Acta Univ. Sapientiae Informatica **15,** 2 (2023) 294–305

DOI: 10.2478/ausi-2023-0018

The eccentricity-based topological indices

Gul OZKAN KIZILIRMAK
Gazi University
Ankara, Turkey
email: gulozkan@gazi.edu.tr

Abstract. The aim of this paper is to obtain some relationships between eccentricity-based topological indices as the eccentric connectivity, connective eccentricity, total eccentricity, second Zagreb eccentricity, first Zagreb eccentricity connectivity, first eccentricity connectivity and first Zagreb eccentricity connectivity of a simple connected graph.

1 Introduction

Let $\mathcal{G}$ denote a graph with k vertices and s edges, which has the vertex and edge sets as $V(\mathcal{G})$ and $E(\mathcal{G})$, respectively. The number of edges connected to vertex i is denoted as the degree of i and shown as $d(i)$. The minimum and maximum vertex degrees are represented by δ and Δ, respectively. In this study, we are interested in simple undirected graph $\mathcal{G}$ which consists of no loops and multiple edges.

In the literature, there are many interesting studies in graph theory related to the distance of any two vertices. The eccentricity $\epsilon(t)$ of a vertex $t \in V(\mathcal{G}\}$ is defined as the maximum distance between t and any other vertex y in $\mathcal{G}$ and shown as $\epsilon(t) = max\{d(t, y) : y \in V(\mathcal{G}\}$. The maximum and minimum eccentricities over all vertices of $\mathcal{G}$ are called the diameter $d = diam(\mathcal{G})$ and the radius $r = rad(\mathcal{G})$ of $\mathcal{G}$, respectively [3, 7].

Key words and phrases: Eccentricity, topological indices

It is known that topological indices can be used to characterize of a graph. One of the most studied indices is the first Zagreb index $\mathcal{M}_1(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} d(r)^2$.

There exist some studies on eccentricity based topological indices in the literature [1, 2, 13, 14]. One of them is the eccentric connectivity index and was introduced by Sharma et al. [12], which was defined as

$$\xi^c(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} d(r)\epsilon(r).$$

Similarly, the connective eccentricity index of a graph $\mathcal{G}$ was defined in [6] and denoted as

$$\xi^{ce}(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} \frac{d(r)}{\epsilon(r)}.$$

Also, the total eccentricity index was introduced by Farooq et al. [4] as:

$$\zeta(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} \epsilon(r),$$

and moreover the first and second Zagreb eccentricity indices were defined in [5] as:

$$E_1(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} \epsilon^2(r).$$

and

$$E_2(\mathcal{G}) = \sum_{rs \in E(\mathcal{G})} \epsilon(r)\epsilon(s).$$

Motivated by the eccentric-connectivity index, the first Zagreb eccentricity connectivity index $\mathcal{M}^1_{\mathcal{ECI}}$, the first eccentricity connectivity index $\mathcal{ECI}^1$ and the first Zagreb eccentricity connectivity index $\mathcal{M}^1_{\mathcal{ECI}^1}$ were introduced in [8] as:

$$\mathcal{M}^1_{\mathcal{ECI}}(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} d^2(r)\epsilon(r).$$

$$\mathcal{ECI}^1(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} d(r)\epsilon^2(r).$$

$$\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G}) = \sum_{r \in V(\mathcal{G})} d^2(r)\epsilon^2(r).$$

In this paper, some relationships between eccentricity-based topological indices are obtained in simple connected graphs.

Now, we give some lemmas. Both of these lemmas are crucial in proving the main results of this paper.

Lemma 1 [10] *If* t_j *and* y_j $(1 \leq j \leq k)$ *are non-negative real numbers, then*

$$\sum_{j=1}^{k}(t_j)^2 \sum_{j=1}^{k}(y_j)^2 - \left(\sum_{j=1}^{k} t_j y_j\right)^2 \leq \frac{k^2}{4}(M_1 M_2 - m_1 m_2)^2,$$

where $M_1 = \max_{1\leq j\leq k}\{t_j\}$, $M_2 = \max_{1\leq j\leq k}\{y_j\}$; $m_1 = \min_{1\leq j\leq k}\{t_j\}$, $m_2 = \min_{1\leq j\leq k}\{y_j\}$.

