 21/17c_{j+1} - 35/1156c_{j+2} - 384\alpha c_j + 6\alpha c_{j-1} + 192\alpha c_{j-2} - 6\alpha c_{j-3} + 6\alpha c_{j+1} + 192\alpha c_{j+2} - 6\alpha c_{j+3}$; $A_1 = 35/6936c_{j-2} - 707/1734c_{j-1} + 707/1734c_{j+1} - 35/6936c_{j+2} + 261\alpha c_{j-1} - 72\alpha c_{j-2} + \alpha c_{j-3} - 261\alpha c_{j+1} + 72\alpha c_{j+2} - \alpha c_{j+3}$; and $A_0 = c_j + 725/388 \cdot 10^{-15}c_{j-1} + 822/295 \cdot 10^{-14}c_{j+2}$.

Assuming that the function $f(x)$ has at least seven continuous derivatives in the interval (x_j, x_{j+1}), then, by applying Taylor's theorem, the value of the function in x_{j+1} is calculated. With the earlier condition on the equality of the interpolation function g with the function f in the k-th interpolation nodes, the coefficients c from Equation 2 are written in the form:

$$c_{j-3}=f(x_{j-3})=\frac{81}{80}\,f^{(6)}\left(x_{j}\right)h^{6}+\frac{81}{40}\,f^{(5)}\left(x_{j}\right)h^{5}+\frac{27}{8}\,f^{(4)}\left(x_{j}\right)h^{4}\\+\frac{9}{2}\,f^{(3)}\left(x_{j}\right)h^{3}+\frac{9}{2}\,f^{(2)}\left(x_{j}\right)h^{2}+3f^{(1)}\left(x_{j}\right)h+f\left(x_{j}\right), \tag{13}$$

$$c_{j-2} = f(x_{j-2}) = \frac{4}{45} f^{(6)}\left(x_j\right) h^6 + \frac{4}{15} f^{(5)}\left(x_j\right) h^5 + \frac{2}{3} f^{(4)}\left(x_j\right) h^4 + \frac{4}{3} f^{(3)}\left(x_j\right) h^3 + 2f^{(2)}\left(x_j\right) h^2 + 2f^{(1)}\left(x_j\right) h + f(x_j), \tag{14}$$

$$c_{j-1} = f(x_{j-1}) = \frac{1}{720} f^{(6)}\left(x_j\right) h^6 + \frac{1}{120} f^{(5)}\left(x_j\right) h^5 + \frac{1}{24} f^{(4)}\left(x_j\right) h^4 + \frac{1}{6} f^{(3)}\left(x_j\right) h^3 + \frac{1}{2} f^{(2)}\left(x_j\right) h^2 + f^{(1)}\left(x_j\right) h + f\left(x_j\right), \tag{15}$$

$$c_j = f\left(x_j\right), \tag{16}$$

$$c_{j+1} = f(x_{j+1}) = \frac{1}{720} f^{(6)}\left(x_j\right) h^6 + \frac{1}{120} f^{(5)}\left(x_j\right) h^5 + \frac{1}{24} f^{(4)}\left(x_j\right) h^4 + \frac{1}{6} f^{(3)}\left(x_j\right) h^3 + \frac{1}{2} f^{(2)}\left(x_j\right) h^2 + f^{(1)}\left(x_j\right) h + f\left(x_j\right), \tag{17}$$

$$c_{j+2} = f(x_{j+2}) = \frac{4}{45} f^{(6)}\left(x_j\right) h^6 + \frac{4}{15} f^{(5)}\left(x_j\right) h^5 + \frac{2}{3} f^{(4)}\left(x_j\right) h^4 + \frac{4}{3} f^{(3)}\left(x_j\right) h^3 + 2f^{(2)}\left(x_j\right) h^2 + 2f^{(1)}\left(x_j\right) h + f\left(x_j\right), \tag{18}$$

$$c_{j+3} = f(x_{j+3}) = \frac{81}{80} f^{(6)}\left(x_j\right) h^6 + \frac{81}{40} f^{(5)}\left(x_j\right) h^5 + \frac{27}{8} f^{(4)}\left(x_j\right) h^4 + \frac{9}{2} f^{(3)}\left(x_j\right) h^3 + \frac{9}{2} f^{(2)}\left(x_j\right) h^2 + 3f^{(1)}(x_j)h + f\left(x_j\right), \tag{19}$$

