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Computing closeness for some graphs

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Abstract. The analysis of networks involves several crucial parameters. In this paper, we consider the closeness parameter, which is based on the total distance between every pair of vertices. Initially, we delve into a discussion about the applicability of the closeness parameter to Mycielski graphs. Our findings are categorized based on the underlying graph's diameter. The formula for calculating the closeness of a Mycielski graph is derived for graphs with a diameter of less than or equal to 4. Furthermore, we establish a sharp lower bound for the closeness of a Mycielski graph when the diameter of the underlying graph is greater than 4. To achieve this, the closeness of the Mycielski transformation of a path graph plays an important role. Additionally, leveraging the obtained results, we examine the closeness of a special planar construction composed of path and cycle graphs, as well as its Mycielski transformation.

1 Introduction

Network science has evolved greatly over the past decade and is now the leading scientific field in the description of complex networks. Therefore, the complex network is a significant research area of complexity science.

Recently, due to the construction of smart cities, complex network applications have been gaining popularity. Complex networks such as traffic networks,

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power grids, social networks, and others can now be observed ubiquitously. These networks bring significant simplicity to our lives. As a result, complex networks, as a novel and dynamic field of scientific research, are increasingly capturing people's attention. They draw substantial inspiration from experimental studies conducted on real-world networks.

Graph theory emerges as an invaluable instrument for deciphering complex networks. By translating network structures into graphs, this theory offers an intuitive and streamlined representation. This renders graph theory a widely adopted tool across contemporary sciences, facilitating the modeling and resolution of real-life quandaries. [13, 20, 24–26, 28].

In a complex network that composed of processing nodes and communication links, it is very important for a network designer to determine which vertices or edges are important. Hereby, centrality is a critical metric because it indicates which vertex is in a sensitive location in an entire network. It has also been widely used in complex network analysis. If we think of a graph as modeling a network, there are many centrality parameters such as closeness centrality, degree centrality, vertex and edge betweenness centrality, residual closeness and etc. which are used to determine the importance of a vertex or an edge in the network including.

The purpose of centrality measures, as closeness or betweenness, determines how centrally a vertex is in a network. There are many studies in the literature on the rapid calculation about centrality index especially on issues related to the solution and calculation of application problems such as social networks, network analysis and determining the best location [14, 16, 18, 19].

Closeness centrality, one of the most studied parameter of complex network through centrality indexes, is applied much from many researchers. The closeness of a vertex is the sum the distances from all other vertices, where the distance from a vertex to another is defined as the length of the shortest path between them. The closeness centrality based on the shortest paths among vertices in the network and it relates how quickly information can spread across the network. If spread of information from one vertex to other vertices can occur rapidly, that node is based on the intuition that it is in an important position.

The closeness centrality takes value between 0 and 1. If closeness value of a vertex approaches to 0 this means indicated vertex far from others. While closeness value of a vertex approaches to 1 this means addressed node is in close proximity to all other vertices.

Closeness centrality concept was first defined in 1948 by Bavelas [5]. Then, a notable definition for closeness defined by Freeman yet it can be utilized solely for connected graphs [12]. After that, Latora and Marchiori [15] provided new definition for point closeness even it can be applied to disconnected graphs. Later, Danglachev introduced a modified closeness definition due to ease of calculation and formulation [8]. Furthermore, Danglachev defined another measure of vulnerability parameter, called as residual closeness. We refer the readers to references about closeness and its varieties in order to get detailed knowledge [1–3, 9, 27].

In this work, we will use Danglachev's closeness parameter. In this definition, the closeness of a graph is defined as: Dangalchev introduced closeness of a vertex definition as $C(u_i) = \sum\limits_{j \neq i} \frac{1}{2^{d(u_i,u_j)}}$ and closeness of the graph is defined as $C = \sum\limits_{i} C(u_i)$ where $d(u_i, u_j)$ denotes *the distance between two vertices* u_i and u_j is shortest path between them.

In this paper, let G be simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. *The open neighborhood*of any vertex in $V(G)$, denoted by $N_G(v) = \{u \in V(G) : (uv) \in E(G)\}$. Also, $deg(u_i)$ denotes *the degree of a vertex* u_i that is cardinality of its neighborhood. *The diameter of* G is largest distance between two vertices in $V(G)$ and represented by $diam(G)$. The complement $\bar{G}$ of a graph G has $V(G)$ as its vertex sets, but two vertex are adjacent in $\bar{G}$ if only if they are not adjacent in G [6, 7].

