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Average distance colouring of graphs

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Abstract. For a graph G with n vertices, average distance $\mu(G)$ is the ratio of sum of the lengths of the shortest paths between all pairs of vertices to the number of edges in a complete graph on n vertices. In this paper, we introduce average distance colouring and find the average distance colouring number of certain classes of graphs.

1 Introduction

For the present study, we consider a graph $G = (V, E)$ with the set of vertices denoted by V and the set of edges E. For terminology and notation not defined here we refer to [1]. The distance between two vertices u and v denoted by $d(u, v)$, is the length of the shortest $u - v$ path, also called a $u - v$ geodesic. The distance between two vertices is considered as the base of the definition of various graph parameters [3]. In this paper, we introduce average distance colouring and obtain average distance colouring number for certain classes of graphs. Note that the distance between two vertices becomes infinite in disconnected graphs therefore, we consider only connected graphs for our study. Although the value of the average distance [6] $\mu(G)$ of any graph G depends on the sum of the distance between every pair of vertices, which generally would keep changing with the change in n, interestingly, we can find

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a constant bound for the value of $\mu(G)$ for various graph classes. We use such classes and study average distance colouring for the same. Before defining *average distance colouring*, we present the definition of average distance of graph G as defined in [2]. For a graph G with n vertices, average distance $\mu(G)$ is the ratio of sum of the lengths of the shortest paths between all pairs of vertices to the number of edges in a complete graph on n vertices. This can also be represented by the following equation.

$$\mu(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V} d(u,v).$$

In our study, we focus on finding the exact value of $\mu(G)$ for certain classes not studied before and use them to colour the related graphs. Note that all graphs considered are connected and are of order at least two.

For a graph G *with average distance* $\mu(G)$*, an average distance colouring of* $G = (V, E)$ *is defined as a function* c *from* V *to the set of non-negative integers, such that for any* $v \in V$*,* $|c(v) - c(u)| \geq 1$ *for all* u *such that* $d(u, v) \leq \lceil \mu(G) \rceil$*. The minimum number of distinct colours required to colour any graph* G *such that* G *admits average distance colouring is called average distance colouring number,* χ_μ*, of* G*.*

Example 1 *Figure 1 shows the average distance colouring of graph* G *whose* $\mu(G) = 1.714$.

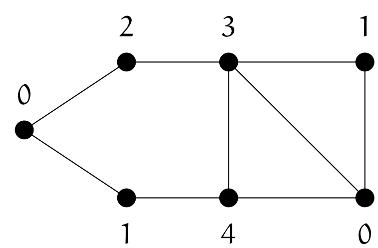

Figure 1: Average distance colouring of graph G with $\mu(G) = 1.714$

From the definition, it follows that all graphs admit average distance colouring. Also, average distance colouring is equivalent to chromatic colouring only for graphs with diameter 1, i.e., *complete graphs*. Further, when $k = \lceil \mu(G) \rceil$, average distance colouring is equivalent to *distance-k colouring* [4, 5]. Before we obtain the bounds and values for average distance colouring number for some specific classes of graphs, we obtain the relation between the value of χ_μ of a graph G and its spanning subgraph H.

Observation 2 *For a graph* G *and its spanning subgraph* H*, such that the average distance of* G *is* $\mu(G)$ *and the average distance of its spanning subgraph* H *is* $\mu(H)$*,* $\mu(G) \leq \mu(H)$*.*

The reason is that the spanning subgraph of graph G on n vertices has a larger value of average distance as the number of edges in the spanning subgraph of the graph will be less as compared to the original graph, thus increasing the distance between the pair of vertices joined by the deleted edge thus leading in the increase of the value of numerator which eventually leads to the increase in the value of average distance. From the above argument, we get the following observation.

