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Eccentric connectivity index in transformation graph G^{xy+}

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Abstract. Let G be a connected graph with vertex set $V(G)$and edge set $E(G)$. The eccentric connectivity index of G is defined as $\sum_{v\in V(G)} \mathrm{ec}(v)\deg(v)$ where $ec(v)$ the eccentricity of a vertex v and $\deg(v)$is its degree and denoted by $\varepsilon^c(G)$. In this paper, we investigate the eccentric connectivity index of the transformation graph G^{xy+}.

1 Introduction

A topological index is a number that describes a molecular structure and is obtained from the associated (hydrogen-depleted) molecular graph. Topological indices are mathematical properties of graphs that are utilized to establish relationships between the structural properties of chemical molecules and their physical attributes. The aforementioned indices are extensively utilized in the fields of quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR), chemical documentation and drug design studies [6, 7, 8, 11, 13, 15].

Key words and phrases: Distance, eccentricity, eccentric connectivity index, transformation graph.

In pharmaceutical research, QSAR data is utilized to identify the most viable compounds with respect to a specific property, thereby reducing the number of compounds that must be synthesized in the process of designing new drugs. Despite the fact that many topological indices have been described, only a small number of them have been used effectively in QSAR investigations. These include Wiener's index, Balaban's index, Hosoya's index, Randic's molecular connectivity index, and the eccentric connectivity index [2, 10, 9, 21]. Eccentricity has been used to create a variety of indexes [3, 12, 4, 5, 14, 20] . Some of these are eccentric connectivity index, graph shape index, and connective eccentricity index. In this study, we discussed the index which is defined, in 1997, by Sharma et al., as eccentric connectivity [14]. The eccentric connectivity index $\varepsilon^c(G)$ of G is defined as $\varepsilon^c(G) = \sum_{v \in V(G)} \mathrm{ec}(v)\deg(v)$.

Consider a simple connected graph denoted by G with its set of vertices represented as $V(G)$ and the set of edges as $E(G)$. The metric that quantifies the distance between two vertices u and v in a graph G, denoted by $d_G(u, v)$, is defined as the minimum number of edges that must be traversed in order to travel from u to v along the shortest path in G. The vertex eccentricity, denoted as $\mathrm{ec}_G(u)$, in a graph G refers to the greatest distance between vertex u and any other vertex in G. The mathematical definition of the diameter, denoted as d, of a graph G is the largest possible value of the eccentricities of all vertices in G. The definition of the radius of a graph G is such that it corresponds to the minimum value of the eccentricities of the vertices that comprise G. In graph theory, a vertex in a graph G is considered to be central if its eccentricity is equivalent to the radius of G. The number of edges that are connected to a vertex $w \in V(G)$ is defined as the degree of the vertex, denoted by $\deg_G(w)$. A graph theory term for a vertex with only one adjacent vertex is a pendant vertex, also known as a leaf vertex, of a given graph G. The open neighborhood and closed neighborhood of a vertex v in a graph G are defined as $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Let the set $N_G^i(v)$ be the set of vertices where the vertex v is at a distance i in the graph G. That is, $N_G^i(v) = \{u \in V(G) \mid d(v, u) = i\}$. Thus, we have $N(v) = N_G(v) = N_G^1(v)$ and $N[v] = N_G[v] = N_G^1(v) \cup \{v\}$ [16].

The vertex set of the complement $\bar{G}$ of a graph G consists of the same vertices as G, but in $\bar{G}$, two vertices are adjacent if and only if they are not adjacent in G. On the other hand, the line graph $L(G)$ of G is a graph whose vertex set is composed of the edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [16].

The transformation graph G^{xyz} is a graph whose vertex set is $V(G) \cup E(G)$, and $s, t \in V(G^{xyz})$. The vertices s and t are adjacent in G^{xyz} if and only if one of the following properties holds [1, 19, 18, 17]:

(P1) Consider $s, t \in V(G)$. If $x = +$, then $t \in N_G(s)$; while if $x = -$, then $t \notin N_G(s)$.
(P2) Consider $s, t \in E(G)$. If $y = +$, then $t \in N_G(s)$; while if $y = -$, then $t \notin N_G(s)$.
(P3) Consider $s \in V(G), t \in E(G)$. If $z = +$, then s is the end-vertex of t; while if $z = -$, then s is not the end-vertex of t.

In this paper, we study about eccentric connectivity index of the transformation graph G^{xy+}. Various notations are employed to enhance the comprehensibility of the proofs of the aforementioned theorems. Consider two arbitrary vertices s and t in the graph G. In the context of graph theory, it is customary to denote the edge between two adjacent vertices s and t in a graph G as e_{st}. Moreover, the aforementioned edge is denoted by the vertex st within the graph G^{xyz}.