The goal of this paper is provide exact formula and sharp lower bound for a Mycielski graph depending on diameter of underlying graph. Mycielski introduced a graph structure that does not contain triangles with large chromatic number. The Mycielski structure, denoted by $\mu(G)$ notation, is defined for a graph $G = (V, E)$ with the vertex set $V(\mu(G)) = V(G) \cup V(G') \cup \{v\}$ where $V(G) = \{v_i : 1 \leq i \leq n\}$ is vertex set of G and $V(G') = \{u_i : 1 \leq i \leq n\}$ is copy of the vertex set $V(G)$ and $E(\mu(G)) = E(G) \cup \{v_i u_j : v_i v_j \in E(G)\} \cup \{u_j v : \forall u_j \in V(G')\}$ [17] (see in Fig. 1). Recently, there has been an increasing interest in studies related to the Mycielski graph and there are many the research papers in the literature about mycielski structures.

For our study, in order to obtain the lower bound of closeness of Mycielski

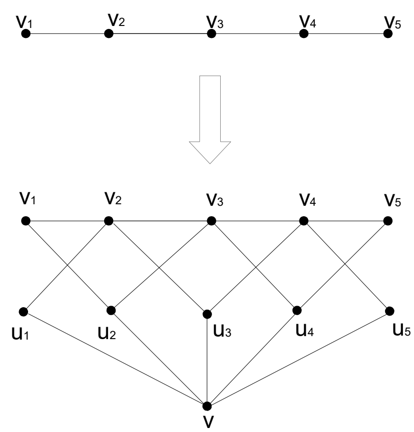

Figure 1: An illustration of a Mycielski graph.

graph, we establish a relationship with the path structure has been the basis. Therefore, first closeness of path mycielskian is provided. Furthermore, we consider Tadpole graph, a construction containing a path. We investigate some results about Tadpole graph and its Mycielski form. In literature, there are some findings about splitting graphs and analogous structure of Mycielski graph [4, 10, 22, 23]. As well as verifying some known basic results with our formula, we also present new general conclusions about closeness of Mycielski graph. Now we state some known lemmas which we use in the proofs of our results.

Theorem 1 [1,8] *The closeness of*

(a) the complete graph K_n *with* n *vertices is* $C(K_n) = \dfrac{n(n-1)}{2}$*;*

(b) the star graph S_n *with* n *vertices is* $C(S_n) = \dfrac{(n-1)(n+2)}{4}$*;*

(c) the path P_n *with* n *vertices is* $C(P_n) = 2n - 4 + \dfrac{1}{2^{n-2}}$ *;*

(d) the cycle C_n *with* n *vertices is* $C(C_n)=\begin{cases} 2n(1-\frac{1}{2^{\lceil n/2 \rceil}}) & \text{if } n \text{ is odd} \\ n(2-\frac{3}{2^{n/2}}) & \text{if } n \text{ is even} \end{cases}$.

Theorem 2 [22] *The closeness of* $\mu(G)$
(a)For $n \geq 4$*; the star graph* $G = S_n$ *is* $C(\mu(G)) = (2(n)^2 + 5n - 3)/2$.

(b)For $n \geq 3$*; the complete graph* $G = K_n$ *is* $C(\mu(G)) = (7n^2 + n)/4$.
(c)For $n \geq 8$*; the cycle graph* $G = C_n$ *is* $C(\mu(G)) = (9n^2 + 77n)/16$.

2 Results about Closeness of Mycielski Graph

In [10] Dangalchev has expressed closeness of splitting graph of G in terms of closeness of G. Analogously, we can apply this process to obtain results about closeness of Mycielski graph depending on diameter of G.

Theorem 3 *Let* G *be* n *order graph and* $\mathrm{diam}(G) \leq 4$. *Then,*

$$C(\mu(G)) = 3C(G) + \frac{n^2 + 7n}{4}.$$

Proof. *To derive the closeness formula of a Mycielski graph with a diameter less than 4, the vertices of the graph can be partitioned into five distinct parts:*

$$\begin{aligned} C(\mu(G)) &= \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(v_i,v_j)} + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} 2^{-d(u_i,v_j)} \\ &+ \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(u_i,u_j)} + 2 \sum_{i=1}^{n} 2^{-d(v,u_i)} + 2 \sum_{i=1}^{n} 2^{-d(v,v_i)} \\ &= C(G) + 2 \sum_{i=1}^{n} 2^{-d(u_i,v_i)} + 2 \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(u_i,v_j)} + \frac{n(n-1)}{4} + 2 \cdot \frac{n}{2} + 2 \cdot \frac{n}{4} \end{aligned}$$