Observation 3 *For two graphs* G_1 *and* G_2 *such that* $\mu(G_1) \geq \mu(G_2)$*,* $\chi_\mu(G_1) \geq \chi_\mu(G_2)$*.*

Theorem 4 *For a graph* G *on* n *vertices with average distance* $\mu(G)$*,* $\chi_\mu = n$ *if and only if* $d = \lceil \mu(G) \rceil$ *where* d *denotes the diameter of* G*.*

Proof. Consider a graph G with diameter $d = \mu(G)$. This implies that the vertices are either distance 1 or 2 apart. In this case, the definition for average distance colouring implies that the pair of vertices u and v at a distance at most $\lceil \mu(G) \rceil$ should receive distinct colours. Since $d = \mu(G)$, the above statement implies that the vertices at most distance d apart should get distinct colours and each pair of vertices are at a distance at most d, Thus we require distinct colours for each vertex.
Let G be a graph on n vertices with $\chi_\mu = n$ and diameter d. We know that $c(u) \neq c(v)$ for all $u, v \in V(G)$. This implies $d(u, v) \leq \lceil \mu(G) \rceil$ for all $u, v \in V(G)$. Since $d = max\ d(u, v)$ over all pairs of $u, v \in V(G)$, $d \leq \lceil \mu(G) \rceil$. Since every pair of vertex gets distinct colour, this implies that every pair of vertices is at most $\lceil \mu(G) \rceil$ distance apart. We know $d(u, v) \leq d$ for all pairs of $u, v \in V(G)$, thus $d \geq \lceil \mu(G) \rceil$. Thus $d = \lceil \mu(G) \rceil$. □

2 Average distance colouring of some classes of graphs

In this section, we obtain the average distance colouring number of certain classes of graphs and show the procedure to colour the same using average distance colouring.

Note that the average distance of any graph can only be one if and only if it is a complete graph. For any graph G where G is not complete, $\mu(G) > 1$. For a complete graph K_n, all vertices are pairwise adjacent and for each $v \in V(K_n)$, $|c(u) - c(v)| \geq 1$ for all u such that $d(u, v) \leq 1$, it is easy to observe that we require n distinct colours to colour the graph which can easily be attained by colouring each vertex with different colours. This implies $\chi_\mu(K_n) \geq n$. We define function c such that $c(v_i) = i-1$, for $i = 1, 2, 3, ..., n$ giving $\chi_\mu(K_n) \leq n$. Using the function c defined above, we require colours $0, 1, 2, ..., n-1$ to colour any complete graph on n vertices, which leads to the following observation.

Observation 5 *For a complete graph* K_n, *for* $n \geq 2$, $\chi_\mu(K_n) = n$.

Next, we consider paths and cycles and obtain the value for average distance colouring number $\chi_\mu(G)$. Before obtaining the result on average distance colouring, we require the following results.

Theorem 6 [2] *The average distance of paths on* n *vertices*

$$\mu(P_n) = \frac{n+1}{3}.$$

Theorem 7 [2] *The average distance of cycles on* n *vertices*

$$\mu(C_n) = \begin{cases} \dfrac{(n+1)}{4} & \textit{if n is odd, and} \\ \dfrac{n^2}{4(n-1)} & \textit{if n is even.} \end{cases}$$

Theorem 8 *For path* P_n *on* n *vertices,* $n \geq 3$, $\chi_\mu(P_n) = \left\lceil \dfrac{n+1}{3} \right\rceil + 1.$

Proof. Consider a path P_n with vertices labelled $v_1, v_2, ..., v_n$. Using Theorem 6, the definition of average distance colouring reduces to the function c from V to a set of non-negative integers such that for any $v \in V$, $|c(u) - c(v)| \geq 1$ for all u such that $d(u, v) \leq \left\lceil \dfrac{n+1}{3} \right\rceil$. We define a colouring c such that

$$c(v_i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod{\left\lceil \frac{n+1}{3} \right\rceil + 1} \\ 1 & \text{if } i \equiv 2 \pmod{\left\lceil \frac{n+1}{3} \right\rceil + 1} \\ 2 & \text{if } i \equiv 3 \pmod{\left\lceil \frac{n+1}{3} \right\rceil + 1} \\ \vdots & \\ \left\lceil \frac{n+1}{3} \right\rceil - 1 & \text{if } i \equiv \left\lceil \frac{n+1}{3} \right\rceil \pmod{\left\lceil \frac{n+1}{3} \right\rceil + 1} \\ \left\lceil \frac{n+1}{3} \right\rceil & \text{if } i \equiv 0 \pmod{\left\lceil \frac{n+1}{3} \right\rceil + 1} \end{cases}$$

