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On connectivity of the semi-splitting block graph of a graph

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Abstract. A graph G is said to be a semi-splitting block graph if there exists a graph H such that $S_B(H) \cong G$. This paper establishes a characterisation of semi-splitting block graphs based on the partition of the vertex set of G. The vertex (edge) connectivity and p-connectedness (p-edge connectedness) of $S_B(G)$ are examined. For all integers a, b with $1 < a < b$, the existence of the graph G for which $\kappa(G) = a$, $\kappa(S_B(G)) = b$ and $\lambda(G) = a, \lambda(S_B(G)) = b$ are proved independently. The characterization of graphs with $\kappa(S_B(G)) = \kappa(G)$ and a necessary condition for graphs with $\kappa(S_B(G)) = \lambda(S_B(G))$ are achieved.

1 Introduction

Graph theory has a wide range of applications in communication networks. An interconnection network can be represented as a simple connected graph

Key words and phrases: semi-splitting block graph of a graph, vertex connectivity, edge connectivity

$G = (V, E)$, where V represents the set of memory modules and E represents the communication links. The vertex cut of a graph G is the set of vertices whose removal gives a disconnected or trivial graph. The minimum cardinality of the vertex cut of the graph is called the vertex connectivity of the graph G, denoted by $\kappa(G)$. If $G - v$ has more than one component then v is a cut vertex of G. A maximal connected subgraph of the graph G which has no cut vertex is called a block of G. The edge cut of a graph G is the set of edges whose removal gives a disconnected graph. The minimum cardinality of the edge cut of the graph is called the edge connectivity of the graph G, denoted by $\lambda(G)$. A graph G is p-connected (p-edge connected) if $\kappa(G) \geq p$ ($\lambda(G) \geq p$).

Many results have been established regarding the connectivity of simple graphs, derived graphs and digraphs over many decades. This paper focuses on analyzing a derived graph's vertex(edge) connectedness, defined in [4], which is stated as follows.

Definition 1 *The semi-splitting block graph* $S_B(G)$ *of a graph of order* n *is a graph with* $V(S_B(G)) = V(G) \cup V_1(G) \cup B(G)$*, where*
$V(G) = \{v_i \mid 1 \leq i \leq n\}$,
$V_1(G) = \{u_i \mid 1 \leq i \leq n, v_i \in V(G)\}$,
$B(G) = \{b_l \mid 1 \leq l \leq k,\ B_l \text{ is a block in } G\}$.

$$E(S_B(G)) = \begin{cases} v_i v_j \mid 1 \leq i, j \leq n, v_i v_j \in E(G) \\ u_i v_j \mid 1 \leq i, j \leq n, v_i v_j \in E(G) \\ v_i b_l \mid 1 \leq i \leq n, 1 \leq l \leq k, v_i \in B_l \text{ in } G \end{cases}$$

where $v_i, v_j \in V(G)$*,* $u_i \in V_1(G)$ *and* $b_l \in B(G)$*.*

Since every edge is a block in a tree, the graph P_3 has 2 blocks say, $B_1 = \{v_1, v_2\}$ and $B_2 = \{v_2, v_3\}$. Figure 1, shows the semi-splitting block graph of P_3.

The study on the planarity of $S_B(G)$ has been carried out extensively in [4]. The scope of this paper is limited to simple, finite and undirected graphs. For terminology in graph theory, refer to [1, 2, 3].

2 Structural properties of $S_B(G)$

In this section, the structural properties of semi-splitting block graph of a graph are examined. If G is a disconnected graph with non trivial components $G_1, G_2, \ldots, G_m$, then $S_B(G)$ has $S_B(G_1), S_B(G_2), \ldots, S_B(G_m)$ as its components.