Since, $d(u_i, u_j) = 2$, $d(v, u_i) = 1$ *and* $d(v, v_j) = 2$. *Also,* $d(u_i, u_j) = 2$. $d(u_i, v_j) = d(v_i, v_j)$ *then* $\sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(u_i,v_j)} = \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(v_i,v_j)} = C(G)$

$$\begin{aligned} &= 3C(G) + 2\frac{n}{4} + \frac{n(n-1)}{4} + n + \frac{n}{2} \\ &= 3C(G) + \frac{n^2 + 7n}{4}. \end{aligned}$$

□

Utilizing previous result we can express the closeness for some special graphs. Also, the results can be compared by formulae in [22] and it can be validated.

Corollary 4 *The Mycielski graph $\mu(S_n)$ of star graph S_n has closeness $C(\mu(S_n)) = \frac{2n^2+5n-3}{2}$ which is proven in [22].*

Proof. *It is known that closeness of star graph*

$$C(S_n) = \frac{(n-1)(n+2)}{4}$$

from [8], we have

$$\begin{aligned} C(\mu(S_n)) &= 3C(S_n) + \frac{n^2+7n}{4} \\ &= 3 \cdot \frac{(n-1)(n+2)}{4} + \frac{n^2+7n}{4} \\ &= \frac{2n^2+5n-3}{2}. \end{aligned}$$

□

Corollary 5 *The closeness of Mycielski complete graph which is proved in [22] is*

$$C(\mu(K_n)) = \frac{7n^2+n}{4}.$$

Proof. *The closeness of complete graph K_n is known from [8]*

$$C(K_n) = \frac{n(n-1)}{2}$$

Then, we get

$$\begin{aligned} C(\mu(K_n)) &= 3C(K_n) + \frac{n^2+7n}{4} \\ &= 3 \cdot \frac{n(n-1)}{2} + \frac{n^2+7n}{4} \\ &= \frac{7n^2+n}{4}. \end{aligned}$$

□

Corollary 6 *The closeness of double star $S_{m,n}$ is*

$$C(S_{m,n}) = \frac{n^2+5n+m^2+5m+mn+4}{4}.$$

Proof. *A double star,* $S_{m,n}$*, can be obtained by joining two star graphs* $K_{1,m}$ *and* $K_{1,n}$ *with an edge. Let,* v *and* w *be two non pendant vertices whose degrees are* $\deg(v) = m + 1$ *and* $\deg(w) = n + 1$*, respectively. Then,* v *adjacent to* m *pendant vertices and* w *adjacent to* n *pendant vertices also* w *and* v *are adjacent. In addition, pendant vertices in* $K_{1,m}$ *and* $K_{1,n}$ *are three distances away also those are 2 distances in themselves. Therefore,*

$$\begin{aligned} C(S_{m,n}) &= 2\sum_{i=1}^{m}\frac{1}{2} + 2\sum_{i=1}^{n}\frac{1}{2} + 2\sum_{i=1}^{n}\frac{1}{2^2} + 2\sum_{i=1}^{m}\frac{1}{2^2} + \sum_{i=1}^{n}\sum_{j=1}^{n-1}\frac{1}{2^2} + \sum_{i=1}^{m}\sum_{j=1}^{m-1}\frac{1}{2^2} \\ &\quad + 2\cdot\frac{1}{2} + 2\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{2^3} \\ &= m + n + \frac{m}{2} + \frac{n}{2} + \frac{n(n-1)}{4} + \frac{m(m-1)}{4} + 1 + \frac{mn}{4} \\ &= \frac{n^2 + 5n + m^2 + 5m + mn + 4}{4}. \end{aligned}$$

□

Since, diameter of double graph is 3, closeness of Mycielski graph of double star can be constructed using Theorem 1.

Corollary 7 *Let* W_n *be wheel graph with* n *vertices. The closeness value of* W_n *is* $C(W_n) = \dfrac{(n-1)(n+4)}{4}$.