$$c_{j+4} = f(x_{j+4}) = \frac{256}{45} f^{(6)}\left(x_j\right) h^6 + \frac{128}{15} f^{(5)}\left(x_j\right) h^5 + \frac{32}{3} f^{(4)}\left(x_j\right) h^4 + \frac{32}{3} f^{(3)}\left(x_j\right) h^3 + 8f^{(2)}\left(x_j\right) h^2 + 4f^{(1)}\left(x_j\right) h + f\left(x_j\right), \tag{20}$$

The expansion of the function f into Taylor series is obtained:

$$f(x) = \frac{s^6}{720} f^{(6)}\left(x_j\right) h^6 + \frac{s^5}{120} f^{(5)}\left(x_j\right) h^5 + \frac{s^4}{24} f^{(4)}\left(x_j\right) h^4 + \frac{s^3}{6} f^{(3)}\left(x_j\right) h^3 + \frac{s^2}{2} f^{(2)}\left(x_j\right) h^2 + s f^{(1)}\left(x_j\right) h + f\left(x_j\right), \tag{21}$$

The interpolation error is:

$$e = f - g = C_7 s^7 + C_6 s^6 + C_5 s^5 + C_4 s^4 + C_3 s^3 + C_2 s^2 + C_1 s + C_0, \tag{22}$$

where are the coefficients:

$$\begin{aligned} C_7 = {} & D_{7,6} h^6 f^{(6)} \left(x_j\right), + D_{7,5} h^5 f^{(5)} \left(x_j\right) + D_{7,4} h^4 f^{(4)} \left(x_j\right) + D_{7,3} h^3 f^{(3)} \left(x_j\right) \\ & + D_{7,2} h^2 f^{(2)} \left(x_j\right) + D_{7,1} h f^{(1)} \left(x_j\right), \end{aligned} \tag{23}$$

$$\begin{aligned} C_6 = {} & D_{6,6} h^6 f^{(6)} \left(x_j\right) + D_{6,5} h^5 f^{(5)} \left(x_j\right) + D_{6,4} h^4 f^{(4)} \left(x_j\right) + D_{6,3} h^3 f^{(3)} \left(x_j\right) \\ & + D_{6,2} h^2 f^{(2)} \left(x_j\right) + D_{6,1} h f^{(1)} \left(x_j\right), \end{aligned} \tag{24}$$

$$C_5 = D_{5,5} h^5 f^{(5)} \left(x_j\right) + D_{5,3} h^3 f^{(3)} \left(x_j\right) + D_{5,1} h f^{(1)} \left(x_j\right), \tag{25}$$

$$\begin{aligned} C_4 = {} & D_{4,6} h^6 f^{(6)} \left(x_j\right) - 1997/385 \cdot 10^{-16} \cdot h^5 f^{(5)} \left(x_j\right) + D_{4,4} h^4 f^{(4)} \left(x_j\right) \\ & - 2209/943 \cdot 10^{-15} \cdot h^3 f^{(3)} \left(x_j\right) + D_{4,2} h^2 f^{(2)} \left(x_j\right) \\ & - 4588/2285 \cdot 10^{-15} \cdot h f^{(1)} \left(x_j\right) - 4843/1206 \cdot 10^{-15} \cdot f \left(x_j\right), \end{aligned} \tag{26}$$

$$\begin{aligned} C_3 = {} & 3838/2825 \cdot 10^{-15} \cdot h^6 f^{(6)} \left(x_j\right) + D_{3,5} h^5 f^{(5)} \left(x_j\right) \\ & + 4631/465 \cdot 10^{-15} \cdot h^4 f^{(4)} \left(x_j\right) + D_{3,3} h^3 f^{(3)} \left(x_j\right) \\ & + 1231/454 \cdot 10^{-14} \cdot h^2 f^{(2)} \left(x_j\right) + D_{3,1} h f^{(1)} \left(x_j\right) + 4843/603 \cdot 10^{-15} \cdot f \left(x_j\right), \end{aligned} \tag{27}$$