Lemma 2 [11] *If* $c_j > 0$, $d_j > 0$, $p > 0$, $j = 1, 2, ..., k$, *then the following inequality holds:*

$$\sum_{j=1}^{k} \frac{c_j^{p+1}}{d_j^p} \geq \frac{\left(\sum\limits_{j=1}^{k} c_j\right)^{p+1}}{\left(\sum\limits_{j=1}^{k} d_j\right)^p}$$

with equality if and only if $\frac{c_1}{d_1} = \frac{c_2}{d_2} = ... = \frac{c_k}{d_k}$.

Lemma 3 [9] *Let* $c_1, c_2, ..., c_k$ *and* $d_1, d_2, ..., d_k$ *be real numbers such that* $c \leq c_j \leq C$ *and* $d \leq d_j \leq D$ *for* $i = 1, 2, ..., k$. *Then there holds*

$$\left|\frac{1}{k}\sum_{j=1}^{k} c_j d_j - \left(\frac{1}{k}\sum_{j=1}^{k} c_j\right)\left(\frac{1}{k}\sum_{j=1}^{k} d_j\right)\right| \leq \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\left(1 - \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\right)(C - c)(D - d).$$

2 Main results

Theorem 4 *Let* $\mathcal{G}$ *be a simple connected graph with* k *vertices. Then we obtain*

$$\mathcal{M}_1(\mathcal{G})E_1(\mathcal{G}) \leq (\xi^c(G))^2 + \frac{k^2}{4}(\Delta d - \delta r)^2.$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof. In Lemma 1, if we take $t_j = d(j)$ and $y_j = \epsilon(j)$, we get

$$\sum_{j=1}^{k}(d(j))^2\sum_{j=1}^{k}(\epsilon(j))^2-\left(\sum_{j=1}^{k}d(j)\epsilon(j)\right)^2 \leq \frac{k^2}{4}(\max(d(j))\max(\epsilon(j))-\min(d(j))\min(\epsilon(j)))^2.$$

By using the definitions of $\mathcal{M}_1(\mathcal{G}), E_1(\mathcal{G})$ and $\xi^{ce}(\mathcal{G})$, we have

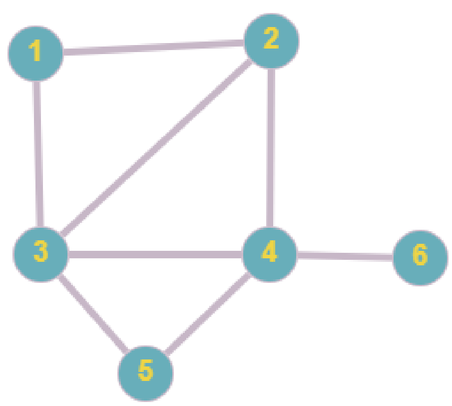
$$\mathcal{M}_1(\mathcal{G})E_1(\mathcal{G})-(\xi^c(\mathcal{G})^2 \leq \frac{k^2}{4}(\max(d(j))\max(\epsilon(j))-\min(d(j))\min(\epsilon(j)))^2.$$

Since $\max(d(j)) = \Delta$, $\max(\epsilon(j)) = d$, $\min(d(j)) = \delta$ and $\min(\epsilon(j)) = r$, we obtain

$$\mathcal{M}_1(\mathcal{G})E_1(\mathcal{G}) \leq (\xi^c(G))^2+\frac{k^2}{4}(\Delta d-\delta r)^2.$$

□

Example 5 *Let $\mathcal{G}$ be a simple connected graph with 6 vertices as follows.*



Then, we get $\mathcal{M}_1(\mathcal{G}) = 50, E_1(\mathcal{G}) = 34$ *and* $\xi^c(G) = 35$.