Proof. *Let* $V(W_n) = \{1, \ldots, n\}$ *be vertex set and* 1 *be center vertex with* $\deg(1) = n - 1$:

$$\begin{aligned} C(W_n) &= 2\sum_{i=1}^{n-1}\frac{1}{2^{-d(1,i)}} + \sum_{i=2}^{n-1}\sum_{\substack{i\sim j,\\ j\neq 1}}\frac{1}{2^{-d(i,j)}} + 2\sum_{\substack{i\nsim j\\ i,j\neq 1}}\frac{1}{2^{-d(i,j)}} \\ &= 2(1(n-1)\frac{1}{2}) + 2\cdot\frac{1}{2}(n-1) + 1\cdot(n-4)\frac{1}{4}(n-1) \\ &= \frac{(n-1)(n+4)}{4} \end{aligned}$$

where $d(1,i) = 1$*, the notation* $i \sim j$ *refers that* i *is adjacent to* j. □

Corollary 8 *Let* $K_{m,n}$ *be complete bipartite graph. The closeness value of* K_n *is* $C(K_{m,n}) = \frac{1}{4}((m+n)^2 - (m+n) + 2mn)$.

Proof. *Let* $V(K_{m,n}) = \{1, 2, ..., m, ...m+n\}$ *be vertex labeling and* $|V_1| = m$ *and* $|V_2| = n$ *be two subset of vertices such that no edge has both endpoints in the same subset:*

$$\begin{aligned}
C(K_{m,n}) &= 2\sum_{i=1}^{m}\sum_{j=m+1}^{m+n}\frac{1}{2^{d(i,j)}} + \sum_{i=1}^{m}\sum_{\substack{j=1\\ i\neq j}}^{m}\frac{1}{2^{d(i,j)}} + \sum_{i=m+1}^{m+n}\sum_{\substack{j=m+1\\ i\neq j}}^{m+n}\frac{1}{2^{d(i,j)}}\\
&= mn + \frac{m(m-1)}{4} + \frac{n(n-1)}{4}\\
&= \frac{1}{4}((m+n)^2 - (m+n) + 2mn).
\end{aligned}$$

□

Corollary 9 *Closeness of Mycielski Double Star, complete bipartite and wheel graphs are*

$$\begin{aligned}
C(\mu(S_{m,n})) &= 3C(S_{m,n}) + \frac{(m+n+2)^2 + 7(m+n+2)}{4}\\
C(\mu(K_{m,n})) &= 3C(K_{m,n}) + \frac{n^2+7n}{4}\\
C(\mu(W_n)) &= \frac{3(n^2+5n-2)}{2}.
\end{aligned}$$

Proof. *The results can be obtained from Theorem 3 and previous corollaries about* $C(S_{m,n})$, $C(K_{m,n})$ *and* $C(W_n)$. □

Theorem 10 *Let* G *be* n *order graph and* $\mathrm{diam}(G) = k > 4$. *Then,*

$$C(\mu(P_{k+1})) \leq C(\mu(G)).$$

Proof. *Let* $\mathrm{diam}(G) = k$, *then the lower bound can be found from* P_{k+1}. *Since, a k-diameter graph includes at least one* P_{k+1}. *Therefore, total closeness value of Mycielski form of a k-diameter graph will be more than closeness value of Mycielski of path graph denoted by* $C(\mu(P_{k+1}))$. *Thus, we have* $C(\mu(P_{k+1})) \leq C(\mu(G))$. □

So, it is necessary to formulate $C(\mu(P_{k+1}))$ value in order to supply sharp lower bound for closeness of Mycielski graph of P_n.

Corollary 11 *The closeness of Mycielski graph path graph* $\mu(P_n)$ *for* $\mathrm{diam}(P_n) = k > 4$ *is*

$$C(\mu(P_n)) = \frac{7n^2 + 91n - 96}{16}.$$

Proof. *In order to calculate closeness of Mycielski graph path graph* $\mu(P_n)$ *for* $\mathrm{diam}(P_n) = k > 4$*, relationship between vertices can be divided into five parts:*

$$\begin{aligned} C(\mu(P_n)) = & \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(v_i,v_j)} + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} 2^{-d(u_i,v_j)} + \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(u_i,u_j)} \\ & + 2 \sum_{i=1}^{n} 2^{-d(v,u_i)} + 2 \sum_{i=1}^{n} 2^{-d(v,v_i)} \\ & = 3 \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(v_i,v_j)} + \frac{n^2 + 7n}{4} \end{aligned}$$

In the Mycielski Graph for G *whose diameter is greater than 4, the value of* $3 \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(v_i,v_j)}$ *is greater than* $3C(G)$*. Since,* $\mathrm{diam}(\mu(G)) = 4$*, and the value of* $2^{-d(v_i,v_j)}$ *in* $C(G)$ *is less than* 2^{-4} *for some pair of vertices. In order to form* $3 \sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(v_i,v_j)}$*, let define a set for* P_n *that contains pair of vertices whose distance greater than 4 and the set denoted by* E_{5^+}.