$$\begin{aligned} C_2 = {} & D_{2,6} h^6 f^{(6)} \left(x_j\right) - 2424/2071 \cdot 10^{-14} \cdot h^5 f^{(5)} \left(x_j\right) + D_{2,4} h^4 f^{(4)} \left(x_j\right) \\ & - 2869/501 \cdot 10^{-14} \cdot h^3 f^{(3)} \left(x_j\right) + D_{2,2} h^2 f^{(2)} \left(x_j\right) \\ & - 4529/578 \cdot 10^{-14} \cdot h f^{(1)} \left(x_j\right) - 1948/359 \cdot 10^{-14} \cdot f \left(x_j\right), \end{aligned} \tag{28}$$

$$\begin{aligned} C_1 = {} & 3095/614 \cdot 10^{-15} \cdot h^6 f^{(6)} \left(x_j\right) + D_{1,5} h^5 f^{(5)} \left(x_j\right) + D_{1,3} h^3 f^{(3)} \left(x_j\right) \\ & + 6377/1696 \cdot 10^{-14} \cdot h^4 f^{(4)} \left(x_j\right) + 651/590 \cdot 10^{-13} \cdot h^2 f^{(2)} \left(x_j\right) \\ & + D_{1,1} h f^{(1)} \left(x_j\right) + 2246/447 \cdot 10^{-14} \cdot f \left(x_j\right), \end{aligned} \tag{29}$$

$$
\begin{aligned}
C_0 = &-1748/705 \cdot 10^{-15} \cdot h^6 f^{(6)}\left(x_j\right) - 1787/241 \cdot 10^{-15} \cdot h^5 f^{(5)}\left(x_j\right) \\
&- 1192/639 \cdot 10^{-14} \cdot h^4 f^{(4)}\left(x_j\right) - 1971/535 \cdot 10^{-14} \cdot h^3 f^{(3)}\left(x_j\right) \\
&- 5332/941 \cdot 10^{-14} \cdot h^2 f^{(2)}\left(x_j\right) - 1465/272 \cdot 10^{-14} \cdot h f^{(1)}\left(x_j\right) \\
&- 2227/749 \cdot 10^{-14} \cdot f\left(x_j\right),
\end{aligned} \tag{30}
$$

where are they: $D_{7,6} = 123/4624+84\alpha$, $D_{7,5} = 381/4624+226\alpha$, $D_{7,4} = 643/3468+360\alpha$, $D_{7,3} = 547/1156 + 840\alpha$, $D_{7,2} = 355/578 + 720\alpha$, $D_{7,1} = 355/289 + 1440\alpha$, $D_{6,6} = -3851/78030-170\alpha$, $D_{6,5} = -861/4624-588\alpha$, $D_{6,4} = -1001/2601-784\alpha$, $D_{6,3} = -4501/3468 - 2520\alpha$, $D_{6,2} = -2485/1734 - 1680\alpha$, $D_{6,1} = -2485/578 - 5040\alpha$, $D_{5,5} = 237/2890 + 366\alpha$, $D_{5,3} = 1505/1734 + 2016\alpha$, $D_{5,1} = 2485/578 + 5040\alpha$, $D_{4,6} = 257/12355 + 108\alpha$, $D_{4,4} = 398/1519 + 640\alpha$, $D_{4,2} = 1446/1009 + 1680\alpha$, $D_{3,5} = 124/4787 + 26\alpha$, $D_{3,3} = 206/2601 - 240\alpha$, $D_{3,1} = -1446/1009 - 1680\alpha$, $D_{2,6} = 41/21013-22\alpha$, $D_{2,4} = -217/3468-216\alpha$, $D_{2,2} = -355/578-720\alpha$, $D_{1,5} = -75/18274 - 30\alpha$, $D_{1,3} = -637/5202 - 96\alpha$, $D_{1,1} = 355/1734 + 240\alpha$.