Theorem 1 [22] *Let* G *be a connected graph with* m *edges. Then,*

$$2m(\text{rad}(G)) \leq \varepsilon^c(G) \leq 2m(\text{diam}(G)).$$

2 Eccentric connectivity index for the graph G^{xy+}

We begin this subsection by determining the eccentric connectivity index of the transformation graph G^{xy+} when G is a specified family of graphs.

Theorem 2 *When* $xyz = + - +$*, let the transformation graph of the graph* G *be* G^{+-+} *and* q *is the number of edges of the graph* G^{+-+}*.*

(a) If $G \cong P_n$ $(n \geq 6)$*, then* $\varepsilon^c(G^{+-+}) = 2n^2 + 6n - 4$*;*

(b) If $G \cong C_n$ $(n \geq 6)$*, then* $\varepsilon^c(G^{+-+}) = 2n^2 + 10n$*;*

(c) If $G \cong K_n$ $(n \geq 4)$*, then* $\varepsilon^c(G^{+-+}) = (n-1)\left(\frac{n^3-5n^2+18n}{2}\right) = 4q$*;*

(d) If $G \cong K_{1,n}$ $(n \geq 3)$*, then* $\varepsilon^c(G^{+-+}) = 10n$*;*

(e) If $G \cong W_{1,n}$ $(n \geq 3)$*, then* $\varepsilon^c(G^{+-+}) = 6n^2 + 10n = 4q$*;*

(f) If $G \cong K_{m,n}$ $(m, n \geq 2)$*, then* $\varepsilon^c(G^{+-+}) = 2mn(mn - m - n + 7)$*;*

Proof. (a) Let x and y be the pendant vertices of the graph G. It is easily seen that $\deg_{G^{+-+}}(x) = \deg_{G^{+-+}}(y) = 2$ and for all $v \in V(G) - \{x, y\}$ $\deg_{G^{+-+}}(v) = 4$. Furthermore, $\deg_{G^{+-+}}(xN_G(x)) = \deg_{G^{+-+}}(yN_G(y)) = n - 1$ and for $uv \in V(\overline{L(G)}) - \{xN_G(x),\ yN_G(y))$ $\deg_{G^{+-+}}(uv) = n - 2$. The eccentricity value of the vertices of the graph is calculated according to the vertices as follows.

- $e(u)$ value of for all $u \in V(G)$:
  Let $A = N_G(u),\ B = V(G) - N_G[u],\ C = uN_G(u),\ D = V(\overline{L(G)}) - C$. The shortest distance between the vertex u and the vertices in $A \cup C$, in $D \cup N_G^2(u)$ and in $V(G) - (N_G[u] \cup N_G^2(u))$ is 1, 2 and 3, respectively. Thus, we get $e(u) = 3$.

- $e(uv)$ value of for all $uv \in V(\overline{L(G)})$:
  Let $A = N_{\overline{L(G)}}(uv)$ and $B = V(L(G)) - N_{\overline{L(G)}}[uv]$. The shortest distance between the vertex uv and the vertices in $A \cup \{u, v\}$, in $N_G(u) - \{v\} \cup N_G(v) - \{u\} \cup N_G(A)$ and in $B = N_{\overline{L(G)}}(A) - N_{\overline{L(G)}}[uv]$ is 1, 2 and 2, respectively. Thus, we get $e(uv) = 2$.
  With the results we found, we get, for $n \geq 6$,

$$\begin{aligned} \varepsilon^c\left(G^{+-+}\right) &= \sum \deg(v)e(v) \\ &= 2 \cdot 2 \cdot 3 + (n-2) \cdot 4 \cdot 3 + 2 \cdot (n-1) \cdot 2 + (n-3)(n-2) \cdot 2 \\ &= 2n^2 + 6n - 4. \end{aligned}$$

(b) We can easily observe that for $\forall v \in V(G)\ \deg_{G^{++}}(v) = 4$ and for $uv \in V(\overline{L(G)})$ $\deg_{G^{+}}(uv) = n - 1$. The eccentricity value of the vertices of the graph is calculated according to the vertices as follows.

- $e(u)$ value of $\forall u \in \overline{V(G)}$: Let $A = N_G(u), B = V(G) - N_G[u], C = uN_G(u)$ and $D = V(\overline{L(G)}) - C$. The shortest distance between the vertex u and the vertices in $A \cup C$, in $D \cup N_G^2(u)$ and in $V(G) - \left(N_G[u] \cup N_G^2(u)\right)$ is 1, 2 and 3, respectively. Thus, we get $e(u) = 3$.

- $e(uv)$ value of $\forall uv \in V(\overline{L(G)})$: Let $A = N_{\overline{L(G)}}(uv)$ and $B = V(\overline{L(G)}) - N_{\overline{L(G)}}[uv]$. The shortest distance between the vertex uv and the vertices in $A \cup \{u, v\}$, in $N_G(u) - \{v\} \cup N_G(v) - \{u\} \cup N_G(A)$ and in $B = N_{\overline{L(G)}}(A) - N_{\overline{L(G)}}[uv]$ is 1, 2 and 2, respectively. Thus, we get $e(uv) = 2$.