$$E_{5^+} = \{(v_i, v_j) : \ |v_i - v_j| \geq 5, \ v_i, v_j \in V(P_n)\}$$

Then $|E_{5^+}| = (n-5)(n-4)$*. The value of* $\sum_{i=1}^{n} \sum_{j \neq i} 2^{-d(v_i,v_j)}$ *increases in the summation of* $C(\mu(P_n))$*, due to the diameter of Mycielski graph. In* $C(P_n)$*, the value* $2 \sum_{i=1}^{n-5} \frac{i}{2^{n-i}}$ *that comes from vertices of* E_{5^+} *will be turn into* $\frac{|E_{5^+}|}{16}$. *Therefore, we get*

$$C(\mu(P_n)) = 3(C(P_n) - 2 \sum_{i=1}^{n-5} \frac{i}{2^{n-i}} + \frac{(n-5)(n-4)}{16}) + \frac{n^2 + 7n}{4}.$$

To calculate the summation, we are going to use geometric summation formula as below:

$$\sum_{i=1}^{n} X^{i-1} = 1 + X + X^2 + ... + X^{n-1} = \frac{X^n - 1}{X - 1}$$

and also differentiating both side of geometric sum, we have

$$\sum_{i=1}^{n}(i-1)X^{i-2} = 1 + 2X + ... + (n-1)X^{n-2} = \frac{nX^{n-1}}{X-1} - \frac{X^n - 1}{(X-1)^2}.$$

Then substitute 2 into the X, we get

$$\sum_{i=1}^{n-5}\frac{i}{2^{n-i}} = \frac{2}{2^n}\sum_{i=1}^{n-5} i \cdot 2^{i-1} = \frac{1}{2^{n-1}}((n-4)2^{n-5} - 2^{n-4} + 1) \tag{1}$$

Using $C(P_n) = 2n - 4 + \frac{1}{2^{n-2}}$ *[8] and the equation 1*

$$C(\mu(P_n)) = \frac{7n^2 + 91n - 96}{16}$$

is obtained. □

Theorem 12 *Let* G *be* n *order graph and* $\mathrm{diam}(G) = k > 4$*. Then,*

$$\frac{7(k+1)^2 + 91(k+1) - 96}{16} \leq C(\mu(G)).$$

Proof. *It can be referred from Theorem 10 and Corollary 11.* □

2.1 Results about Tadpole graph

In previous section, we have obtained result for Mycielski graph of G ,whose diameter greater than 4, based on the Mycielski of path graph. In this section, we will investigate results about Tadpole graph and its Mycielski form. Tadpole graph is special planar graph which contains path and cycle graphs as a subgraph. Therefore, results will be benefited from closeness of path and cycle graphs.

Definition 13 *Tadpole graph, denoted by* $T_{n,m}$*, is a graph obtained by identifying a vertex of the cycle graph* C_n *with a pendant vertex of the path graph* P_m*. An example of the illustration of the Tadpole graph can be seen in Figure 2. Truszczynski called these graphs as Dragon [21] and Koh et. al called these forms as Tadpole graphs [11].*

Theorem 14 *Let* $T_{n,m}$ *be a Tadpole graph contains* C_n *and* P_m*. Closeness of Tadpole graph in terms of n is:*

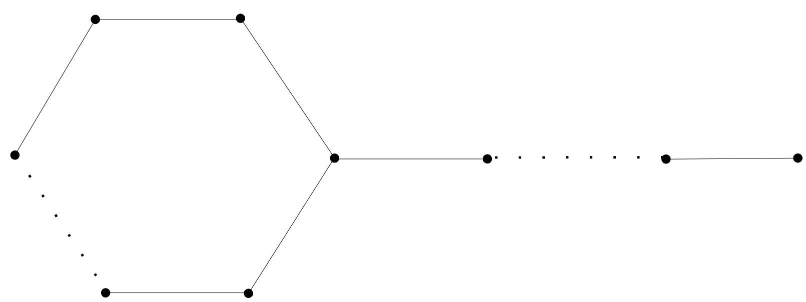

Figure 2: An illustration of a Tadpole graph.