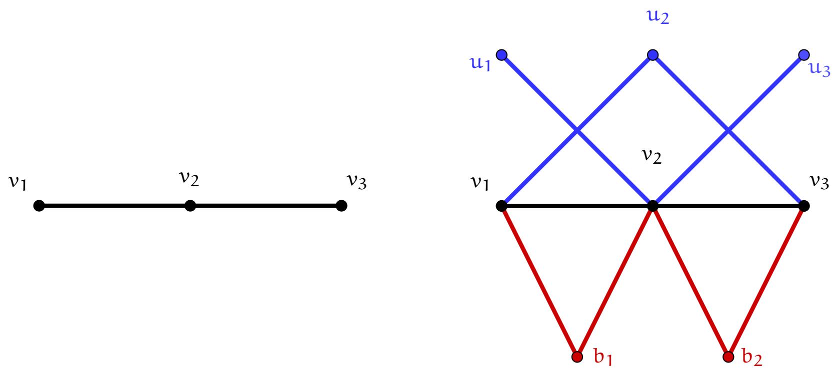

Figure 1: P_3 and $S_B(P_3)$

Theorem 2 *Let* $S_B(G)$ *be the semi-splitting block graph of* G *of order* n, $n \geq 2$, *and* k *blocks. Then,*

(i) *For each* $u_i \in V_1(G)$, $\deg_{S_B(G)}(u_i) = \deg_G(v_i)$, $v_i \in V(G)$, $1 \leq i \leq n$,

(ii) *For each* $b_l \in B(G)$, $\deg_{S_B(G)}(b_l) = |V(B_l)|$, *where* $1 \leq l \leq k$ *and* $V(B_l) \subset V(G)$,

(iii) *For each* $v_i \in V(G)$, $\deg_{S_B(G)}(v_i) = 2\deg_G(v_i) + s$, *where* s *is the number of blocks containing* v_i *in* G *and* $1 \leq i \leq n$.

Proof. Clearly, $N_{S_B(G)}(v_i) = N_G(v_i) \cup \{u_j : v_j \in N_G(v_i)\} \cup \{b_l : v_i \in B_l \text{ in } G\}$. $N_{S_B(G)}(u_i) = \{v_j : v_j \in N_G(v_i)\}$ and $N_{S_B(G)}(b_l) = V(B_l)$, where $V(B_l) \subset V(G)$. Hence the theorem follows. □

Corollary 3 *For a non-cut vertex* v_m *of the graph* G,
$\deg_{S_B(G)}(v_m) = 2\deg_G(v_m) + 1$.

Corollary 4 *If* G *is a block of order* n, $n \geq 2$, *then*

(i) *For each* $v_i \in V(G)$, $\deg_{S_B(G)}(v_i) = 2\deg_G(v_i) + 1$, $1 \leq i \leq n$,

(ii) *For* $b_1 \in B(G)$, $\deg_{S_B(G)}(b_1) = n$.

Remark 5 $S_B(G)$ *is always non-regular, for any graph* G, *for all* $v_i \in V(G)$, $\deg_{S_B(G)}(v_i) > \deg_{S_B(G)}(u_i)$, $1 \leq i \leq n$.

The characterization of semi-splitting block graph based on the partition of the vertex set of the graph is given in the following theorem.

Theorem 6 *The following statements are equivalent.*

1. *A graph* G *of order* n *is a semi-splitting block graph.*

2. *The vertex set of a graph* G *can be partitioned into three subsets namely* V_1, V_2, V_3 *such that*

   (a) i. *There is a bijective mapping* $f : V_1 \rightarrow V_2$ *such that* $f(v_1) = v_2$*, where* $v_1 \in V_1, v_2 \in V_2$*.*

   ii. $N(v_2) = N(v_1) \cap V_1$

   (b) *For each* $v_3 \in V_3$*,* $< N(v_3) >$ *is a block of* $< V_1 >$*.*

Proof. $(1) \implies (2)$. Let G be a semi-splitting block graph of order n. Then for some H, $G \cong S_B(H)$. By definition, adjacency between two vertices in $S_B(H)$ is as follows:

I. adjacent vertices in H are adjacent in $S_B(H)$.

II. for each vertex v_i of $V(H)$, a new vertex u_i being adjacent to $N_H(v)$ is added.

III. for each block in H, a new vertex b_l adjacent to all the vertices of the respective block is added.

Let $V_1 = V(H)$, $V_2 = \{u_i\}_{i=1}^{|V(H)|}$ and $V_3 = \{b_l\}_{l=1}^{k}$. For each $v_i \in V_1$, let $u_i \in V_2$ be the corresponding new vertex added in $S_B(H)$. Then, $f : V_1 \rightarrow V_2$ is a bijective mapping such that $f(v_i) = u_i$ and $N(u_i) = N(v_i) \cap V_1$. Since, each $b_l \in V_3$ is adjacent only to vertices of the corresponding block, $N(b_l) = B_l$, where $1 \leq l \leq k$ and B_l is a unique block of $< V_1 >$.
$(2) \implies (1)$. Suppose (2) is true. Let $H =< V_1 >$. Then, $G \cong S_B(H)$.
Therefore, G is a semi-splitting block graph.
Hence the theorem. □

Theorem 7 *For any graph* G *of order* n*,* $n \geq 2$*, with* k *blocks,* $\delta(S_B(G)) = \min\{|V(B_l)|, \delta(G)\}$*, where* B_l*,* $1 \leq l \leq k$ *is a block of* G*.*

Proof. It is evident from Theorem 2 that in $S_B(G)$, for any $v_i \in V(G)$, $\deg(u_i) = \deg_G(v_i)$ and $\deg(v_i) > 2\deg_G(v_i)$. The following cases are considered.

Case 1 Suppose G is a block. By Corollary 4, $\deg(b_1) = n > \delta(G)$ in $S_B(G)$. Therefore, $\delta(S_B(G)) = \delta(G)$.

Case 2 Suppose G is not a block. Then there exist at least two blocks in G. This implies that $|V(B_l)| \geq 2$, for $1 \leq l \leq k$ and $k \geq 2$.
Thus, $\delta(S_B(G)) = min\{\delta(G), |V(B_l)|\}$.

Therefore, it inferred that, $\delta(S_B(G)) = min\{\delta(G), |V(B_l)|\}$.
Hence the theorem. □

Corollary 8 *For any graph* G*,* $\delta(S_B(G)) = \delta(G)$ *if and only if* G *is a block or* G *is not a block with* $|V(B_l)| \geq \delta(G)$ *for all* $1 \leq l \leq k$ *.*

Corollary 9 *For integers* a, b *with* $a > b > 1$*, there exists a graph* G *with* $\delta(G) = a$ *and* $\delta(S_B(G)) = b$ *if and only if there exists at least one block* B_l *in* G *whose* $|V(B_l)| < \delta(G)$*, where* $1 \leq l \leq k$*.*

3 Connectedness of $S_B(G)$

In this section, the vertex (edge) connectedness of semi-splitting block graph of a connected graph is examined. Let $v_a \in V(G)$ such that $deg(v_a) = \delta(G)$. Since, $\kappa(G) \leq \lambda(G) \leq \delta(G)$, $N(v_a) = \{v_s | 1 \leq s \leq \delta(G)\}$ is a vertex cut of G and $Y = \{(v_a, v_s) | v_s \in N(v_a)\}$ is an edge cut in G.

Theorem 10 *If* G *is a block with* $deg(v_a) = \delta(G)$*, then the following statements are true in* $S_B(G)$*.*

1. $N(u_a)$ *is a vertex cut.*

2. $Y' = \{(u_a, v_s) | v_s \in N_G(v_a)\}$ *is an edge cut.*

Proof. Consider G to be a block such that $deg(v_a) = \delta(G)$. By Theorem 2 and Corollary 8, $deg(u_a) = \delta(S_B(G))$. As $\kappa(S_B(G)) \leq \lambda(S_B(G) \leq \delta(S_B(G))$ and $N(u_a) = N_G(v_a)$, $N(u_a)$ is a vertex cut and $Y' = \{(u_a, v_s) | v_s \in N(u_a)\}$ is an edge cut in $S_B(G)$.
Hence the theorem. □

Let $S = \{v_j | 1 \leq j \leq t\}$ be a minimum vertex cut of G. As G is an induced subgraph of $S_B(G)$, S is the subset of a vertex cut of $S_B(G)$. The following theorem gives the vertex connectivity of $S_B(G)$.