  With the results we found, we get, for $n \geq 6$,

  $\varepsilon^c\left(G^{+-+}\right) = \sum \deg(v)e(v) = n \cdot 4 \cdot 3 + n \cdot (n-1) \cdot 2 = n^2 + 10n.$

(c) It can easily be observed that for $\forall v \in V(G)\, \deg_{G^{+-+}}(v) = 2(n-1)$ and for $uv \in V(\overline{L(G)})\, \deg_{G^{+-+}}(uv) = \frac{n^2-5n+10}{2}$. The eccentricity value of the vertices of the graph is calculated according to the vertices as follows.

- $e(u)$ value of $\forall u \in V(G)$: Let $A = N_G(u), B = V(G) - N_G[u] = \varnothing, C = uN_G(u)$ and $D = V(\overline{L(G)}) - C$. The shortest distance between the vertex u and the vertices in $A \cup C$ and in $D = N_{\overline{L(G)}}(C)$ is 1 and 2, respectively. Thus, we get $e(u) = 2$.

- $e(uv)$ value of $\forall uv \in V(\overline{L(G)})$: Let $A = N_{\overline{L(G)}}(uv), B = V(\overline{L(G)}) - N_{\overline{L(G)}}[uv], C = N_G(uv)$ and $D = V(G) - C$. The shortest distance between the vertex uv and the vertices in $A \cup C$, in $B = N_{\overline{L(G)}}(A) - N_{\overline{L(G)}}[uv]$ and D is 1, 2 and 1, respectively. Thus, we get $e(uv) = 2$. With the results we found, we get, for $n \geq 4$,

$$\begin{aligned}\varepsilon^c\left(G^{+-+}\right) &= \sum \deg(v)e(v) = n.2(n-1).2 + \frac{n(n-1)}{2}.\frac{n^2-5n+10}{2}.2 \\ &= (n-1)\left(\frac{n^3-5n^2+18n}{2}\right)\end{aligned}$$

Furthermore, since the graph's vertices have an eccentricity value of 2, according to Theorem 1, $\varepsilon^c\left(G^{+-+}\right) = 4q$ where q is the number of edges of the graph G^{+-+}.

(d) Let c be the central vertex of the graph G. It is easily seen that $\deg_{G^{+-+}}(c) = 2n$ and for $\forall v \in V(G) - \{c\} \quad \deg_{G^{+-+}}(v) = 2$. Since the structure of $\overline{L(K_{1,n})}$ consists of n isolated vertices, $\deg_{G^{+-+}}(cN_G(c)) = 2$. Also, $e(c) = 1$ and for $\forall w \in V(G^{+-+}) - \{c\}, e(w) = 2$.

With the results we found, we get, for $n \geq 3$,
$\varepsilon^c\left(G^{+-+}\right) = \sum \deg(v)e(v) = 2 \cdot n + 2 \cdot n \cdot 2 \cdot 2 = 10n.$

(e) Let c be the central vertex of the graph G. It is easy to see that $\deg_{G^{+-+}}(c) = 2n$ and for $\forall v \in V(G) - \{c\} \deg_{G^{+-+}}(v) = 6$. For the vertices corresponding to the edges connecting the central vertex and the vertices on the cycle graph, $\deg_{G^{+-+}}(cN_G(c)) = n$ and for $\forall uv \in V(\overline{L(G)}) - \{cN_G(c)\} \deg_{G^+}(uv) = 2n-3$. The graph's vertices have an eccentricity value of 2.

With the results we found, we get, for $n \geq 3$,

$$\varepsilon^c\left(G^{+-+}\right) = \sum \deg(v)e(v) = 1 \cdot 2n \cdot 2 + n \cdot 6 \cdot 2 + n \cdot n \cdot 2 + n \cdot (2n-3) \cdot 2 = 6n^2 + 10n.$$

Furthermore, since the graph's vertices have an eccentricity value of 2, according to Theorem 1, $\varepsilon^c\left(G^{+-+}\right) = 4q$ where q is the number of edges of the graph G^{+-+}.

(f) While the degree of m vertices in the graph G is $\deg_{G^{+-+}}(v) = 2n$, the degree of n vertices is $\deg_{G^{+-+}}(v) = 2m$. Since an edge in the graph G is connected to $(m-1)+(n-1)$ edges, each vertex in the graph $L(G)$ is adjacent to $(m-1)+(n-1)$ vertices. Therefore, in the graph $\overline{L(G)}$, each vertex is adjacent to $mn-1-(m+n-2)$ vertices. Therefore, for $\forall uv \in V(\overline{L(G)})$ $\deg_{G^{+-+}}(uv) = mn-m-n+3$. The graph's vertices have an eccentricity value of 2.