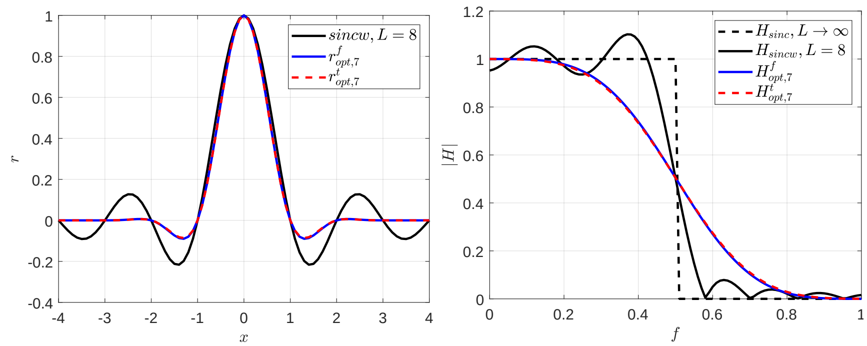
Minimization of the interpolation error e (Equation 22) is done by choosing the appropriate kernel parameter α. This means that the coefficients of the Equations 23 - 30 should be equal to zero. In this way, a system of 27 equations with one unknown was formed. In that case, it is not possible to find a unique solution, and, therefore, the least squares method (LSM) was applied. As a result of applying LSM, optimal kernel parameter was calculated: $\alpha^f_{opt,7} = -22/27931$.

In Figure 1.a shows the time forms of: a) ideal windows interpolation kernel *sincw*, length $L = 8$, which is windowed using a rectangular window, on the segment (-4, 4); b) the seventh-order polynomial 1P kernel $r^f_{opt,7}$, which is optimized in the spectral domain [20], with the criterion of minimizing the ripple of the spectral characteristics ($\alpha^f_{opt,7} = -71/83232$), and c) the seventh-order polynomial 1P kernel, which is optimized in the time domain (Section 3), ($\alpha^f_{opt,7} = -22/27931$). In Figure 1.b shows the spectral characteristics: a) ideal interpolation kernel *sinc*, H_{sinc} ($L \to \infty$), b) windowized ideal kernel H_{sincw}, ($L = 8$), c) 1P kernel $H^f_{opt,7}$, that is optimized in the spectral domain, and d) 1P kernel $H^t_{opt,7}$, that is optimized in the time domain.

4 Experimental Results and Analysis

4.1 Experiment

The precision of the convolutional interpolation was evaluated by experiment. The convolutional interpolation was realized by applying the polynomial, one-parameter



a) b)

Figure 1: a) Time forms of: the ideal interpolation kernel *sinc*, seventh-order polynomial 1P kernel optimized in the spectral domain $r^f_{opt,7}$ and time domain $r^t_{opt,7}$, on the interval (-4, 4); b) Spectral characteristics of: ideal interpolation kernel H_{sinc}, windowized ideal kernel H_{sincw}, 1P kernel optimized in the spectral domain $H^f_{opt,7}$, and 1P kernel optimized in the time domain $H^t_{opt,7}$.

(1P), interpolation kernels, namely: a) third-order kernel where the optimization of the parameter was performed in the spectral and time domain ($\alpha^{f,t}_{opt,3} = -0.5$) [19], [20], b) fifth-order kernel where the optimization of the parameter was performed in the spectral and time domain ($\alpha^{f,t}_{opt,5} = -3/64$) [20], [23], c) seventh-order kernel where the optimization of the parameter was performed in the spectral domain ($\alpha^{f}_{opt,7} = -71/83232$) [20], and d) seventh-order kernel where the optimization of the parameter was performed in the time domain ($\alpha^{t}_{opt,7} = -22/27931$) (Section III). For the purpose of comparative analysis of the interpolation precision, the mean square error MSE as a measure of the interpolation precision, was used.

The algorithm for calculating the interpolation precision, which is MSE, and the optimal kernel parameter α_{opt}, for the test function *f*, was implemented in the following steps:

Input: r - interpolation kernel, L - length of interpolation kernel, α_L and α_H kernel parameter boundaries, $\Delta\alpha$ kernel parameter step, f - test function, K_L and K_H - segment boundaries, h - sampling period, Δx - interpolation period

Output: MSE_{min}, α_{opt}

Step 1: The test function f is sampled on the segment (K_L, K_H) in N interpolation nodes with a uniform sampling period h, where $N = (K_H - K_L - L)/h$.

