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E-super arithmetic graceful labelling of $H_i(m, m)$, $H_i^{(1)}(m, m)$ and chain of even cycles

S. ANUBALA
Department of Mathematics,
Sri Kaliswari College
(Autonomous),Sivakasi,
Virudhunagar, TamilNadu, India.
email: anubala.ias@gmail.com

V. RAMACHANDRAN
PG and Researh Department of
Mathematics,
Mannar Thirumalai Naicker
College(Autonomous), Pasumalai,
Madurai, TamilNadu,India.
email: me.ram111@gmail.com

Abstract. E-super arithmetic graceful labelling of a graph G is a bijection f from the union of the vertex set and edge set to the set of positive integers $(1, 2, 3, \ldots |V(G) \cup E(G)|)$ such that the edges have the labels from the set $\{1, 2, 3, \ldots, |E(G)|\}$ and the induced mapping f^* given by $f^*(uv) = f(u) + f(v) - f(uv)$ for $uv \in E(G)$ has the range $\{|V(G) \cup E(G)| + 1, |V(G) \cup E(G)| + 2, \ldots, |V(G)| + 2|E(G)|\}$

In this paper we prove that $H_i(m, m)$ and $H_i^{(1)}(m, m)$ and chain of even cycles $C_{4,n}$, $C_{6,n}$ are E-super arithmetic graceful.

1 Introduction

Rosa [9] in 1967, called a function f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [3] subsequently called such labelling graceful.

Key words and phrases: E-super,$H_i(m, m)$, $H_i^{(1)}(m, m)$, $C_{4,n}$, $C_{6,n}$.

In 1970 Kotzig and Rosa [5] defined a *magic valuation of a graph* $G(V, E)$ as a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant (called the magic constant).

Acharya and Hegde [1] have defined (k, d)-arithmetic graphs.
Let G be a graph with q edges and let k and d be positive integers. A labelling f of G is said to be (k, d)-arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge xy are $k, k + d, k + 2d, \ldots, k + (q - 1)d$. The case where $k = 1$ and $d = 1$ was called additively graceful by Hegde [4].

J. A. Gallian [2] surveyed numerous graph labelling methods.

V. Ramachandran and C. Sekar [8] introduced (1,N)-arithmetic labelling.

A labelling of $G(V, E)$ is said to be E-super if $f(E(G)) = \{1, 2, 3, \ldots, |E(G)|\}$.

MacDougall, Slamin, Miller and Wallis [6] introduced the notion of a vertex-magic total labelling in 1999. For a graph $G(V, E)$ an injective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a vertex-magic total labeling if there is a constant k, called the magic constant such that for every vertex v, $f(u) + \Sigma f(vu) = k$ where the sum is taken over all vertices u adjacent to v.

Marimuthu and Balakrishnan [7] defined a graph $G(V, E)$ to be edge magic graceful if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p + q\}$ such that $|f(u) + f(v) - f(uv)|$ is a constant for all edges uv of G.

We define a graph $G(p, q)$ to be ***E-super arithmetic graceful*** if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p + q\}$ such that $f(E(G)) = \{1, 2, \ldots, q\}$, $f(V(G)) = \{q + 1, q + 2, \ldots, q + p\}$ and the induced mapping f^* given by $f^*(uv) = f(u) + f(v) - f(uv)$ for $uv \in E(G)$ has the range $\{p + q + 1, p + q + 2, \ldots, p + 2q\}$.

In this paper we prove that $H_i(m, m)$ and $H_i^{(1)}(m, m)$ and $C_{4,n}$,$C_{6,n}$ are E-super arithmetic graceful.

2 Preliminaries

Definition 1 *A connected graph is* ***highly irregular*** *if each of its vertices is adjacent only to vertices with distinct degrees. Let* H *denote the bipartite graph of order* $n = 2m$, $m \geq 2$ *having partite sets,* $V_1 = \{u_1, u_2, \ldots, u_m\}$ *and* $V_2 = \{v_1, v_2, \ldots, v_m\}$ *and edge set* $E(H) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq m + 1 - i\}$ *with* $deg_H(u_i) = deg_H(v_i) = m + 1 - i$ *for* $i = 1, 2, \ldots, m$.
H *is a irregular graph of order* $n = 2m$, $m \geq 2$. *Let us denote this graph as* $H_i(m, m)$.

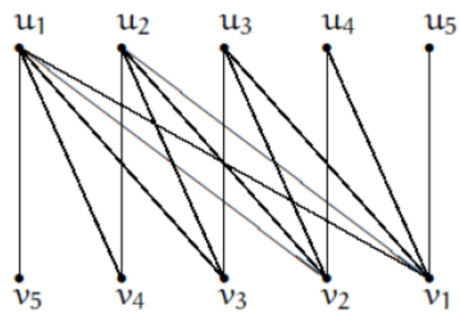

Figure 1: $H_i(5,5)$ – highly irregular graph of order 10

Definition 2 *By subdividing the edge* u_2v_{m-1} *of* $H_i(m,m)$ *for* $m \geq 4$*, we obtain a highly irregular graph of order* $2m+1 \geq 9$*. Denote this graph by* $H_i^{(1)}(m,m)$*.*