With the results we found, we get, for $m, n \geq 2$

$$\begin{aligned}\varepsilon^c\left(G^{++}\right) &= \sum \deg(v)e(v)\\ &= m\cdot 2n\cdot 2 + n\cdot 2m\cdot 2 + m\cdot n(mn-m-n+3)\cdot 2\\ &= 2mn(mn-m-n+7).\end{aligned}$$

Furthermore, since the graph's vertices have an eccentricity value of 2, according to Theorem 1, $\varepsilon^c\left(G^{++}\right) = 4q$ where q is the number of edges of the graph G^{+-+}.

The theorem is thus proved. □

Theorem 3 *When $xyz = -++$, let the transformation graph of the graph G be G^{-++} and q is the number of edges of the graph G^{-++}.*

(a) If $G \cong P_n$ $(n \geq 6)$, then $\varepsilon^c(G^{-++}) = 2n^2 + 10n - 18$;

(b) If $G \cong C_n$ $(n \geq 6)$, then $\varepsilon^c(G^{-++}) = 2n^2 + 10n$;

(c) If $G \cong K_n$ $(n \geq 3)$, then $\varepsilon^c(G^{-++}) = 2n^2(n-1) = 4q$;

(d) If $G \cong W_{1,n}$ $(n \geq 3)$, then $\varepsilon^c(G^{-++}) = 4n^2 + 26n$;

(e) If $G \cong K_{m,n}$ $(m, n \geq 2)$, then $\varepsilon^c(G^{-++}) = 2(m+n)(mn+m+n-1) = 4q$.

Proof. (a) Let x and y be the pendant vertices of the graph G. It is easily seen that for $\forall v \in V(\bar{G})$ $\deg_{G^{++}}(v) = n-1$. Furthermore, $\deg_{G^{-++}}(xN_G(x)) = \deg_{G^{-++}}(yN_G(y)) = 3$ and for $uv \in V(\overline{L(G)}) - \{xN_G(x), yN_G(y)\}$ $\deg_{G^{-++}}(uv) = 4$. The eccentricity value of the vertices of the graph is calculated according to the vertices as follows.

- $e(u)$ value of $\forall u \in V(\bar{G})$: Let $A = N_{\bar{G}}(u), B = V(\bar{G}) - N_{\bar{G}}[u], C = N_{L(G)}(u)$ and $D = V(L(G)) - C$. The shortest distance between the vertex u and the vertices in $A \cup C$ and in $B \cup D$ is 1 and 2, respectively. Thus, we get $e(u) = 2$.

- $e(uv)$ value of $\forall uv \in V(L(G))$: Let $A = N_{\bar{G}}(uv), B = V(\bar{G}) - N_{\bar{G}}(uv)$ and $C = N_{L(G)}(uv)$. The shortest distance between the vertex uv and the vertices in $A \cup C$, in B and in $V(L(G)) - \left\{N_{L(G)}[uv] \cup N^2_{L(G)}(uv)\right\}$ is 1, 2 and 3, respectively. Thus, we get $e(uv) = 3$.

  With the results we found, we get, for $n \geq 6$, we have

$$\begin{aligned}\varepsilon^c\left(G^{-++}\right) &= \sum \deg(v)e(v) = n\cdot(n-1)\cdot 2 + 2\cdot 3\cdot 3 + (n-1-2)\cdot 4\cdot 3 \\ &= 2n^2 + 10n - 18.\end{aligned}$$

(b) It is easily seen that for $\forall v \in V(\bar{G})$ $\deg_{G^{-++}}(v) = n-1$ and for $\forall uv \in L(P_n)$ $\deg_{G^{-++}}(uv) = 4$. As in Theorem 2 (a), we get eccentricity value for $\forall u \in V(\bar{G})$ and $\forall uv \in V(L(G))$ is $e(u) = 2$ and $e(uv) = 3$, respectively.
With the results we found, we get, for $n \geq 6$, we have
$\varepsilon^c\left(G^{-++}\right) = \sum \deg(v)e(v) = n(n-1)2 + n\cdot 4.3 = 2n^2 + 10n.$

(c) We can easily observe that for $\forall v \in V(\bar{G})$ $\deg_{G^{-++}}(v) = n-1$ and for all $uv \in L(G)$ $\deg_{G^{-++}}(uv) = 2n-2$. It is also seen that he eccentricity value of $\forall u \in V\left(G^{-++}\right)$is $e(u) = 2$. With the results we found, we get, for $n \geq 5$,
$\varepsilon^c\left(G^{-++}\right) = \sum \deg(v)e(v) = n(n-1)2 + \frac{n(n-1)}{2}2(n-1)2 = 2n^2(n-1).$
Furthermore, since the graph's vertices have an eccentricity value of 2, according to Theorem1, $\varepsilon^c\left(G^{-++}\right) = 4q$ where q is the number of edges of the graph G^{-++}.