This function gives an average distance colouring with $\chi_\mu(P_n) \leq \left\lceil \frac{n+1}{3} \right\rceil + 1$. Due to average distance colouring constraint the vertices which are at distance $\left\lceil \frac{n+1}{3} \right\rceil$ from v_1 cannot have same colour thus we require $\left\lceil \frac{n+1}{3} \right\rceil + 1$ distinct colour to colour any path of length n giving $\chi_\mu(P_n) \geq \left\lceil \frac{n+1}{3} \right\rceil + 1$. Hence, the result. □

Theorem 9 *For cycles on* n *vertices,*

$$\chi_\mu(C_n) \leq \begin{cases} \left\lceil \frac{(n+1)}{4} \right\rceil + 1 + r & \textit{if n is odd, and} \\ \left\lceil \frac{n^2}{4(n-1)} \right\rceil + 1 + r & \textit{if n is even.} \end{cases}$$

where r is the remainder obtained after dividing n *by* $\lceil \mu(C_n) \rceil$.

Proof. Consider a cycle C_n with n vertices ordered $v_1, v_2, ..., v_n$ such that v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and v_1 adjacent to v_1.
Using Theorem 7, the definition for average distance colouring reduces to a colouring c such that $|c(u) - c(v)| \geq 1$ for all u such that $d(u, v) \leq \lceil \mu(C_n) \rceil$ for every $v \in V$, where $\mu(C_n) = \frac{(n+1)}{4}$ if n is odd, and $\mu(C_n) = \frac{n^2}{4(n-1)}$ if n is even.

To colour the cycle, we will use the following function till a certain value of i which is given in the following cases.

$$c(v_i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod{\lceil \mu(C_n) \rceil + 1} \\ 1 & \text{if } i \equiv 2 \pmod{\lceil \mu(C_n) \rceil + 1} \\ 2 & \text{if } i \equiv 3 \pmod{\lceil \mu(C_n) \rceil + 1} \\ \vdots & \\ \vdots & \\ \lceil \mu(C_n) \rceil - 1 & \text{if } i \equiv \mu(C_n) \pmod{\lceil \mu(C_n) \rceil + 1} \\ \lceil \mu(C_n) \rceil & \text{if } i \equiv 0 \pmod{\lceil \mu(C_n) \rceil + 1} \end{cases} \tag{1}$$

Further, we will consider the following cases.

Case 1: When n divided by $\lceil \mu(C_n) \rceil + 1$ leaves no remainder.
In this case, the function c defined as in Equation (1) is used for all $1 \leq i \leq n$ thus, giving the average distance colouring of cycle with $\chi_\mu \leq \lceil \mu(C_n) \rceil + 1$.

Case 2: When n divided by $\lceil \mu(C_n) \rceil + 1$ leaves a remainder.
In this case, we obtain the remainder (say r) after dividing n by $\lceil \mu(C_n) \rceil + 1$. For $1 \leq i \leq n - r$, we use function c defined as in Equation (1) to colour the vertices. For the remaining r vertices, we define c given as in Equation (2)

$$c(v_i) = \begin{cases} \lceil \mu(C_n) \rceil + 1 & \text{if } i = n - r + 1 \\ \lceil \mu(C_n) \rceil + 2 & \text{if } i = n - r + 2 \\ \lceil \mu(C_n) \rceil + 3 & \text{if } i = n - r + 3 \\ \vdots & \\ \vdots & \\ \lceil \mu(C_n) \rceil + (r - 1) & \text{if } i = n - r + (r - 1) \\ \lceil \mu(C_n) \rceil + (r) & \text{if } i = n - r + (r) \end{cases} \tag{2}$$

which gives $\chi_\mu \leq \mu(C_n) + r + 1$.