Since $\Delta = 4, d = 3, \delta = 1$ and $r = 2$, *we obtain* $\mathcal{M}_1(\mathcal{G})E_1(\mathcal{G}) = 1700$ *and* $(\xi^c(G))^2+\frac{k^2}{4}(\Delta d-\delta r)^2 = 1989$. *Thus the inequality in Theorem 4 is satisfied.*

Theorem 6 *If $\mathcal{G}$ is a simple connected graph with k vertices and s edges, then we get*

$$\xi^{c}(\mathcal{G})\xi^{ce}(\mathcal{G}) \leq 4s^2 + \frac{k^2}{4}\left(\Delta\sqrt{\frac{d}{r}} - \delta\sqrt{\frac{r}{d}}\right)^2.$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof. In Lemma 1, we let $t_j = \sqrt{d(j)\epsilon(j)}$ and $y_j = \sqrt{\frac{d(j)}{\epsilon(j)}}$ to get

$$\sum_{j=1}^{k} d(j)\epsilon(j) \sum_{j=1}^{k} \frac{d(j)}{\epsilon(j)} - \left(\sum_{j=1}^{k} d(j)\right)^2 \leq \frac{k^2}{4}\left(\Delta\sqrt{\frac{d}{r}} - \delta\sqrt{\frac{r}{d}}\right)^2.$$

Since $\left(\sum_{j=1}^{k} d(j)\right)^2 = 4s^2$ and from the definitions of $\xi^{c}(\mathcal{G})$ and $\xi^{ce}(\mathcal{G})$, we get

$$\xi^{c}(\mathcal{G})\xi^{ce}(\mathcal{G}) \leq 4s^2 + \frac{k^2}{4}\left(\Delta\sqrt{\frac{d}{r}} - \delta\sqrt{\frac{r}{d}}\right)^2.$$

□

Theorem 7 *Let $\mathcal{G}$ be a simple connected graph with k vertices and s edges. Then we have*

$$\left|\frac{1}{k}\xi^{c}(\mathcal{G}) - \frac{2s}{k^2}\zeta(\mathcal{G})\right| \leq \frac{1}{4}\left(1 - \frac{1+(-1)^{k+1}}{2k^2}\right)(d-r)(\Delta-\delta).$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof.

By using $r \leq \epsilon(j) \leq d$ and $\delta \leq d(j) \leq \Delta$ and choosing $c_j = \epsilon(j)$ and $d_j = d(j)$ in Lemma 3, we get

$$\left|\frac{1}{k}\sum_{j=1}^{k} \epsilon(j)d(j) - \left(\frac{1}{k}\sum_{j=1}^{k} \epsilon(j)\right)\left(\frac{1}{k}\sum_{j=1}^{k} d(j)\right)\right| \leq \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\left(1 - \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\right)(d-r)(\Delta-\delta).$$

Using the definitions of $\xi^c(\mathcal{G})$ and $\zeta(\mathcal{G})$, we get

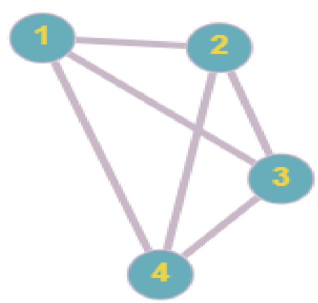
$$\left|\frac{1}{k}\xi^c(\mathcal{G}) - \frac{2s}{k^2}\zeta(\mathcal{G})\right| \leq \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\left(1 - \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\right)(d-r)(\Delta-\delta).$$

Since $\left\lfloor\frac{k}{2}\right\rfloor\left(1-\frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\right) = \frac{k}{4}\left(1 - \frac{1+(-1)^{k+1}}{2k^2}\right)$, we obtain

$$\left|\frac{1}{k}\xi^c(\mathcal{G}) - \frac{2s}{k^2}\zeta(\mathcal{G})\right| \leq \frac{1}{4}\left(1 - \frac{1+(-1)^{k+1}}{2k^2}\right)(d-r)(\Delta-\delta).$$