(d) Let c be the central vertex of the graph G. It is easy to see that $\deg_{G^{-++}}(c) = n$ and for $\forall v \in V(\bar{G}) - \{c\}\deg_{G^{-++}}(v) = n$. For the vertices corresponding to the edges connecting the central vertex and the vertices on the cycle graph, $\deg_{G^{-++}}(cN_G(c)) = n+3$ and for $\forall uv \in V(L(G)) - \{cN_G(c)\}\deg_{G^{-++}}(uv) = 6$. The graph's vertices have an eccentricity value of 2.
With the results we found, we get, for $n \geq 3$,

$\varepsilon^c\left(G^{-++}\right) = \sum \deg(v)e(v) = (n+1)\cdot n\cdot 2 + n(n+3)2 + n\cdot 6\cdot 3 = 4n^2 + 26n.$

(e) In the graph $\bar{G}$, the degree of each vertex v is $\deg_{G^{-++}}(v) = m + n - 1$. Since an edge in the graph G is connected to $(m-1)+(n-1)$ edges, each vertex in the graph $L(G)$ is adjacent to $(m-1)+(n-1)$ vertices. Therefore, for $\forall uv \in V(L(G))$ $\deg_{G^{-++}}(uv) = (m-1)+(n-1)+2 = m+n$. The graph's vertices have an eccentricity value of 2.

With the results we found, we get, for $m, n \geq 2$,

$\varepsilon^c\left(G^{-++}\right) = \sum \deg(v)e(v) = 2(m+n)(mn+m+n-1)$.

Furthermore, since the graph's vertices have an eccentricity value of 2, according to Theorem 1, $\varepsilon^c\left(G^{-++}\right) = 4q$ where q is the number of edges of the graph G^{-++}.

The theorem is thus proved. □

Theorem 4 *When $xyz = --+$, let the transformation graph of the graph G be G^{--+}, and q is the number of edges of the graph G^{--+}.*

(a) If $G \cong P_n$ $(n \geq 3)$, then $\varepsilon^c\left(G^{--+}\right) = 4\left(n^2 - 2n + 2\right) = 4q$;

(b) If $G \cong C_n$ $(n \geq 4)$, then $\varepsilon^c\left(G^{--+}\right) = 4\left(n^2 - n\right) = 4q$;

(c) If $G \cong K_n$ $(n \geq 4)$, then $\varepsilon^c\left(G^{--+}\right) = n(n-1)(n-7)(n+2)/2 = 4q$;

(d) If $G \cong K_{1,n}$ $(n \geq 3)$, then $\varepsilon^c\left(G^{--+}\right) = 2n^2 + 6n = 4q$;

(e) If $G \cong W_{1,n}$ $(n \geq 4)$, then $\varepsilon^c\left(G^{--+}\right) = 8n^2 - 4n = 4q$.

Proof. (a) Let x and y be the pendant vertices of the graph G. It is easily seen that for $\forall v \in V(\bar{G})$ $\deg_{G^{--+}}(v) = n-1$. Furthermore, $\deg_{G^{--+}}(xN_G(x)) = \deg_{G^{--+}}(yN_G(y)) = (n-2-1)+2 = n-1$ and for $\forall uv \in V(\overline{L(G)}) - \{xN_G(x), yN_G(y)\} \deg_G(uv) = n-2$. The graph's vertices have an eccentricity value of 2.

With the results we found, we get, for $n \geq 3$, we have

$$\begin{aligned}\varepsilon^c\left(G^{--+}\right) &= \sum \deg(v)e(v) = n \cdot (n-1) \cdot 2 + 2 \cdot (n-1) \cdot 2 + (n-3)(n-2)2 \\ &= 4\left(n^2 - 2n + 2\right).\end{aligned}$$

(b) We can easily observe that for $\forall v \in V\left(G^{--+}\right) \deg_{G^{--+}}(v) = n-1$. The graph's vertices have an eccentricity value of 2. With the results we found, we get, for $n > 3$,

$$\varepsilon^c\left(G^{--+}\right) = \sum \deg(v)e(v) = 2n(n-1)2 = 4n(n-1).$$

(c) We get for $\forall v \in V(\bar{G})$ $\deg_{G^{--+}}(v) = n-1$ and $\forall uv \in V(\overline{L(G)})$ $\deg_{G^{--+}}(uv) = \left(n^2-5n+10\right)/2$. The graph's vertices have an eccentricity value of 2. With the results we found, we get, for $n > 3$,

$$\begin{aligned}\varepsilon^c\left(G^{--+}\right) &= \sum \deg(v)e(v) = n(n-1)2 + n(n-1)/2.\left(n^2-5n+10\right)/2.2 \\ &= n(n-1)(n-7)(n+2)/2.\end{aligned}$$