On substituting the value of $\mu(C_n)$ for odd and even number of vertices we get,

$$\chi_\mu(C_n) \leq \begin{cases} \left\lceil \frac{(n+1)}{4} \right\rceil + 1 + r \text{ if n is odd, and} \\ \left\lceil \frac{n^2}{4(n-1)} \right\rceil + 1 + r \text{ if n is even.} \end{cases}$$

□

For the next result, we consider complete bipartite graphs. The following result gives the χ_μ of complete bipartite graph $K_{r,s}$.

Theorem 10 *For a complete bipartite graph* $K_{r,s}$*,* $\chi_\mu(K_{r,s}) = r + s$*.*

Proof. Consider a complete bipartite graph $K_{r,s}$ with two partite sets A and B consisting of r and s vertices respectively. The vertices in A and B can be ordered as $v_1, v_2, v_3, ..., v_r$ and $u_1, u_2, u_3, ..., u_s$ respectively. Vertices in the different partite sets are distance one apart, therefore $\sum_{i=1}^{r} d(v_1, u_i) = s$. Similarly, for r vertices in A we get

$$\sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} d(v_i, u_j) = rs. \quad (3)$$

Also, vertices in the same partite set are distance two apart, therefore, on considering partite set A we obtain

$$\left.\begin{aligned} \sum_{i=2}^{r} d(v_1, v_i) &= 2(r-1) \\ \sum_{i=3}^{r} d(v_2, v_i) &= 2(r-2) \\ &\vdots \\ \sum_{i=r-1}^{r} d(v_{r-2}, v_i) &= 2(2) \\ \sum_{i=r}^{r} d(v_{r-1}, v_i) &= 2(1) \end{aligned}\right\} \quad (4)$$

Adding all the equations in Expression (4),

$$\sum_{u,v \in V(A)} d(u, v) = 2\{1 + 2 + ... + (r-1)\} = r(r-1). \quad (5)$$

Similarly, for partite set B

$$\sum_{u,v \in V(B)} d(u,v) = 2\{1+2+...+(s-1)\} = s(s-1). \quad (6)$$

Adding Equations (3), (5), and (6), we obtain

$$\sum_{u,v \in V(G)} d(u,v) = rs + r(r-1) + s(s-1). \quad (7)$$

Given that the total number of vertices in $K_{r,s} = r+s$, and by the definition of average distance of graph, we get

$$\mu(K_{r,s}) = \frac{2(rs + r(r-1) + s(s-1))}{(r+s)(r+s-1)}. \quad (8)$$

Next, we claim that for $r, s \geq 1$, $\mu(K_{r,s}) \leq 2$.
If possible, let $\mu(K_{r,s}) > 2$.

$$\frac{2(rs + r(r-1) + s(s-1))}{(r+s)(r+s-1)} > 2$$

$$\implies \frac{2(r+s)(r+s-1) - 2(rs + r(r-1) + s(s-1))}{(r+s)(r+s-1)} < 0.$$

On simplification, we get $rs < 0$ which is a contradiction since r and s are greater than 0. Therefore, $\mu(K_{r,s}) \leq 2$.

Hence, it reduces the definition of average distance colouring as the function c from V to a set of non-negative integers such that for any $v \in V$, $|c(u) - c(v)| \geq 1$ for all u such that $d(u,v) \leq 2$. For a complete bipartite graph, every pair of vertices is either distance one or two apart which implies that the colour given to each vertex must be unique, giving $\chi_\mu(K_{r,s}) \geq r+s$. Further, this can be attained by using the following colouring c defined by $c(v_i) = i-1$ for $1 \leq i \leq r$ and $c(u_i) = c(v_r) + i$ for $1 \leq i \leq s$ given for graph $K_{r,s}$ with vertices of first and second partite set labelled as $v_1, v_2, ..., v_r$ and $u_1, u_2, ..., u_s$ respectively. The above function c gives $\chi_\mu(K_{r,s}) \leq r+s$,thus proving that $\chi_\mu(K_{r,s}) = r+s$.