Theorem 11 *For a connected graph* G *with order* n*,* $n \geq 2$*,*

$$\kappa(S_B(G)) = \begin{cases} min\{\delta(G), 2\kappa(G) + 1\} & when\ G\ is\ a\ block \\ min\{2, \delta(G)\} & otherwise \end{cases}$$

Proof. Let G be a connected graph of order $n \geq 2$. The following cases are considered:

Case 1 Suppose G is not a block.
Let v_c be a cut vertex in G. In $S_B(G)$, the vertices of $N_G(v_c)$ are adjacent to v_c and u_c. Thus, u_c is the cut vertex in $S_B(G)$ if and only if v_c is a pendant vertex in G. In all other cases, removing the vertices $\{v_c, u_c\}$ disconnects the graph $S_B(G)$. Thus, $\kappa(S_B(G)) = 1$, if $\delta(G) = 1$ and $\kappa(S_B(G)) = 2$, otherwise.
Hence, $\kappa(S_B(G)) = \min\{2, \delta(G)\}$.

Case 2 Suppose G is a block.
In $S_B(G)$, there exists exactly one block vertex b_1 adjacent to all the vertices of G. Thus, $S' = S \cup \{u_j, b_1 | 1 \leq j \leq t\}$ and by Theorem 10, $N(u_a)$ are vertex cuts in $S_B(G)$. Here, $|S'| = 2\kappa(G) + 1$ and $|N(u_a)| = \delta(S_B(G)) = \delta(G)$.
Hence, $\kappa(S_B(G)) = \min\{\delta(G), 2\kappa(G) + 1\}$.

Hence the theorem. □

Corollary 12 *For a connected graph* G, $\kappa(S_B(G)) = \kappa(G)$ *if and only if* $\kappa(G) = \delta(G)$.

Proof. Suppose $\kappa(G) = \delta(G)$.
Suppose G is a block. As $\delta(G) < 2\delta(G) + 1$, by Theorem 11, $\kappa(S_B(G)) = \delta(G) = \kappa(G)$. If G is not a block, then $\kappa(G) = 1$. By Theorem 11, $\kappa(S_B(G)) = \delta(G) = \kappa(G)$. The converse can also be proved in the same manner.
Hence the theorem. □

The following theorem gives the necessary and sufficient condition for the existence of a graph whose vertex connectivity is a and the vertex connectivity of its $S_B(G)$ is b, for all a, b such that $1 < a < b$.

Theorem 13 *For integers* a, b *with* $1 < a < b$, *there exists a graph* G *with* $\kappa(G) = a$ *and* $\kappa(S_B(G)) = b$ *if and only if* $b \leq 2a + 1$.

Proof. Assume that $b \leq 2a + 1$. Consider G_1 and G_2 as any two connected block graphs each of minimum degree b. The following assumptions are made.

1. $V(G_1) = \{v_{1_r} | 1 \leq r \leq s, s > b\}$ and
   $V(G_2) = \{v_{2_w} | 1 \leq w \leq t, t > b\}$.