FOR $\alpha = \alpha_L : \Delta\alpha : \alpha_H$

Step 2: The test function f is interpolated in the K interpolation points, with an interpolation period Δx, where $K = (K_H - K_L - L)/\Delta x$. Interpolation was performed using convolutional interpolation:

$$g_{k,\alpha} = \sum_{i=n-L/2}^{n+L/2} f_n \cdot r_\alpha(k-i), n \leq k \leq n+1, \tag{31}$$

where f_n is the n-th sample of the test function on the segment (K_L, K_H), r_α is the interpolation kernel with the kernel parameter α, and L is the length of the kernel.

Step 3: In each interpolation point k, the interpolation error $e_{k,\alpha} = |f_k - g_{k,\alpha}|$ was calculated.

Step 4: Mean squared error:

$$MSE_\alpha = \frac{1}{K}\sum_{n=1}^{K} e_{k,\alpha}^2, \tag{32}$$

is calculated.

END α

Step 5: The minimum root mean square error MSE was calculated:

$$MSE_{min} = \min(MSE_\alpha), \tag{33}$$

and the optimal kernel parameter, that corresponds to the minimum interpolation error, was calculated:

$$\alpha_{opt} = \arg\min_\alpha(MSE_\alpha). \tag{34}$$

Applying the described algorithm to each test function f_i, where $i = 1, \ldots, M$, the mean interpolation error $\overline{\mathrm{MSE}} = 1/M \cdot \sum_{i=1}^{M} MSE_i$, was calculated. In this way, it is possible to perform a comparative analysis of the precision of interpolation, for all tested interpolation kernels. The test functions used in the experiment are:

$$f_1(x) = 1.5\sin\left(\frac{x}{2\pi}\right) + \sin\left(\frac{x^2}{2\pi}\right), \tag{35}$$

$$f_2(x) = 10^{-3}(x-10)(x-15)(x-35)\sin\left(\frac{x}{\pi}\right). \tag{36}$$

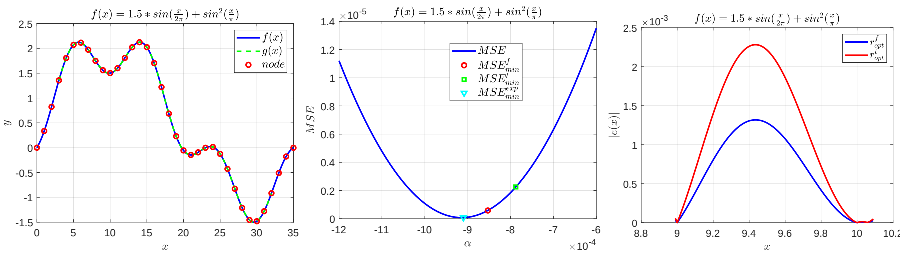
$$f_3(x) = e^{-\frac{x}{2\pi}} \sin\left(\frac{4x}{\pi}\right), \tag{37}$$

$$f_4(x) = \sin\left(\frac{x}{3\pi}\right) \cdot \sin\left(\frac{2x}{\pi}\right). \tag{38}$$

The experiment was carried out with parameters: $K_L = 0$, $K_H = 35$, $h = 1$, $\Delta x = 0.01$, and $M = 4$. The results are presented using graphs and tables.