□

For the next result, we consider a unicyclic graph $S_k + e$ obtained by adding a single edge between two pendant vertices of the star graph S_k shown in Figure 2.

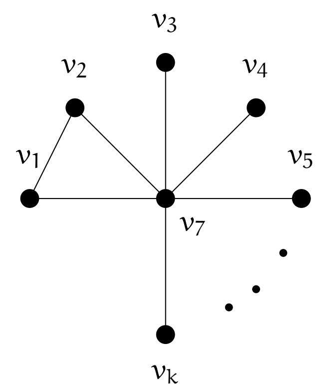

Figure 2: Graph $S_k + e$

Theorem 11 *For a graph obtained by joining the two pendant vertices of the star by an edge,* $S_k + e$*,* $\chi_\mu(S_k + e) = k + 1$.

Proof. Consider a graph S_k+e with its vertices labelled as follows. Let the central vertex be labelled as v and the pendant vertices be labelled as $v_1, v_2, ..., v_k$ and the edge e is drawn between the vertices labelled v_{k-1} and v_k. The sum of the distance from vertex v_1 to other vertices given by t_1 is given by the following equation.

$$t_1 = 2(k-1) + 1. \tag{9}$$

Similarly, the sum of the distance from vertex v_2 to other vertices given by t_2 is

$$t_2 = 2(k-2) + 1. \tag{10}$$

On generalising Equations (9) and (10), we get the sum of distance from vertices $v_1, v_2, ..., v_{k-2}$ to other vertices denoted by t_m for $1 \leq m \leq k-2$ respectively. Therefore $\sum\limits_{i=1}^{k-2} t_m = 2\left[(k-1) + (k-2) + ... + 2\right] + k - 2$ which can be further simplified to $\left(k^2 - 4\right)$.
Further, the sum of distance between the vertices of the triangle formed by vertices v, v_{k-1} and v_k will be 3. Therefore, the sum of distance between any two pair of vertices is

$$\sum_{u,v \in V} d(u,v) = k^2 - 1$$

Given that the total number of vertices in $S_k + e = k+1$, and by the definition of average distance of graph, we get

$$\mu(S_k + e) = \frac{2(k^2 - 1)}{k(k+1)}.$$

Next, we claim $\mu(S_k + e) \leq 2$. If possible, let $\mu(S_k + e) > 2$.

$$\implies \frac{2(k^2 - 1)}{k(k+1)} > 2$$
$$\implies k < k - 1.$$

which is impossible. Therefore, $\mu(S_k + e) \leq 2$. Using the above inequation, the definition of average distance colouring reduces to the function c from V to set of non-negative integers such that for any $v \in V$, $|c(u) - c(v)| \geq 1$ for all u such that $d(u, v) \leq 2$. Since the diameter of the graph is 2, we know that each vertex should get a distinct colour to satisfy the given constraint for colouring giving $\chi_\mu(S_k + e) \geq k+1$. This can be obtained by using the function c which assigns integers to the vertices of the graph defined by $c(v_i) = i - 1$ for the vertices of the graph labelled v_1 and $v_2, v_3, ..., v_{k+1}$ representing the central vertex and pendant vertices respectively with an edge drawn between each v_i for $2 \leq i \leq k+1$ and v_1. Also, there exists an edge between v_2 and v_3. Since the total number of vertices of the graph is $k+1$, $\chi_\mu(S_k + e) \leq k+1$. Hence the result.

□

On further increasing the number of partitions of vertices from two to r such that no two vertices in the same partition have an edge, we get a complete multipartite graph. In the next result, we obtain the value for χ_μ for a complete multipartite graph.