(d) Let c be the central vertex of the graph G. It is easy to see that $\deg_{G^{--+}}(c) = n$ and for $\forall v \in V(\bar{G}) - \{c\} \deg_{G^{--+}}(v) = n$. Since $\overline{L(K_{1,n})}$ contains n isolated peaks, for $\forall uv \in V(\overline{L(G)}) \deg_{G^{--+}}(uv) = 2$. The graph's vertices have an eccentricity value of 2. With the results we found, we get, for $n \geq 3$,

$$\varepsilon^c\left(G^{--+}\right) = \sum \deg(v)e(v) = (n+1)\cdot n \cdot 2 + n \cdot 2 \cdot 2 = 2n^2 + 6n.$$

(e) Let c be the central vertex of the graph G. It is easy to see that $\forall v \in V(\bar{G})$ $\deg_{G^{--+}}(v) = n$. For the vertices corresponding to the edges connecting the central vertex and the vertices on the cycle graph, $\deg_{G^{--+}}\left(cN_G(c)\right) = n$ and for $\forall uv \in V(\overline{L(G)}) - \{cN_G(c)\}$ $\deg_{G^{--+}}(uv) = 2n-3$. The graph's vertices have an eccentricity value of 2.

With the results we found, we get, for $n \geq 4$,

$$\varepsilon^c\left(G^{--+}\right) = \sum \deg(v)e(v) = (n+1)\cdot n\cdot 2 + n\cdot n\cdot 2 + n\cdot(2n-3)2 = 2n(4n-2).$$

Because of the form of the graph G^{--+} $(G \cong P_n, C_n, K_n, K_{1,n}, W_{1,n})$, it can be easily seen from above that the graph's vertices have an eccentricity value of 2. According to Theorem 1, we get $\varepsilon^c\left(G^{--+}\right) = 4q$ where q is the number of edges of the graph G^{--+}.

The theorem is thus proved. □

Theorem 5 *When* $xyz = +++$*, let the transformation graph of the graph* G *be* G^{+++} *and* q *is the number of edges of the graph* G^{+++}.

(a) If $G \cong C_n$ $(n \geq 3)$*, then* $\varepsilon^c\left(G^{+++}\right) = 8n\left\lceil \frac{n}{2} \right\rceil$;

(b) If $G \cong P_n$ $(n \geq 5)$*, then* $\varepsilon^c\left(G^{+++}\right) = 6(n-1)^2$;

(c) If $G \cong K_n$ $(n \geq 3)$*, then* $\varepsilon^c\left(G^{+++}\right) = 2\left(n^2+n\right)(n-1) = 4q$;

(d) If $G \cong K_{1,n}$ $(n \geq 3)$*, then* $\varepsilon^c\left(G^{+++}\right) = 2n^2 + 8n$;

(e) If $G \cong W_{1,n}$ $(n \geq 5)$*, then* $\varepsilon^c\left(G^{+++}\right) = 2n^2 + 46n$.

Proof. (a) The graph G^{+++} is a 4-regular graph. We get that the graph's vertices have an eccentricity value of $\lceil \frac{n}{2} \rceil$. With the results we found, we get, for $n \geq 3$, $\varepsilon^c(G^{++}) = \sum \deg(v)e(v) = 8n \lceil \frac{n}{2} \rceil$.

(b) Let x and y be the pendant vertices of the graph G. It is easily seen that $\deg_{G^{+++}}(x) = \deg_{G^{+++}}(y) = 2$ and for $\forall v \in V(G) - \{x, y\}$ $\deg_{G^{+++}}(v) = 4$. Furthermore, $\deg_{G^{+++}}(xN_G(x)) = \deg_{G^{+++}}(yN_G(y)) = 3$ and for $\forall uv \in V(L(G)) - \{xN_G(x), yN_G(y)\}$ $\deg_{G^{+++}}(uv) = 4$. There occur two cases depending on n for the graph's vertices' eccentricity values.

To make the proof clearer, let the n vertices of G be $v_1, v_2, v_3 \ldots v_{n-1,}, v_n$ and the $n-1$ vertices of $L(G)$ be $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n$.

Case 1. For n even. By the definition of the eccentricity value, for every vertex of G, it can easily be observed that,

$$e\left(v_{\frac{n}{2}}\right) = e\left(v_{\frac{n}{2}+1}\right), e\left(v_{\frac{n}{2}-1}\right) = e\left(v_{\frac{n}{2}+2}\right), \ldots, e(v_2) = e(v_{n-1}), e(v_1) = e(v_n).$$

Thus, we have $e(v_i) = e\left(v_{n-(i-1)}\right) = n - i$, where $i \in \{1, 2, \ldots, n/2\}$.
For every vertex of $L(G)$, we receive the following equalities.

$e\left(v_{\frac{n}{2}-1}v_{\frac{n}{2}}\right) = e\left(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}}\right), e\left(v_{\frac{n-4}{2}}v_{\frac{n-2}{2}}\right) = e\left(v_{\frac{n+4}{2}}v_{\frac{n+6}{2}}\right), \ldots, e(v_2v_3) = e(v_{n-2}v_{n-1}), e(v_1v_2) = e(v_{n-1}v_n)$. These value are $e(v_jv_{j+1}) = e\left(v_{n-j}v_{n-(j-1)}\right) = n - j$, where $j \in \{1, 2, \ldots, (n/2) - 1\}$ and $e\left(v_{\frac{n}{2}}v_{\frac{n}{2}+1}\right) = n - \frac{n}{2}$.