□

Example 8 *Let's consider $\mathcal{G} = K_4$ complete graph as follows.*



We can calculate as $\xi^c(\mathcal{G}) = 12$ and $\zeta(\mathcal{G}) = 4$. Since $\Delta = 3, d = 1, \delta = 3$ and $r = 1$, we obtain

$$\left|\frac{1}{k}\xi^c(\mathcal{G}) - \frac{2s}{k^2}\zeta(\mathcal{G})\right| = 0$$

and

$$\frac{1}{4}\left(1 - \frac{1+(-1)^{k+1}}{2k^2}\right)(d-r)(\Delta-\delta) = 0.$$

Hence, the equality holds.

Theorem 9 *If $\mathcal{G}$ is a simple connected graph, then we obtain*

$$\mathcal{M}^1_{\mathcal{ECI}}(\mathcal{G})\zeta(\mathcal{G}) \geq (\xi^c(\mathcal{G}))^2.$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof. In Lemma 2, letting $c_j = \epsilon(j)d(j)$, $d_j = \epsilon(j)$ and $p = 1$ gives

$$\sum_{j=1}^{k} \frac{(\epsilon(j)d(j))^2}{\epsilon(j)} \geq \frac{\left(\sum\limits_{j=1}^{k} \epsilon(j)d(j)\right)^2}{\sum\limits_{j=1}^{k} \epsilon(j)}.$$

So, we have

$$\mathcal{M}^1_{\mathcal{ECI}}(\mathcal{G}) \geq \frac{(\xi^c(\mathcal{G}))^2}{\zeta(\mathcal{G})}.$$

Hence, it follows that

$$\mathcal{M}^1_{\mathcal{ECI}}(\mathcal{G})\zeta(\mathcal{G}) \geq (\xi^c(\mathcal{G}))^2.$$

□

Theorem 10 *Let $\mathcal{G}$ be a simple connected graph with k vertices. Then we have*

$$E_1(\mathcal{G}) \leq \frac{1}{k\delta^4}(\mathcal{M}^1_{\mathcal{ECI}}(\mathcal{G}))^2 + \frac{k}{4\delta^4}(d\Delta^2 - r\delta^2)^2.$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof. In Lemma 1, we choose $t_j = \epsilon(j)$ and $y_j = (d(j))^2$ to get

$$\sum_{i=1}^{k}(\epsilon(j))^2 \sum_{j=1}^{k}((d(j))^2)^2 - \left(\sum_{j=1}^{k} \epsilon(j)d(j))^2\right)^2 \\ \leq \frac{k^2}{4}(\max(\epsilon(j))\max((d(j))^2) - \min(\epsilon(j))\min((d(j))^2)^2.$$

Since $\sum\limits_{j=1}^{k}((d(j))^2)^2 \geq k\delta^4$, we have

$$k\delta^4 E_1(\mathcal{G}) - (\mathcal{M}^1_{\mathcal{ECI}}(\mathcal{G}))^2 \leq \frac{k^2}{4}(d\Delta^2 - r\delta^2)^2.$$

Hence, we obtain

$$E_1(\mathcal{G}) \leq \frac{1}{k\delta^4}(\mathcal{M}^1_{\mathcal{ECI}}(\mathcal{G}))^2 + \frac{k}{4\delta^4}(d\Delta^2 - r\delta^2)^2.$$

□

Theorem 11 *If $\mathcal{G}$ is a simple connected graph with k vertices, then we get*

$$\left|\frac{1}{k}\mathcal{M}^{1}_{\mathcal{ECI}}(\mathcal{G})-\frac{1}{k^{2}}\zeta(\mathcal{G})\mathcal{M}_{1}(\mathcal{G})\right|\leq\frac{1}{4}(1-\frac{1+(-1)^{k+1}}{2k^{2}})(d-r)(\Delta^{2}-\delta^{2}).$$

The equality holds for $\mathcal{G}\cong K_{n}$.