Theorem 12 *For a complete multipartite graph* $K_{m_1, m_2, ..., m_r}$,

$$\chi_\mu(K_{m_1, m_2, ..., m_r}) = m_1 + m_2 + ... + m_r.$$

Proof. Consider a complete multipartite graph G with r partite sets namely $A_1, A_2, ..., A_r$ such that $|A_i| = m_i$ for $1 \leq i \leq r$. Vertex in A_i partite set is given by $\{v_{i1}, v_{i2}, ..., v_{im_i}\}$ for $1 \leq i \leq r$.
Using the fact that the vertices in the same partite set are distance 2 apart, for the partite set A_1, we get

$$\sum_{u,v \in V(A_1)} d(u, v) = 2(1 + 2 + ... + (m_1 - 1)).$$

In general, the sum of distances between vertices belonging to the same partite set is given as

$$\sum_{u,v \in V(A_i)} d(u,v) = 2(1+2+...+(m_i-1)), \text{ for } 1 \leq i \leq r.$$

Also, vertices belonging to different partite sets are distance 1 apart, therefore the sum of the distance from any vertex of A_1 to vertices of partite set $A_2, A_3, ...A_r$ is given by $m_1(m_2+m_3+m_4+...+m_r)$. Similarly, the sum of the distance from any vertex of A_i to vertices of other partite sets $A_{i+1}, A_{i+2}, ...A_r$ is given by $m_i(m_{i+1}+m_{i+2}+...+m_r)$.
Therefore, the sum of distances between vertices belonging to the same partite set for $K_{m_1,m_2,...,m_r}$ is given by $(say\ s_1)$

$$s_1 = m_1(m_1-1)+m_2(m_2-1)+...m_r(m_r-1) \tag{11}$$

and the sum of the distance between vertices taken from a different partite set is given by $(say\ s_2)$

$$s_2 = m_1(m_2+m_3+...+m_r)+m_2(m_3+m_4+...+m_r)+...+m_{r-1}m_r. \tag{12}$$

Adding Equations (11) and (12), we get

$$\begin{aligned}\sum_{u,v \in V(G))} d(u,v) = {} & m_1(m_1-1)+m_2(m_2-1)+...+m_r(m_r-1)\\ & +m_1(m_2+m_3+...+m_r)\\ & +m_2(m_3+m_4+...+m_r)+...+m_{r-1}m_r\end{aligned} \tag{13}$$

which can be simplified to

$$\begin{aligned}\sum_{u,v \in V(G))} d(u,v) = {} & (m_1^2+m_2^2+...+m_r^2)-(m_1+m_2+...+m_r)\\ & + \sum_{1 \leq i \leq r-1} m_i m_{i+1}+m_i m_{i+2}+...+m_i m_{i+(r-i)}.\end{aligned} \tag{14}$$

Since the number of vertices in a complete multipartite graph is $m_1+m_2+...+m_r$, we get

$$\begin{aligned}\mu(K_{m_1,m_2,...,m_r}) = {} & 2\frac{(m_1^2+m_2^2+...+m_r^2)-(m_1+m_2+...+m_r)}{(m_1+m_2+...+m_r)(m_1+m_2+....+m_r-1)}\\ & +2\frac{\sum_{i=1}^{r-1}(m_i m_{i+1}+m_i m_{i+2}+...+m_i m_{i+(r-i)})}{(m_1+m_2+...+m_r)(m_1+m_2+....+m_r-1)}.\end{aligned}$$

We claim that for $r \geq 2$ and $n \geq 2$, value of $\mu(K_{m_1,m_2,...,m_r}) \leq 2$.
If possible, let $\mu(K_{m_1,m_2,...,m_r}) > 2$.
$\implies 2 - \mu(K_{m_1,m_2,...,m_r}) < 0$

$$2 - 2\frac{(m_1^2 + m_2^2 + ... + m_r^2) - (m_1 + m_2 + ... + m_r)}{(m_1 + m_2 + ... + m_r)(m_1 + m_2 + ... + m_r - 1)} + 2\frac{\sum_{i=1}^{r-1}(m_i m_{i+1} + m_i m_{i+2} + ... + m_i m_{i+(r-i)})}{(m_1 + m_2 + ... + m_r)(m_1 + m_2 + ... + m_r - 1)} < 0.$$