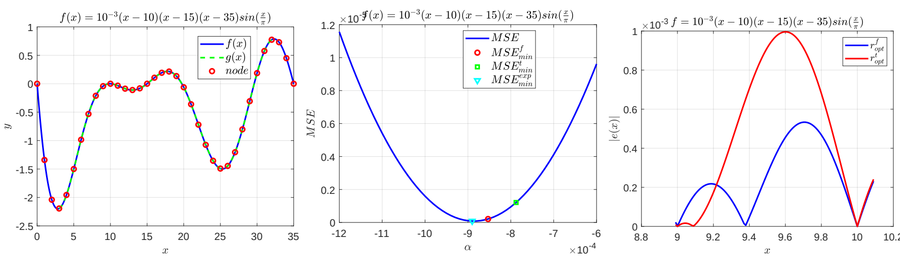
4.2 Results

Time forms of the test functions f, interpolation functions g and interpolation nodes are shown in: a) Figure 2.a (f_1, Equation 35), b) Figure 3.a (f_2, Equation 36), c) Figure 4.a (f_3, Equation 37) and d) Figure 5.a (f_4, Equation 38). Interpolation errors MSE, (Equation 32), depending on the kernel parameter α, are shown with a blue line on: a) Figure 2.b Fig 2.b (f_1), b) Figure 3.b Fig 3.b (f_2), c) Figure 4.b Fig 4.b (f_3) and d) Figure 5.b Fig. 5.b (f_4). The values of minimum interpolation errors are marked on the same MSE graph, for cases when the kernel parameter is optimized in: a) spectral domain (MSE^f_{min} , marker: '●'), b) time domain (MSE^t_{min}, marker: '■') and c) obtained experimentally (MSE^{exp}_{min}, marker: '▼'). The absolute interpolation errors e (Equation 22) on the segment (9, 10), when convolutional interpolation is performed with a seventh-order polynomial 1P kernel, which is optimized in a) spectral domain (r^f_{opt}, α^f_{opt} = -71/83232) [20] and b) time domain (r^t_{opt}, α^t_{opt} = -22/27931) (Section III), are shown in: a) Figure 2.c Fig. 2.c (f_1), b) Figure 3.c Fig. 3.c (f_2), c) Figure 4.c Fig. 4.c (f_3) and d) Figure 5.c Fig. 5.c (f_4). In Table 1 shows the minimum MSE values for the case of interpolation with a seventh-order polynomial kernel optimized in the spectral domain (MSE^f_{min}) and the time domain (MSE^t_{min}). In addition, in order to perform a comparative analysis, the minimum value of MSE, for the case of interpolation with a polynomial kernel: a) of the third-order $MSE^{f,t}_{min,3}$ ($\alpha^{f,t}_{min,3} = -1/2$) [19], [20], and b) of the fifth-order $MSE^{f,t}_{min,5}$ ($\alpha^{f,t}_{min,5} = -3/64$) [20], [23], where the optimal parameter values are equal in both the spectral and time domains, are shown. Applying the algorithm (Section 4.1), the minimum interpolation error MSE (Equation 33) and the optimal values of the kernel parameters α (Equation 34), for all test functions, were experimentally determined: a) f_1 ($MSE^{exp}_{min,f_1} = -7.8387 \cdot 10^{-8}$, $\alpha^{exp}_{opt,f_1} = -10/10989$), b) f_2 ($MSE^{exp}_{min,f_2} = 8.4264 \cdot 10^{-8}$, $\alpha^{exp}_{opt,f_2} = -5/5618$), c) f_3 ($MSE^{exp}_{min,f_3} = 1.1154 \cdot 10^{-7}$, $\alpha^{exp}_{opt,f_3} = -14/14433$), and d) f_4 ($MSE^{exp}_{min,f_4} = 1.1989 \cdot 10^{-7}$, $\alpha^{exp}_{opt,f_4} = -6/6383$).



a) b) c)

Figure 2: a) Interpolated signal $f_1(x)$, interpolation function $g_1(x)$ and interpolation nodes *n*; b) Interpolation errors MSE, depending on the kernel parameter α, and c) absolute interpolation errors *e* on the segment (9, 10).



a) b) c)

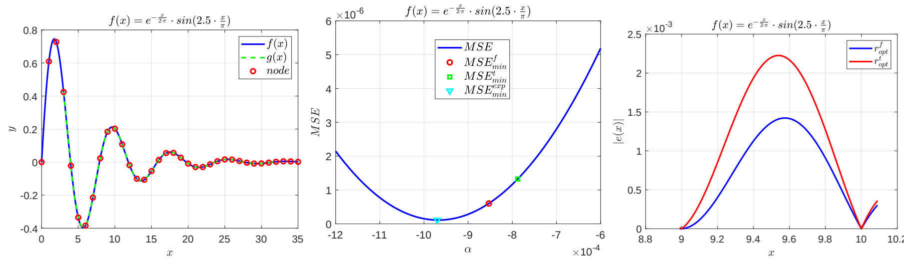
Figure 3: a) Interpolated signal $f_2(x)$, interpolation function $g_2(x)$ and interpolation nodes *n*; b) Interpolation errors MSE, depending on the kernel parameter α, and c) absolute interpolation errors *e* on the segment (9, 10).