Proof. It is known that $r\leq\epsilon(j)\leq d$ and $\delta^{2}\leq(d(j))^{2}\leq\Delta^{2}$. So, we let $c_{j}=\epsilon(j)$ and $d_{j}=(d(j))^{2}$ in Lemma 3, then

$$\left|\frac{1}{k}\sum_{j=1}^{k}\epsilon(j)(d(j))^{2}-\left(\frac{1}{k}\sum_{j=1}^{k}\epsilon(j)\right)\left(\frac{1}{k}\sum_{j=1}^{k}(d(j))^{2}\right)\right|$$
$$\leq\frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\left(1-\frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\right)(d-r)(\Delta^{2}-\delta^{2}).$$

Using the definitions of $\mathcal{M}^{1}_{\mathcal{ECI}}(\mathcal{G})$, $\zeta(\mathcal{G})$ and $\mathcal{M}_{1}(\mathcal{G}), we get$

$$\left|\frac{1}{k}\mathcal{M}^{1}_{\mathcal{ECI}}(\mathcal{G})-\frac{1}{k^{2}}\zeta(\mathcal{G})\mathcal{M}_{1}(\mathcal{G})\right|\leq\frac{1}{k}\left(\frac{k}{4}\left(1-\frac{1+(-1)^{k+1}}{2k^{2}}\right)\right)(d-r)(\Delta^{2}-\delta^{2}).$$

Thus, we obtain

$$\left|\frac{1}{k}\mathcal{M}^{1}_{\mathcal{ECI}}(\mathcal{G})-\frac{1}{k^{2}}\zeta(\mathcal{G})\mathcal{M}_{1}(\mathcal{G})\right|\leq\frac{1}{4}\left(1-\frac{1+(-1)^{k+1}}{2k^{2}}\right)(d-r)(\Delta^{2}-\delta^{2}).$$

□

Example 12 *Let's consider $\mathcal{G}=K_{4}$ complete graph in Example 8. Since $\Delta=3, d=1, \delta=3$ and $r=1$,the right side of the inequality in Theorem 11 is 0. Since $\mathcal{M}^{1}_{\mathcal{ECI}}(\mathcal{G})=36, \zeta(\mathcal{G})=4$ and $\mathcal{M}_{1}(\mathcal{G})=36$, we have*

$$\left|\frac{1}{k}\mathcal{M}^{1}_{\mathcal{ECI}}(\mathcal{G})-\frac{1}{k^{2}}\zeta(\mathcal{G})\mathcal{M}_{1}(\mathcal{G})\right|=0$$

Hence, the equality holds.

Theorem 13 *If $\mathcal{G}$ is a simple connected graph with k vertices, then we obtain*

$$\mathcal{M}_{1}(\mathcal{G})\leq\frac{1}{kr^{4}}(\mathcal{ECI}^{1}(\mathcal{G}))^{2}+\frac{k}{4r^{4}}(d^{2}\Delta-r^{2}\delta)^{2}.$$

The equality holds for $\mathcal{G}\cong K_{n}$.

Proof. In Lemma 1, we choose $t_j = (\epsilon(j))^2$ and $y_j = d(j)$ and get

$$\sum_{j=1}^{k}(\epsilon(j))^4 \sum_{j=1}^{k}(d(j))^2 - \left(\sum_{j=1}^{k}(\epsilon(j))^2 d(j)\right)^2 \leq \frac{k^2}{4}(\max(\epsilon(j))^2 \max(d(j)) - \min(\epsilon(j))^2 \min(d(j))^2.$$

Then,

$$kr^4\mathcal{M}_1(\mathcal{G}) - (ECI^1(G))^2 \leq \frac{k^2}{4}(d^2\Delta - r^2\delta)^2.$$

So, we obtain

$$\mathcal{M}_1(\mathcal{G}) \leq \frac{1}{kr^4}(\mathcal{ECI}^1(\mathcal{G}))^2 + \frac{k}{4r^4}(d^2\Delta - r^2\delta)^2.$$