Since for all $1 \leq i \leq r$, $m_i \geq 1$ therefore, the denominator is always greater than 0. Therefore, multiplying both side by $(m_1 + m_2 + ... + m_r)(m_1 + m_2 + ... + m_r - 1)$, and expanding the summation we get
$2(m_1+m_2+...+m_r)(m_1+m_2+...+m_r-1)-2(m_1^2+m_2^2+...+m_r^2)+2(m_1+m_2+...+m_r)-2m_1(m_2+m_3+...+m_r)-2m_2(m_3+m_4+...m_r)-...-2m_{r-1}(m_r) < 0$.
On solving the above inequation, we get
$2(m_1+m_2+...+m_r)+m_2m_1+m_3m_1+m_3m_2+...+m_rm_1+m_rm_2+..+m_rm_{r-2} < 0$. Since each of m_i for $1 \leq i \leq r$ is greater than zero, the above in equation is not possible. Therefore, $\mu(K_{m_1,m_2,...,m_r}) \leq 2$.

The above arguments reduce the definition of average distance colouring as the function c from V to a set of non-negative integers such that for any $v \in V$, $|c(u) - c(v)| \geq 1$ for all u such that $d(u, v) \leq 2$. For a complete multipartite graph, every pair of vertices is either distance one or two apart, which implies that the colour given to each vertex must be unique giving $\chi_\mu(K_{m_1,m_2,...,m_r}) \geq m_1+m_2+...+m_r$. This bound can be achieved by colouring the vertices of the graph using the function c defined as follows $c(v_i) = i - 1$ for $1 \leq i \leq m_1 + m_2 + ... + m_r$ where the vertices of m_j^{th} partite set are labelled $v_{m_1+m_2+..+m_{j-1}+1}, v_{m_1+m_2+..+m_{j-1}+2}, ..., v_{m_1+m_2+..+m_{j-1}+m_j}$ for $2 \leq j \leq r$ and vertices of the first partite set are labelled $v_1, v_2, ..., v_{m_1}$ which gives $\chi_\mu(K_{m_1,m_2,...,m_r}) \leq m_1 + m_2 + ... + m_r$. Hence the result. □

For the next result, we consider double star $B_{s,t}$ such that $s \leq t$ with s and t number of pendant vertices as shown in Figure 3. In this case, the diameter of the graph considered for study has a diameter of three.

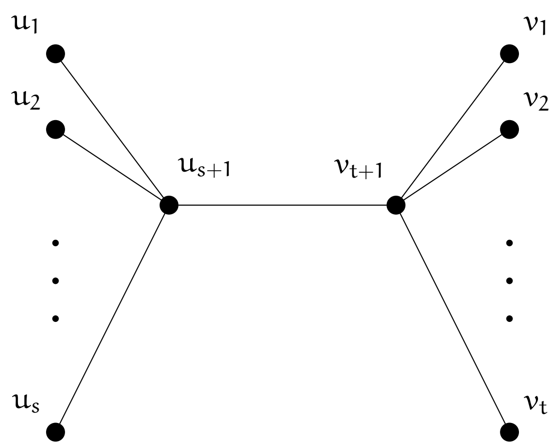

Figure 3: Double star $B_{s,t}$

Theorem 13 *For a double star* $B_{s,t}$*, with* s *and* t *number of pendant number of vertices,*

$$\chi_\mu(B_{s,t}) = \begin{cases} (t+2) \textit{ for } \begin{cases} s=1 \textit{ and } t \geq 1 \textit{ or } t=1 \textit{ and } s \geq 1 \\ s=2 \textit{ and } t=3 \end{cases} \\ s+t+2 \quad \textit{otherwise.} \end{cases}$$

Proof. Consider a double star $B_{s,t}$ with vertices labelled as shown in Figure 3. Note that the structure of the graph is symmetric, we consider only one of the cases to prove the result by assuming $s \leq t$. Using the result of complete bipartite graph, we get the sum of distances between vertices $\{u_1, u_2, ..., u_s, u_{s+1}\}$ (say sum_1).

$$sum_1 = (s + s(s-1)) \tag{15}$$

Similarly, sum of distances between the vertices $\{v_1, v_2, ..., v_t, v_{t+1}\}$ (say sum_2) is given by the following equation.

$$sum_2 = (t + t(t-1)). \tag{16}$$

Also,

$$\sum_{1 \leq i \leq s} d(u_i, v_{t+1}) = 2s. \tag{17}$$

Similarly,

$$\sum_{1 \leq i \leq t} d(u_{s+1}, v_i) = 2t. \tag{18}$$

Also, note that

$$d(u_{s+1}, u_{t+1}) = 1 \tag{19}$$

.