- *if n is odd:*

$$C(T_{n,m}) = 2n(1-\frac{1}{2^{[n/2]}}) + (2m-4+\frac{1}{2^{m-2}}) + 2(2-2^{2-\frac{n+1}{2}})(1-(\frac{1}{2})^{m-1})$$

- *if n is even:*

$$C(T_{n,m}) = 2n(1-\frac{1}{2^{[n/2]}})+(2m-4+\frac{1}{2^{m-2}})+2(2-2^{2-\frac{n}{2}}+2^{-n/2})(1-(\frac{1}{2})^{m-1}).$$

Proof. *Closeness of $T_{n,m}$ can be think as three parts. Closeness of C_n and closeness of P_m and closeness value which comes from relationship between vertices in C_n and vertices in P_m, let it be denoted by $C(C_n - P_m)$*

$$\begin{aligned} C(T_{n,m}) &= C(C_n) + C(P_m) + 2C(C_{n-}P_m) \\ &= 2n(1-\frac{1}{2^{[n/2]}}) + (2m-4+\frac{1}{2^{m-2}}) + 2C(Cn - P_m). \end{aligned}$$

Closeness of P_m and C_n are known [8]. It is need to find $C(C_{n-}P_m)$. Assume that, v_1 is a vertex as intersection point of C_n and P_m. Let divide C_n into exactly two pieces. However, form of division depends on whether n is odd or even.

Case 1: Let n be even and labeling of C_n be $\{v_1, v_2, ..., v_n\}$. Therefore, the closeness of v_1 in C_n can be calculated as

$$2\sum_{i=2}^{n/2} \frac{1}{2^{i-1}} + \frac{1}{2^{n/2}}.$$

Since, there are $(n-2)/2$ *symmetric vertices in* C_n *whose distance from* v_1 *to* v_i *can be calculated as* $(i-1)$ *and there is one vertex whose distance from* v_1 *is* $n/2$.

Also, let vertices of P_m *be labeled as* $\{v_1, v_{\bar{2}}, ..., v_{\overline{m}}\}$. *Distance of* $v_{\bar{j}}$, $j = 2, ...m$, *to* v_1 *equal to*

$$\frac{1}{2^{j-1}}(2\sum_{i=2}^{n/2}\frac{1}{2^{i-1}}+\frac{1}{2^{n/2}}).$$

In general

$$\sum_{j=1}^{m-1}\frac{1}{2^j}((2\sum_{i=2}^{n/2}\frac{1}{2^{i-1}}+\frac{1}{2^{n/2}}))=2(2-2^{2-(n/2)}+2^{-n/2})(1-\frac{1}{2^{m-1}}).$$

Case 2: Similarly it can be done for odd n *values. Using same labeling as in Case 1, the closeness of* v_1 *in* C_n is :

$$2\sum_{i=2}^{n+1/2}\frac{1}{2^{i-1}}.$$

Because of this, C_n *can be divided into exact two equal part and distance from* v_1 *to* $v_i \in V(C_n)$ *can be calculated as* $(i-1)$.

Also, distance of $v_{\bar{j}}$, $j = 2, ...m$, *to* v_1 *equal to*

$$\sum_{j=1}^{m-1}\frac{1}{2^j}.(2\sum_{i=2}^{n+1/2}\frac{1}{2^{i-1}})=2(2-2^{2-\frac{n+1}{2}})(1-(\frac{1}{2})^{m-1}).$$

Therefore, we have

- *if* n *is odd:*

$$C(T_{n,m})=2n(1-\frac{1}{2^{[n/2]}})+(2m-4+\frac{1}{2^{m-2}})+2(2-2^{2-\frac{n+1}{2}})(1-(\frac{1}{2})^{m-1})$$

- *if* n *is even:*

$$C(T_{n,m})=2n(1-\frac{1}{2^{[n/2]}})+(2m-4+\frac{1}{2^{m-2}})+2(2-2^{2-\frac{n}{2}}+2^{-n/2})(1-(\frac{1}{2})^{m-1}).$$

□

In previous results, we had talked about closeness value of $\mu(P_n)$ and special variant of P_n and C_n, named Tadpole graph. Similarly, we can ready to investigate Mycielski of Tadpole graph using the closeness result of $\mu(P_n)$ and $T_{n,m}$. Mycielski of Tadpole graph $T_{n,m}$ has $2(n+m)+1$ vertices and its diameter always equal to 4 regardless from $diam(T_{n,m})$. However, $C(\mu(T_{n,m}))$ should be taken hand according to diameter of $T_{n,m}$.

Let $V(\mu(T_{n,m}))$ be vertex set of $T_{n,m}$ including $V(T_{n,m})$, $V(T'_{n,m})$ and w.