4.3 Analysis of results

Based on the experimental results shown in Figures 2 - Figure 5 and Table 1, it is concluded that the precision of interpolation with the the polynomial seventh-order 1P kernel, whose optimal parameter is determined by optimization in the time domain, is higher, compared to:

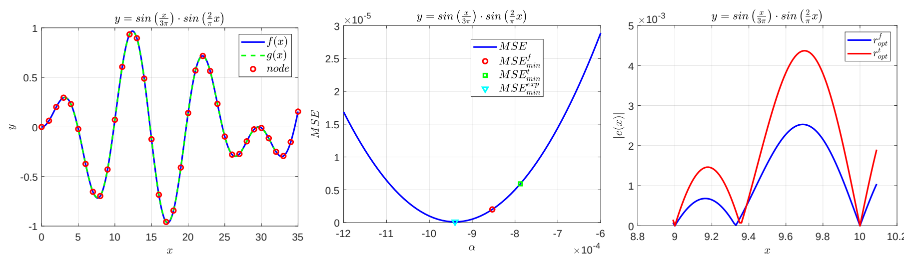
a) third-order 1P kernel $\overline{MSE^{f,t}_{min,3}}/\overline{MSE^{t}_{min,7}} = 3.319 \cdot 10^{-6}/1.3995 \cdot 10^{-6} = 2.3714$ times, and b) fifth order 1P kernel $\overline{MSE^{f,t}_{min,5}}/\overline{MSE^{t}_{min,7}} = 1.5557 \cdot 10^{-6}/1.3995 \cdot 10^{-6} = 1.1115$ times. Based on the experimental results related to the mini-mum interpolation errors of all test functions, the experimental mean value of the interpolation error was determined: $\overline{MSE^{exp}_{min}} = \sum_{k=1}^{4} MSE^{exp}_{min,f_k} = 9.8520 \cdot 10^{-8}$.

The absolute of the interpolation errors in relation to the experimental error are:



a) b) c)

Figure 4: a) Interpolated signal $f_3(x)$, interpolation function $g_3(x)$ and interpolation nodes n; b) Interpolation errors MSE, depending on the kernel parameter α, and c) absolute interpolation errors e on the segment (9, 10).



a) b) c)

Figure 5: a) Interpolated signal $f_4(x)$, interpolation function $g_4(x)$ and interpolation nodes n; b) Interpolation errors MSE, depending on the kernel parameter α, and c) absolute interpolation errors e on the segment (9, 10).

a) third-order 1P kernel $\Delta MSE^{f,t}_{min,3} = |\overline{MSE^{f,t}_{min,3}} - \overline{MSE^{exp}_{min}}| = |3.319 \cdot 10^{-6} - 9.8520 \cdot 10^{-8}| = 3.2205 \cdot 10^{-6}$,

b) fifth-order 1P kernel $\Delta MSE^{f,t}_{min,5} = |\overline{MSE^{f,t}_{min,5}} - \overline{MSE^{exp}_{min}}| = |1.5557 \cdot 10^{-6} - 9.8520 \cdot 10^{-8}| = 1.4572 \cdot 10^{-6}$, and

c) seventh-order 1P $\Delta MSE^{t}_{min} = |\overline{MSE^{t}_{min,7}} - \overline{MSE^{exp}_{min}}| = |1.3995 \cdot 10^{-6} - 9.8520 \cdot 10^{-8}| = 1.3010 \cdot 10^{-6}$. In this way, the efficiency of the seventh order kernel is indicated.

By analyzing the interpolation error for the case of applying the seventh-order kernel, it can be concluded that the interpolation error, when interpolating using a kernel that is optimized in the time domain, compared to a kernel that is optimized in the spectral domain, is greater $\overline{MSE^{t}_{min,7}}/\overline{MSE^{f}_{min}} = 1.3995 \cdot 10^{-6}/0.85211 \cdot 10^{-6} =$

Table 1: Minimum of the interpolation errors MSEs, when the interpolation of the test function f is performed by the polynomial interpolation kernel r: a) third-order ($MSE^{f,t}_{opt,3}$), b) fifth-order ($MSE^{f,t}_{opt,5}$), c) seventh-order ($MSE^{f}_{opt,7}$), optimized in the spectral domain and d) seventh-order ($MSE^{t}_{opt,7}$), optimized in the time domain.