Now, the distance between any two vertices each taken from set $\{u_1, u_2, ..., u_s\}$ and $\{v_1, v_2, ..., v_t\}$ is 3. This gives the following set of equations.

$$\left.\begin{aligned} \sum_{1\le i\le t} d(u_1, v_i) &= 3t \\ \sum_{1\le i\le t} d(u_2, v_i) &= 3t \\ &\vdots \\ \sum_{1\le i\le t} d(u_s, v_i) &= 3t \end{aligned}\right\} \tag{20}$$

Adding equations from (15) to (20) we get,

$$\sum_{u,v\in V(B_{s,t})} d(u, v) = s + s(s-1) + t + t(t-1) + 2(s+t) + 3st + 1.$$

Since the total number of vertices in double star $B_{s,t}$ is $s + t + 2$, we obtain

$$\mu(B_{s,t}) = \frac{2(s^2 + t^2 + 3st + 2s + 2t + 1)}{(s+t+2)(s+t+1)}.$$

Next, We examine the value of s and t for which $\mu(B_{s,t}) \le 2$.

$$\frac{2(s^2 + t^2 + 3st + 2s + 2t + 1)}{(s+t+2)(s+t+1)} \le 2$$

On simplification we get,

$$s + t + 1 \ge st$$

It is easy to verify that the above inequation holds true in the following two cases.

Case 1: When either s or t is equal to 1 and the other variable assumes any value greater than or equal to 1.

Case 2: When one of the variables is equal to two and the other is equal to three. To colour the double star for the above cases, we consider a double star with vertices $u_1, u_2, ...u_{s+1}, v_1, v_2, ...v_{t+1}$ as shown in Figure 3. We know $\mu(B_{s,t}) \leq 2$ reducing the definition of average distance colouring as the function c from V to set of non-negative integers such that for any $v \in V$, $|c(u)-c(v)| \geq 1$ for all u such that $d(u, v) \leq 2$.

Subcase 1: When $s < t$.

We define a colouring c such that $c(v_{t+1}) = 0, c(v_i) = i$ for $1 \leq i \leq t$. Further, $c(u_{s+1}) = (t+1)$ given that $d(u_{s+1}, v_t) = 2$ and $c(u_i) = i$ for $1 \leq i \leq s$. This function gives the same set of colours to pendant vertices and since we know $s < t$, we have t distinct colours assigned. Also, v_{t+1} and u_{s+1} get different colour by using the above-defined function. Thus, $\chi_\mu \leq t+2$.

Since double star consists of two stars with two non-pendant vertices joined by an edge, using Theorem 10, we require minimum $t+2$ colours knowing $s < t$ giving $\chi_\mu \geq t+2$. Thus, $\chi_\mu = t+2$.

Subcase 2: When $s = t = 1$.

In this case, we get a P_4, which requires 3 distinct colours to colour it with average distance colouring protocol. This is attained by colouring four consecutive vertices with colours $1, 0, 2, 1$.

For the remaining values of s and t, $\mu(B_{s,t}) > 2$, which implies that we require $s+t+2$ colours using Theorem 4. □

Further, we consider a graph obtained by joining k-copies of K_s with one common vertex termed windmill graph.

3 Conclusion

In this paper, we have considered the average distance of the graph which gives the approximate distance between any two vertices in the graph and introduce the concept of average distance colouring of graphs. We study average distance colouring number $\chi_\mu(G)$ for certain networks. We have already worked on the condition where a graph would require n distinct colours for it to admit average distance colouring. It would be interesting to identify the bound for k characterise graphs with $\chi_\mu(G) = k$ where k is any positive integer.

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