Theorem 15 *Let* $T_{n,m}$ *be a Tadpole graph contains* C_n *and* P_m *and* $diam(T_{n,m}) < 4$. *Closeness of Mycielski of Tadpole graph is*

$$C(\mu(T_{n,m})) = 3C(T_{n,m}) + \frac{(n+m-1)^2 + 7(n+m-1)}{4}.$$

Proof. *It can be acquired from Theorem 3.* □

Before giving result about $C(\mu(T_{n,m}))$ when $diam(T_{n,m}) > 4$, some useful findings will be investigated in order to get rid of expressional burden of $C(\mu(T_{n,m}))$. In Mycielski graph, closeness value of some vertex pairs turns to $\frac{1}{2^4}$ due to form of it. In order to calculate the value of $C(\mu(G))$ when $diam(G) > 4$, the closeness value of vertex pair with distance 5 or more should be removed from $C(G)$ and $\frac{1}{2^4}$ should be added as the number of subtracted value instead.

Let total closeness value of the pair of vertices in $T_{n,m}$ whose distance between them greater than 4 be excess closeness, denoted by $C_{ex}(T_{n,m})$ and the number of the vertex pair that has closeness value smaller than $\frac{1}{2^4}$, denoted by $|V_{ex}(T_{n,m})|$.

Once it comes to calculating $C_{ex}(T_{n,m})$ and $|V_{ex}(T_{n,m})|$, $T_{n,m}$ can be thought as being divided into two parts as upper and lower as illustrated in the Figure 3. Then, it can be examine in two cases.

Case 1: If n is odd then there is a path having $m + \frac{n-1}{2}$ vertices on upper side. Thus, $(m+\frac{n-1}{2}-5)(m+\frac{n-1}{2}-4)$ pair of vertices with distance 5 or more comes from the upper part. Because of repeated pair of vertices, the lower part can be evaluated as

$$2.\sum_{i=1}^{(n-1)/2}(m+i-5) = (m-5)(n-1) + (\frac{n^2-1}{2}).$$

Because of this, there are $(n-1)/2$ vertices in lower part of cycle whose distance can de greater or equal to 5 to vertices in P_m. Hence, if n is odd;

$$|V_{ex}(T_{n,m})| = (m+\frac{n-1}{2}-5)(m+\frac{n-1}{2}-4) + (m-5)(n-1) + (\frac{n^2-1}{2}).$$

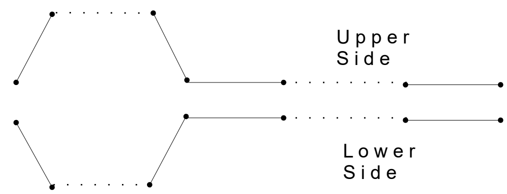

Figure 3: A Tadpole graph divided into two parts

According to proof of Corollary 11, total closeness value of the pair of vertices in P_k whose distance greater or equal than 5 had been calculated as

$$2\sum_{i=1}^{k-5}\frac{i}{2^{k-i}}=\frac{2}{2^{k-1}}((k-4)2^{k-5}-2^{k-4}+1).$$

Then, substitute $m+\frac{n-1}{2}$ into k:

$$\frac{2}{2^{m+\frac{n-1}{2}-1}}((m+\frac{n-1}{2}-4)2^{m+\frac{n-1}{2}-5}-2^{m+\frac{n-1}{2}-4}+1) \quad (2)$$

obtained from upper side. In order to hinder repeated value coming from lower part, it should be subtracted the value of $C_{ex}(P_m)$ from the value in equation 2

$$C_{ex}(P_m)=\frac{1}{2^{m-2}}((m-4)2^{m-5}-2^{m-4}+1).$$

Therefore, we have

$$C_{ex}(T_{n,m})=\frac{1}{2^{m+\frac{n-1}{2}-3}}((m+\frac{n-1}{2}-4)2^{m+\frac{n-1}{2}-5}-2^{m+\frac{n-1}{2}-4}+1).$$

$$-\frac{1}{2^{m-2}}((m-4)2^{m-5}-2^{m-4}+1)$$

Case 2: If n is even then there is a path having $m+\frac{n}{2}$ vertices. Thus, $(m+\frac{n}{2}-5)(m+\frac{n}{2}-4)$ pair of vertices with distance 5 or more comes from the upper part. As in case 1, if we consider the repetitions as:

$$|V_{ex}(T_{n,m})| = (m+\frac{n}{2}-5)(m+\frac{n}{2}-4)+(m-5)(n-2)+(\frac{n(n-2)}{2}).$$