r	$\alpha^{f,t}_{opt,3}$	$\alpha^{f,t}_{opt,5}$	$\alpha^{f}_{opt,7}$	$\alpha^{t}_{opt,7}$
$MSE(x10^{-6})$	$MSE^{f,t}_{opt,3}$	$MSE^{f,t}_{opt,5}$	$MSE^{f}_{opt,7}$	$MSE^{t}_{opt,7}$
f_1	2.8483	1.0957	0.58436	1.1543
f_2	0.87537	0.45616	0.20944	1.2105
f_3	2.6753	1.0536	0.60248	1.0253
f_4	6.8772	3.6171	2.0122	2.2082
$\overline{MSE}$	3.319	1.5557	0.85211	1.3995

1.6425 times. The mean value of the optimal kernel parameter for all test functions is $\overline{\alpha^{exp}_{opt}} = \sum_{k=1}^{4} \alpha^{exp}_{opt,f_k} = -6/6469 = -9.275 \cdot 10^{-4}$. The absolute error of the estimation of the optimal kernel parameters, which were obtained as a result of optimization in: a) spectral domain $\Delta\alpha^{f}_{opt} = |\alpha^{f}_{opt,7} - \overline{\alpha^{exp}_{opt}}| = |-71/83232 - (-6/6469)| = 2/26859 = 7.4463 \cdot 10^{-5}$, and b) time domain $\Delta\alpha^{t}_{opt} = |\alpha^{t}_{opt,7} - \overline{\alpha^{exp}_{opt}}| = |-22/27931 - (-6/6469)| = 27/15742 = 19/135865 = 1.3985 \cdot 10^{-4}$. It can be seen that the absolute error in determining the optimal value of the kernel parameter is smaller when the optimization is performed in the spectral domain.

Based on the conducted analysis, as well as the fact is $\alpha^{t}_{opt,7} \approx \alpha^{f}_{opt,7}$, it is concluded that the optimal choice is the seventh-order polynomial kernel with kernel parameter $\alpha^{f,t}_{opt,7} = -71/83232$. The kernel constructed in this way is suitable for practical application, i.e. implementation in real-time systems.

5 Conclusion

In this paper, the optimization of the seventh-order polynomial convolutional interpolation 1P kernel is described. The optimization of the kernel, which was realized in the time domain, implied the selection of the optimal value of the kernel parameter α. The optimization criterion was the minimization of the interpolation error e, which is defined as the difference between the interpolated function f and the interpolation function g. With the condition that the interpolated function f has at least seven continuous derivatives in the interval where the interpolation is performed, the interpolation error e is developed in the Taylor series up to the seventh term. By minimizing the first seven terms of the Taylor series, the optimal value of the kernel parameter, α_{opt}, is calculated. In order to calculate the optimal kernel parameter, a system of 27 equations with one unknown was formed. In this case, there is no unique solution, and, therefore, the least squares method (LSM) was applied. As a result of applying LSM, the optimal kernel parameter was calculated: $\alpha_{opt} = -22/27931$.

The validity of the proposed optimal kernel parameter was experimentally tested. For the purposes of the experiment, four test functions, whose shape is complex, were created. Each test function is interpolated by convolutional interpolation, using third-order and fifth-order interpolation 1P kernels, whose kernel parameters are calculated in both the spectral and time domains, as well as with a seventh-order kernel that is optimized in the time domain. The results of the experiment show that the precision of the interpolation, which was calculated using MSE, when the seventh-order kernel was applied, is higher than the third-order (2.3714 times), and the fifth-order (1.1115 times). However, the interpolation error of the seventh-order kernel, which is optimized in the time domain, compared to the seventh-order kernel, which is optimized in the spectral domain, is greater by 1.6425 times. With the fact that the optimal parameters, calculated in time ($\alpha^{t}_{opt,7} = -22/27931 = 7.8765 \cdot 10^{-4}$) and spectral ($\alpha^{f}_{opt,7} = -71/83232 = -8.53037 \cdot 10^{-4}$) domains are approximately equal ($\alpha^{t}_{opt,7} \approx \alpha^{f}_{opt,7}$), based on experimental results, it is possible to give a recommendation for the implementation of the seventh-order kernel with the kernel parameter $\alpha^{f,t}_{opt,7} = -71/83232$ in the real-time system.

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