□

Theorem 14 *Let $\mathcal{G}$ be a simple connected graph with k vertices and s edges. Then we have*

$$\left|\frac{1}{k}\mathcal{ECI}^1(\mathcal{G}) - \frac{2s}{k^2}E_1(\mathcal{G})\right| \leq \frac{1}{4}\left(1 - \frac{1+(-1)^{k+1}}{2k^2}\right)(d^2 - r^2)(\Delta - \delta).$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof. By using $r^2 \leq (\epsilon(j))^2 \leq d^2$ and $\delta \leq d(j) \leq \Delta$. We let $c_j = (\epsilon(j))^2$ and $d_j = d(j)$ in Lemma 3, then

$$\left|\frac{1}{k}\sum_{j=1}^{k}(\epsilon(j))^2 d(j) - \left(\frac{1}{k}\sum_{j=1}^{k}(\epsilon(j))^2\right)\left(\frac{1}{k}\sum_{j=1}^{k}d(j)\right)\right| \leq \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\left(1 - \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\right)(d^2 - r^2)(\Delta - \delta).$$

By using the definitions of $\mathcal{ECI}^1(\mathcal{G})$ and $E_1(\mathcal{G})$, we obtain

$$\left|\frac{1}{k}\mathcal{ECI}^1(\mathcal{G}) - \frac{1}{k}E_1(\mathcal{G})\frac{2s}{k}\right| = \left|\frac{1}{k}\mathcal{ECI}^1(\mathcal{G}) - \frac{2s}{k^2}E_1(\mathcal{G})\right| \leq \frac{1}{4}\left(1 - \frac{1+(-1)^{k+1}}{2k^2}\right)(d^2 - r^2)(\Delta - \delta).$$

□

Theorem 15 *If $\mathcal{G}$ is a simple connected graph with k vertices, then we obtain*

$$\frac{k^2}{4}(3r^4\delta^4 - d^2\Delta^2(d^2\Delta^2 - 2r^2\delta^2)) \le (\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G}))^2.$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof. If we choose $t_j = (\epsilon(j))^2$ and $y_j = (d(j))^2$ in Lemma 1, we have

$$\sum_{i=j}^{k}(\epsilon(j))^4 \sum_{j=1}^{k}(d(j))^4 - \left(\sum_{j=1}^{k}(\epsilon(j))^2(d(j))^2\right)^2$$
$$\le \frac{j^2}{4}(\max(\epsilon(j))^2 \max(d(j))^2 - \min(\epsilon(j))^2 \min(d(j))^2)^2.$$

So, we get

$$kr^4\delta^4 - (\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G}))^2 \le \frac{k^2}{4}(d^2\Delta^2 - r^2\delta^2)^2.$$

After simplifying the above expression, we get the desired result as

$$\frac{k^2}{4}(3r^4\delta^4 - d^2\Delta^2(d^2\Delta^2 - 2r^2\delta^2)) \le (\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G}))^2.$$

□

Theorem 16 *If $\mathcal{G}$ is a simple connected graph with k vertices, then we get*

$$\left|\frac{1}{k}\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G}) - \frac{1}{k^2}E_1(\mathcal{G})\mathcal{M}_1(\mathcal{G})\right| \le \frac{1}{4}\left(1 - \frac{1+(-1)^{k+1}}{2k^2}\right)(d^2 - r^2)(\Delta^2 - \delta^2).$$

The equality holds for $\mathcal{G} \cong K_n$.

Proof. We use the inequalities $r^2 \le (\epsilon(j))^2 \le d^2$ and $\delta^2 \le (d(j))^2 \le \Delta^2$. In Lemma 3, we choose $c_j = (\epsilon(j))^2$ and $d_j = (d(j))^2$, we get

$$\left|\frac{1}{k}\sum_{j=1}^{k}(\epsilon(j))^2(d(j))^2 - \left(\frac{1}{k}\sum_{j=1}^{k}(\epsilon(j))^2\right)\left(\frac{1}{k}\sum_{j=1}^{k}(d(j))^2\right)\right|$$
$$\le \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\left(1 - \frac{1}{k}\left\lfloor\frac{k}{2}\right\rfloor\right)(d^2 - r^2)(\Delta^2 - \delta^2).$$