Since n is even, we cannot divide C_n into two equal parts. Thus, we get

$$\begin{aligned}
C_{ex}(T_{n,m}) &= C_{ex}(P_{m+\frac{n}{2}}) + C_{ex}(P_{m+\frac{n-2}{2}}) - C_{ex}(P_m) \\
&= \frac{1}{2^{m+\frac{n}{2}-2}}((m+\frac{n}{2}-4)2^{m+\frac{n}{2}-5} - 2^{m+\frac{n}{2}-4}+1) \\
&+ \frac{1}{2^{m+\frac{n-2}{2}-2}}((m+\frac{n-2}{2}-4)2^{m+\frac{n-2}{2}-5} - 2^{m+\frac{n-2}{2}-4}+1) \\
&- \frac{1}{2^{m-2}}((m-4)2^{m-5} - 2^{m-4}+1).
\end{aligned}$$

Theorem 16 *Let $T_{n,m}$ be a Tadpole graph contains C_n and P_m and $\mathrm{diam}(T_{n,m}) >$ 4. Closeness of Mycielski of Tadpole graph is*

$$C(\mu(T_{n,m})) = 3(C(T_{n,m}) - C_{ex}(T_{n,m}) + \frac{|V_{ex}(T_{n,m})|}{2^4}) + \frac{(n+m-1)^2+7(n+m-1)}{4}.$$

Proof. *Let $V(\mu(T_{n,m})) = \{V(T_{n,m}),\ V(T'_{n,m}),\ w\}$ where copy of Tadpole graph denoted by $T'_{n,m}$. According to form of Mycielski graph, it is known that $\mathrm{diam}(\mu(T_{n,m})) =$ 4. However, the diameter of $T_{n,m}$ is greater than 4 in this case. Thus, we have*

$$3C(T_{n,m}) + \frac{(n+m-1)^2+7(n+m-1)}{4} < C(\mu(T_{n,m})).$$

Let v_i, v_j be vertices in $T_{n,m}$ provided that distance between them greater than 4 in $T_{n,m}$. Therefore, the value of $2^{-d(v_i,v_j)}$ turns into 2^{-4} in the $\mu(T_{n,m})$. It is also valid for copy vertices u_i, u_j in $T'_{n,m}$.

$$\begin{aligned}
C(\mu(T_{n,m})) &= \sum_{i=1}^{m+n-1}\sum_{j\neq i} 2^{-d(v_i,v_j)} + 2\sum_{i=1}^{m+n-1}\sum_{j=1}^{m+n-1} 2^{-d(u_i,v_j)} \\
&+ \sum_{i=1}^{m+n-1}\sum_{j\neq i} 2^{-d(u_i,u_j)} + 2\sum_{i=1}^{m+n-1} 2^{-d(w,u_i)} + 2\sum_{i=1}^{m+n-1} 2^{-d(w,v_i)} \\
&= 3\sum_{i=1}^{m+n-1}\sum_{j\neq i} 2^{-d(v_i,v_j)} + \frac{(n+m-1)^2+7(n+m-1)}{4}
\end{aligned}$$

Since, the distance $d(v_i, v_j) = d(u_i, v_j)$ *whenever* $i \neq j$ *and* $d(u_i, v_i) = 2 = d(u_i, u_j)$ *and also* $d(w, u_i) = 1$, $d(w, v_i) = 2$. *Whereas* $\sum_{i=1}^{m+n-1} \sum_{j \neq i} 2^{-d(v_i, v_j)}$ *is equal to* $C(T_{n,m})$, *in Mycielski graph this value will be increased. Even so, the value of* $\sum_{i=1}^{m+n-1} \sum_{j \neq i} 2^{-d(v_i, v_j)}$ *can be expressed in terms of* $C(T_{n,m})$.

$$= 3(C(T_{n,m}) - C_{ex}(T_{n,m}) + \frac{|V_{ex}(T_{n,m})|}{2^4}) + \frac{(n+m-1)^2 + 7(n+m-1)}{4}.$$

□

3 Conclusion

In this article, closeness of Mycielski graph has taken into consideration depending on diameter of original graph. For the case of diameter less than 4, the outcome is expressed in terms of closeness of original graph. For special graphs whose diameter less than 4, results calculated in [22] verified with our expression. Furthermore, a sharp lower bound has provided case of diameter greater than 4. This lower bound equal to closeness of Mycielskian of path, calculated by us. In addition, closeness of Tadpole graph and its Mycielski form is evaluated by utilizing closeness of path graph P_n, considering whether n is even or odd.

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