2. Let $deg(v_{x_y}) = b$, where $v_{x_y} \in V(G_1 \cup G_2)$.

The graph G is formed from G_1 and G_2 by adding (v_{1_q}, v_{2_q}) new edges such that $v_{1_q}, v_{2_q} \neq v_{x_y}$ and $1 \leq q \leq a$. Here, G is a block with $deg(v_{x_y}) = b$. The removal of vertices v_{1_q}, $1 \leq q \leq a$, disconnects the graph G. Thus, $X = \{v_{1_q} | 1 \leq q \leq a\}$ and $N_G(v_{x_y})$ are the vertex cuts of G which implies, $\kappa(G) = min\{|X|, |N_G(v_{x_y})|\}$. As $|X| = a$ and $|N_G(v_{x_y})| = b > a$, we get $\kappa(G) = |X| = a$. Hence, $X' = X \cup \{u_{1_q}, b_1 | 1 \leq q \leq a\}$ and by Theorem 10, $N(u_{x_y})$ are the vertex cuts in $S_B(G)$, where $deg(u_{x_y}) = \delta(S_B(G))$. Therefore, $\kappa(S_B(G)) = min\{|X'|, |N(u_{x_y})|\}$. Since, $|X'| = 2a+1$ and $|N(u_{x_y})| = b \leq 2a+1$, we conclude that $\kappa(S_B(G)) = b$.
On the contrary, consider $b > 2a + 1$. Let G be a graph as defined above, then $\kappa(G) = a$ and $\kappa(S_B(G)) = min\{|X'|, |N(u_{x_y})|\}$. Here, $|X'| = 2a + 1$ and $|N(u_{x_y})| = b > 2a + 1$, which implies that $\kappa(S_B(G)) = 2a + 1$.
Therefore, $\kappa(S_B(G)) \neq b$.
Hence the theorem. □

Let $\lambda(G) = t$, i.e., $T = \{(v_e, v_f) | (v_e, v_f) \in E(G), e \neq f\}$ be a minimum edge cut of G. Since G is an induced subgraph of $S_B(G)$, T is the subset of an edge cut of $S_B(G)$. The edge connectivity of $S_B(G)$ is discussed in the next theorem.

Theorem 14 *For a connected graph of order* n, $n \geq 2$,

1. *If* G *has a bridge, then* $\lambda(S_B(G)) = min\{2, \delta(G)\}$.
2. *If G is a block, then* $\lambda(S_B(G)) = \delta(G)$.

Proof. Let G be a connected graph with $n \geq 2$. The following cases are considered:

Case 1 Suppose G has a bridge.
Let $e_m = \{v_g, v_h\}$ be a bridge. As every bridge is a block, let B_m be a block with $V(B_m) = \{v_g, v_h\}$. Thus, $U = \{(b_m, v_g), (b_m, v_h)\}$ is an edge cut in $S_B(G)$, where $b_m \in V(S_B(G)))$ and $|U| = 2$. By Theorem 7, e_m is a bridge in $S_B(G)$ if and only if $\delta(G) = 1$, which implies that $\lambda(S_B(G)) = 1$, when $\delta(G) = 1$. Suppose $\delta(G) \geq 2$, then $\delta(S_B(G)) \geq 2$. Hence, U is the minimum edge cut in $S_B(G)$, for $\delta(G) \geq 2$. Therefore, it is concluded that $\lambda(S_B(G)) = min\{2, \delta(G)\}$.

Case 2 Suppose G is a block.
Then, $\lambda(G) \geq 2$. In $S_B(G)$, b_1 is the only block vertex such that $|N(b_1)| = n$. Thus, $T' = T \cup \{(u_e, v_f), (v_e, u_f), (b_1, v_i) | 1 \leq i \leq n\}$ and by Theorem 10, $Y' = \{(u_a, v_k) | v_k \in N_G(v_a)\}$ are edge cuts in $S_B(G)$. Hence, $\lambda(S_B(G)) = \{|T'|, |Y'|\}$. Here, $|T'| = 3\lambda(G) + n$ and $|Y'| = \delta(G) < n$. Therefore, $\lambda(S_B(G)) = \delta(G)$.

Hence the theorem. □

Note that from Theorem 14 (2), $\lambda(G) = a \geq 2$ and $\lambda(S_B(G)) = \delta(G) \geq a$. This leads to the following corollary.