With the results we found, we get, for $n \geq 6$,

$$\begin{aligned} \varepsilon^c\left(G^{+++}\right. &= 2\left(2(n-1) + \sum_{i=n/2}^{n-2} 4i\right) + 2\left(3(n-1) + \sum_{j=(n+2)/2}^{n-2} 4j\right) + 4\frac{n}{2} \\ &= 6n^2 - 12n + 6 = 6(n-1)^2. \end{aligned}$$

Case 2. For n odd. The eccentricity values for every vertex of G are

$$e\left(v_{\frac{n-1}{2}}\right) = e\left(v_{\frac{n+1}{2}+1}\right), e\left(v_{\frac{n-3}{2}}\right) = e\left(v_{\frac{n+1}{2}+2}\right), \ldots, e(v_2) = e(v_{n-1}),$$

$e(v_1) = e(v_n)$.

It is easy to see that $e(v_i) = e\left(v_{n-(i-1)}\right) = n - i$ where $i \in \left\{1, 2, \ldots, \frac{n-1}{2}\right\}$ and $e\left(v_{\frac{n+1}{2}}\right) = n - \frac{n+1}{2}$. Since the vertices in the $L(G)$ subgraph are even with degrees, the eccentricity values of the vertices are as in Case 1.

With the results we found, we get, for $n \geq 5$,

$$\varepsilon^c\left(G^{+++}\right) = 2\left(2(n-1) + \sum_{i=(n+1)/2}^{n-2} 4i\right) + 4\frac{n-1}{2} + 2\left(3(n-1) + \sum_{i=(n+1)/2}^{n-2} 4i\right)$$
$$= 6n^2 - 12n + 6 = 6(n-1)^2.$$

(c) We get for $\forall v \in V(G)$ $\deg_{G^{+++}}(v) = 2(n-1)$ and $\forall uv \in V(L(G))$ $\deg_{G^{+++}}(uv) = 2(n-1)$. The graph's vertices have an eccentricity value of 2. With the results we found, we get, for $n \geq 3$,

$$\varepsilon^c\left(G^{+++}\right) = \sum \deg(v)e(v) = 2\left(n^2 + n\right)(n-1).$$

Furthermore, since the graph's vertices have an eccentricity value of 2, according to Theorem 1, $\varepsilon^c\left(G^{+++}\right) = 4q$ where q is the number of edges of the graph G^{+++}.

(d) Let c be the central vertex of the graph G. It is easy to see that $\deg_{G^{+++}}(c) = 2n$ and for $\forall v \in V(G) - \{c\}$ $\deg_{G^{+++}}(v) = 2$. Since the structure of the $L(G)$ subgraph is a complete graph with n vertices, for $\forall uv \in V(L(G))$ $\deg_{G^{+++}}(uv) = n + 1$. Also, it is easily seen that $e(c) = 1$ and for $\forall v \in V\left(G^{+++}\right) - \{c\}$ $e(v) = 2$.

With the results we found, we get, for $n \geq 3$,

$$\varepsilon^c\left(G^{+++}\right) = \sum \deg(v)e(v) = 2n + n \cdot 2 \cdot 2 + n \cdot (n+1)2 = 2n^2 + 8n.$$

(e) The vertex set of G^{+++} can be partitioned into four subsets as

$V_1\left(G^{+++}\right)$; central vertex c of G. $\deg_{G^{+++}}(c) = 2n$.

$V_2\left(G^{+++}\right) = V(G) - \{c\}$. For $\forall u \in V_2\left(G^{+++}\right)$, $\deg_{G^{+++}}(u) = 6$.

$V_3\left(G^{+++}\right)$: the $\{cN_G(c)\}$ vertices in $L(G)$. For $\forall xy \in V_3\left(G^{+++}\right)$ $\deg_{G^{+++}}(xy) = n + 3$.

Furthermore, the vertices of $V_3\left(G^{+++}\right)$ are a complete graph in themselves. $V_4\left(G^{+++}\right)$; the vertices in $L(G)$ formed by the edges of the graph C_n. For $\forall xy \in V_4\left(G^{+++}\right)$ $\deg_{G^{+++}}(xy) = 6$.

The eccentricity values of the vertices are as follows: If the central vertex c, then for $\forall u \in V_2(G^{+++})$ $d(c,u) = 1$, for $\forall xy \in V_3(G^{+++})$ $d(c,xy) = 1$ and for $\forall xy \in V_4(G^{++})$ $d(c,xy) = 2$. Thus, we get $e(c) = 2$.