Hence, we obtain

$$\left|\frac{1}{k}\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G})-\frac{1}{k^2}E_1(\mathcal{G})\mathcal{M}_1(\mathcal{G})\right|\leq\frac{1}{4}\left(1-\frac{1+(-1)^{k+1}}{2k^2}\right)(d^2-r^2)(\Delta^2-\delta^2).$$

□

Example 17 *Let's consider the graph in Example 5. Then, we have* $\mathcal{M}_1(\mathcal{G})=50, E_1(\mathcal{G})=34$ *and* $\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G})=225$.
Since $\Delta=4, d=3, \delta=1$ and $r=2$, *we get*

$$\left|\frac{1}{k}\mathcal{M}^1_{\mathcal{ECI}^1}(\mathcal{G})-\frac{1}{k^2}E_1(\mathcal{G})\mathcal{M}_1(\mathcal{G})\right|=9.72$$

and

$$\frac{1}{4}\left(1-\frac{1+(-1)^{k+1}}{2k^2}\right)(d^2-r^2)(\Delta^2-\delta^2)=18.75$$

Thus, the inequality in Theorem 16 is satisfied.

References

[1] A. Alqesmah, A. Saleh, R. Rangarajan, A. Yurttaş, İ. N. Cangul, Distance eccentric connectivity index of graphs, *Kyungpook Math. J.* **61** (2021) 61–74. ⇒295

[2] K.C. Das, Comparison between zagreb eccentricity indices and the eccentric connectivity index, the second geometric-arithmetic index and the graovac-ghorbani index. *Croat. Chem. Acta* **89** (2016) 505–510. ⇒295

[3] R. C. Entringer, D. E. Jackson, D. A.Snyder, Distance in graphs, *Czech. Math. J.* **26** (1976) 283–296. ⇒294

[4] R. Farooq, M. A. Malik, On some eccentricity based topological indices of nanostar dendrimers, *Optoelectron. Adv. Mater. Rapid. Commun.* **9** (2015) 842–849. ⇒295

[5] D. Vuki evi, A. Graovac, Note on the Comparison of the First and Second Normalized Zagreb Eccentricity Indices, *Acta Chim. Slov* **57** (2010) 524–528. ⇒295

[6] S. Gupta, M. Singh, A.K. Madan, Connective eccentricity index: a novel topological descriptor for predicting biological activity, *J.Mol. Graph. Model.* **18** (2000) 18–25. ⇒295

[7] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total pi-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538. ⇒294

[8] S. M. Hosamani , S. S. Shirakol‡, M. V. Kalyanshetti, İ. N. Cangul, New eccentricity based topological indices of total transformation Graphs †arXiv:2008.10194v1 [math.CO] 24 Aug 2020 ⇒295

[9] X. Li, R. N. Mohapatra, R. S. Rodriguez, Grüss-type inequalities, *J. Math. Anal. Appl.* **267** (2002) 434–443. ⇒296

[10] N. Ozeki, On the estimation of inequalities by maximum and minimum values, Journal of College Arts and Science, **5** (1968) 199–203. (in Japanese) ⇒296

[11] J. Radon, Uber die absolut additiven mengenfunktionen, ¨ Wiener-Sitzungsber, (1913) 1295–1438. ⇒296

[12] V. Sharma, R. Goswami, A.K. Madan, Eccentric connectivity index: a novel highly discriminating topological descriptor for structure–property and structure–activity studies, *J. Chem. Inf. Comput. Sci* **37** (1997) 273–282. ⇒295

[13] K. Xu, Y. Alizadeh, K.C. Das, On two eccentricity-based topological indices of graphs, *Discrete Appl. Math.* **233** (2017) 240–251. ⇒295

[14] K. Xu, K.C. Das, H. Liu, Some extremal results on the connective eccentricity index of graphs, *J. Math. Anal. Appl.* **433** (2016) 803–817. ⇒295

Received: August 11, 2023 • Revised: November 28, 2023