Corollary 15 *For integers* a, b *with* $1 < a \leq b$*, there exists a graph* G *with* $\lambda(G) = a$ *and* $\lambda(S_B(G)) = b$*.*

Theorem 16 *Let* G *be a connected graph which is bridgeless and not a block. If* G *has* $T = \{(v_c, v_f) | 1 \leq f \leq t, v_c \text{ is a cut vertex}\}$ *as a minimum edge cut, then* $\lambda(S_B(G)) = \min\{\delta(S_B(G)), 3\lambda(G) + 1\}$*.*

Proof. Consider G to be a connected graph which is bridgeless and not a block. Let T be a minimum edge cut. Thus in $S_B(G)$, $T' = T \cup \{(v_c, u_f), (u_c, v_f), (v_c, b_x)\}$, where $v_c, v_f \in B_x$ and by Theorem 10, $Y' = \{(v_a, v_s) | v_s \in N(v_a), deg(v_a) = \delta(G)\}$ are edge cuts. Here, $|T'| = 3\lambda(G) + 1$.
Therefore, $\lambda(S_B(G)) = \min\{\delta(S_B(G)), 3\lambda(G) + 1\}$. □

For any block $B_l, 1 \leq l \leq k$, in G, $|V(B_l)| \geq 2$ if G has a bridge and $|V(B_l)| \geq 3$, otherwise. The following theorem gives a necessary condition for $S_B(G)$ for which its vertex and edge connectivity are equal.

Theorem 17 *If* $\Delta(G) \leq 3$*, then* $\kappa(S_B(G)) = \lambda(S_B(G))$*.*

Proof. Let G be a connected graph of order $n \geq 2$ and $\Delta(G) \leq 3$. The following cases are considered:

Case 1 Suppose $\delta(G) \leq 2$.
By Theorem 11, $\kappa(S_B(G))) = \delta(G)$. When G is bridgeless and is not a block, by Theorem 14, $\lambda(S_B(G)) = \delta(G)$. Consider a graph G which is bridgeless and has a cut vertex then, $\lambda(G) \geq 2$. Since $\lambda(G) \leq \delta(G)$, $\lambda(G) = \delta(G) = 2$. As $|V(B_l)| \geq 3$, for all $1 \leq l \leq k$, by Theorem 7, $\delta(S_B(G)) = \delta(G)$ in $S_B(G)$. Since $\lambda(S_B(G)) \leq \delta(S_B(G))$, it implies that $\lambda(S_B(G)) \leq 2$. The graph G being an induced subgraph of $S_B(G)$, $\lambda(G) \leq \lambda(S_B(G))$ and $\lambda(S_B(G)) = 2 = \delta(G)$.
Therefore, $\kappa(S_B(G)) = \lambda(S_B(G))$.

Case 2 Since $\Delta(G) = 3$, G is 3-regular. If G is a block, then by Theorems 11 and 14, $\kappa(S_B(G)) = \lambda(S_B(G)) = \delta(G)$. When G is 3-regular, G has a bridge if and only if G has a cut vertex. So, if G is not a block, then G has a bridge. Thus, by Theorems 11 and 14, $\kappa(S_B(G)) = \lambda(S_B(G)) = 2$.

Therefore, $\kappa(S_B(G)) = \lambda(S_B(G))$.
Hence the theorem. □

A graph G is p-connected (p-edge connected) if and only if every pair of vertices is joined by at least p vertex (edge) disjoint paths. Also, every p-connected graph is p-edge connected. Further, G is p-edge connected if and only if each of its blocks is p-edge connected. It follows that G is p-connected if and only if each of its blocks is p-edge connected. So, $\lambda(B_l) \geq k$, which implies $\delta(B_l) \geq k$, where $1 \leq l \leq k$. Now, the p-connectedness (p-edge connected) of $S_B(G)$ is discussed.

Theorem 18 *If G is p-connected (p-edge connected) with $p \geq 1$, then $S_B(G)$ is also p-connected (p-edge connected).*

Proof. Let G be a p-connected (p-edge connected) graph with $p \geq 1$. To prove $S_B(G)$ is p-connected (p-edge connected) it is enough to show that between any two vertices of $S_B(G)$, there exist p-vertex (edge) disjoint paths. The following cases are considered:

Case 1 Let $v_a, v_b \in V(G)$
Then, v_a and v_b have at least p vertex (edge) disjoint paths between them. Since G is an induced subgraph of $S_B(G)$, in $S_B(G)$ there exist at least p disjoint paths between the vertices v_a and v_b.