If $v \in V_2(G^{+++})$, then $d(c,v) = 1$, for $\forall u \in V_2(G^{+++}) - N_G[v]$ $d(v,u) = 2$, for $\forall xy \in V_3(G^{++})$ $d(v,xy) \leq 2$ and for $\forall xy \in V_4(G^{+++})$ $d(v,xy) \leq 3$. Hence, we have $e(v) = 3$. Similarly, we have $e(xy) = 2$ for $\forall xy \in V_3(G^{+++})$ and $e(xy) = 3$ for $\forall xy \in V_4(G^{+++})$. With the results we found, we get, for $n \geq 5$, $\varepsilon^c(G^{+++}) = \sum \deg(v)e(v) = 4n + 18n + n(n+3)2 + 18n = 2n^2 + 46n$. $\square$

References

[1] A. Aytaç and T. Turacı, Vulnerability Measures of Transformation Graph, *Inter. Journal of Found. of Comp. Sci.*, **26**, 6 (2015), 667–675. ⇒113

[2] A. T. Balaban, Highly discriminating distance-based topological index, *Chem. Phys. Lett.*, **89**, (1982), 399–404. ⇒112

[3] Z.N. Berberler and M.E. Berberler, Edge eccentric connectivity index of nanothorns, *Bulgarian Chemical Communications*, **48**, 1 (2016), 165–170 ⇒112

[4] Z.N. Berberler and M.E. Berberler, The Edge Geometric-Arithmetic Index of an Infinite Class of Dendrimers, *Journal of Computational and Theoretical Nanoscience*, **10**, 9 (2013), 2207–2208. ⇒112

[5] Z.N. Berberler and M.E. Berberler, The Modified Eccentric Connectivity Index of Nanocomposites, *Journal of Computational and Theoretical Nanoscience*, **12**, 9 (2015), 2208–2212. ⇒112

[6] W. Gao and M. R. Farahani, Degree-based indices computation for special chemical molecular structures using edge dividing method, *Appl. Math. Nonlinear Sci.* **1**, (2016), 94–117. ⇒111

[7] W. Gao, W. F. Wang and Farahani M. R., Topological indices study of molecular structure in anticancer drugs, *Journal of Chemistry*, (2016). ⇒111

[8] S. Gupta, M. Singh and A. K. Madan, Application of Graph Theory: Relationship of Eccentric Connectivity Index and Wiener's Index with Anti-inflammatory Activity, *J. Math. Anal. Appl.*, **266**, 2 (2002), 259–268. ⇒111

[9] W. Harry, Structural Determination of Paraffin Boiling Points , *Journal of the American Chemical Society*, **69**, 1 (1947), 17–20. ⇒112

[10] H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, *Bulletin of the Chemical Society of Japan*, **44**, 9 (1971), 2332–2339. ⇒112

[11] O. Ivanciuc, T. Ivanciuc and A.T. Balaban, Vertex-and edge-weighted molecular graphs and derived structural descriptors, *Topological Indices and Related Descriptors in QSAR and QSPR*, CRC Press, 1999. ⇒111

[12] Z.N. Odabaş, The Edge Eccentric Connectivity Index of Dendrimers, *Journal of Computational and Theoretical Nanoscience*, **10**, 4 (2013), 783–784. ⇒112

[13] S. Sardana and A.K. Madan, Predicting anti-HIV activity of TIBO derivatives: a computational approach using a novel topological descriptor, *J. Mol. Model.*, **8**, (2002), 258–265. ⇒111

[14] V. Sharma, R. Goswami, A.K. Madan, Eccentric Connectivity Index: A Novel Highly Discriminating Topological Descriptor for Structure-Property and Structure-Activity Studies, *J. Chem. Inf. Comput. Sci.*, **37**, (1997), 273–282. ⇒112

[15] R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim, 2008. ⇒111

[16] D.B. West, *Introduction to Graph Theory*, Prentice Hall, NJ, 2001. ⇒112

[17] B. Wu and J. Meng, Basic properties of total transformation graphs, *J. Math. Study*, **34**, 2 (2001), 109–116. ⇒113

[18] B. Wu, L. Zhang and Z. Zhang, The transformation graph G^{xyz} when $xyz = -++$, *Discrete Mathematics*, **296**, (2005), 263–270. ⇒113

[19] L. Xu and B. Wu, Transformation graph, *Discrete Mathematics*, **308**, (2008), 5144–5148. ⇒113

[20] X. Xu and Y. Guo, The Edge Version of Eccentric Connectivity Index, *International Mathematical Forum*, **7(6)**, (2012), 273–280 . ⇒112

[21] L. Xueliang and S. Yongtang, A survey on the Randic index, *MATCH Communications in Mathematical and in Computer Chemistry*, **59**, 1 (2008), 127–156. ⇒112

[22] B. Zhou and Z. Du, On eccentric connectivity index, *MATCH Commun. in Math.and in Computer Chem.*, **63**, (2010), 181–198. ⇒113

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