Case 2 Let $v_a \in V(G)$ and $u_b \in V_1(G)$
Since $p \leq \kappa(G) \leq \delta(G)$, there exist at least p vertices adjacent to v_a. Assume that $v_{a_1}, v_{a_2}, \ldots, v_{a_p}$ are the vertices adjacent to v_a in G. Then in $S_B(G)$, by Case 1 that there exist p vertex (edge) disjoint paths between v_a and v_{a_i} $(1 \leq i \leq p)$. In addition, the vertices $u_{a_i} \in V_1(G)$, $1 \leq i \leq p$, are adjacent to the vertex v_a. As $u_b \in V_1(G)$, by Theorem 6, $N(u_b) = N(v_b) \cap V(G)$ for some $v_b \in V(G)$. Since $|N(v_b)| \geq p$, let $v_{b_1}, \ldots, v_{b_p} \in V(G)$ such that $v_{b_i} \in N(v_b)$ for $1 \leq i \leq p$, which implies in $S_B(G)$, $v_{b_i} \in N(u_b)$. As $v_a, v_{b_i} \in V(G)$, by Case 1 in $S_B(G)$ there exist p vertex (edge) disjoint paths between them. In conclusion, the vertices v_a, u_b are joined by p vertex (edge) disjoint paths between them in $S_B(G)$.

Case 3 Let $v_a \in V(G)$, $b_m \in B(G)$.
As defined in Case 2, let $v_{a_i} \in N(v_a)$, $1 \leq i \leq p$. Since $\delta(B_m) \geq k$, we have $|V(B_m)| > k$. Let $v_{x_1}, \ldots, v_{x_p} \in V(G)$ such that $v_{x_i} \in N(b_m)$, $1 \leq i \leq p$. It follows from Case 1 that in $S_B(G)$, the vertices v_a and v_{x_i} have p vertex (edge) disjoint paths between them as $v_a, v_{x_i} \in V(G)$. Therefore, in $S_B(G)$, the vertices v_a, b_m are joined by p vertex (edge) disjoint paths between them.

Case 4 Let $u_a, u_b \in V_1(G)$
As defined in Case 2, let $v_{a_i} \in N(u_a)$ and $v_{b_i} \in N(u_b)$, $1 \leq i \leq p$. Since $v_{a_i}, v_{b_i} \in V(G)$ and by Case 1, the theorem follows.

Case 5 Let $u_a \in V_1(G)$, $b_m \in B(G)$
It follows from Case 2 and 3, let $v_{a_i} \in N(u_a)$ and $v_{x_i} \in N(b_m)$, $1 \leq i \leq p$, respectively. As $v_{a_i}, v_{x_i} \in V(G)$ and by Case 1, the theorem follows.

Case 6 Let $b_m, b_s \in B(G)$
It follows from Case 3, let $v_{x_i} \in N(b_m)$ and $v_{y_i} \in N(b_s)$, $1 \leq i \leq p$. As $v_{x_i}, v_{y_i} \in V(G)$ and by Case 1, the theorem follows.

Hence the theorem. □

If G is p-connected, then $G + K_1$ is $(p+1)$-connected. This result leads to the following theorem.

Theorem 19 *If G is p-connected, then the following statements are true:*

1. $S_B(G + K_1)$ *is* $(p+1)$*-connected.*
2. $S_B(G) + K_1$ *is* $(p+1)$*-connected.*

Proof. Consider the graph G to be a p-connected. Then, $G + K_1$ is $(p+1)$-connected. Also, by Theorem 18, $S_B(G)$ is p-connected.
Hence the theorem follows. □

Conclusion

The structural properties of $S_B(G)$ have been investigated which helped to determine the results on the vertex and edge connectivity of $S_B(G)$. The semi-splitting block graph has been characterized based on its vertex set. A necessary condition for p-connectedness (p-edge connectedness) of $S_B(G)$ has been established. The scope of future work is to characterize graphs whose $S_B(G)$ is $(p+1)$-connected ($(p+1)$-edge connected).

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