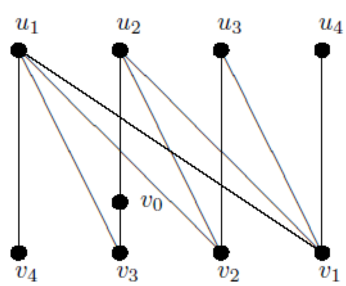

Figure 2: $H_i^{(1)}(4,4)$ – highly irregular graph of order 9

Definition 3 *Let* C_{2k} *be an even cycle. Consider* n *copies of* C_{2k}*. A chain of even cycles* C_{2k} *denoted by* $C_{2k,n}$ *is obtained by identifying the vertex* u_{k+1} *of each copy of* C_{2k} *with the vertex* u_1 *of the successive copy of* C_{2k}*.*
$C_{2k,n}$ *has* $(2k-1)n+1$ *vertices and* $2kn$ *edges.*
$C_{2k,n}$ *has* $(k-1)n$ *upper nodes* $u_1, u_2, \ldots, u_{(k-1)n}$ *;*
$(k-1)n$ *lower nodes* $w_1, w_2, \ldots, w_{(k-1)n}$ *and* $(n+1)$ *middle nodes* $v_1, v_2, \ldots, v_{n+1}$*.*

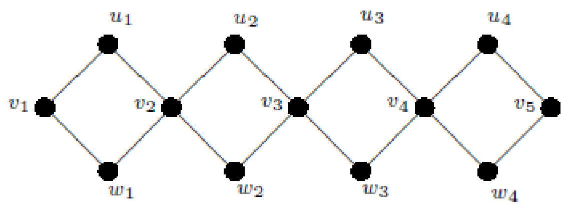

Figure 3: $C_{4,4}$

3 Main results

Theorem 4 $H_i(m,m)$ *is E-super arithmetic graceful for* $m \geq 2$.

Proof. Let $G = H_i(m,m)$. G has $2m$ vertices and $\binom{m+1}{2} = \frac{m(m+1)}{2}$ edges.
Define $f : V(G) \cup E(G) \longrightarrow \left\{1, 2, \ldots, 2m + \frac{m(m+1)}{2}\right\}$ as follows:
$f(u_i) = \binom{m+1}{2} + i, \qquad i = 1, 2, \ldots, m.$
$f(v_i) = \binom{m+1}{2} + 2m + 1 - i, \quad i = 1, 2, \ldots, m$
$f(u_i v_j) = \binom{m+1}{2} + i - \binom{i+j}{2}, \; i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, (m+1) - i$
Clearly f is a bijection from $V \cup E$ to $\left\{1, 2, \ldots, \binom{m+1}{2} + 2m\right\}$ where
$f(E) = \left\{1, 2, \ldots, \binom{m+1}{2}\right\}$.
Also

$$f^*(E(H_i(m,m))) = \left\{\binom{m+1}{2} + 2m + 1, \binom{m+1}{2} + 2m + 3, \ldots, 2\left[\binom{m+1}{2} + m\right]\right\}$$

Therefore, $H_i(m,m)$ for $m \geq 2$ is E-super arithmetic graceful. □

Example 5 *E-super arithmetic graceful labelling of* $H_i(6,6)$.

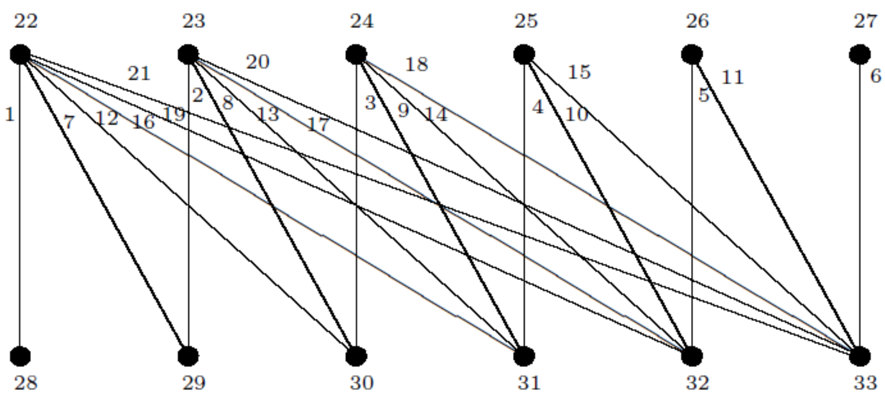

Figure 4: $H_i(6,6)$

Theorem 6 $H_i^{(1)}(m,m)$ *for* $m \geq 4$ *is E-super arithmetic graceful.*

Proof. $H_i^{(1)}(m,m)$ for $m \geq 4$ has $2m+1$ vertices and $\binom{m+1}{2}+1$ edges.
Define

$f(u_i) = \binom{m+1}{2} + 1 + i, \quad i = 1,2,\ldots,m.$
$f(v_0) = \binom{m+1}{2} + m + 2,$
$f(v_i) = \binom{m+1}{2} + 2m + 3 - i, \quad i = 1,2,\ldots,m$
$f(u_1v_1) = 1$
$f(u_1v_m) = 2$
$f(u_2v_0) = 3$
$f(v_0v_{m-1}) = m + 2$
$f(u_iv_{m+1-i}) = i + 1, \quad i = 3,4,\ldots m$
For $i = 1;$ $\quad j = 2,3,\ldots,m-1$ and
for $i = 2,3,\ldots,m,$ $\quad j = 1,2,\ldots,(m+1)-i$
$f(u_iv_j) = \binom{m+1}{2} + 2 + i - \binom{i+j}{2}$

Clearly f is a bijection from $V \cup E$ to $\left\{1,2,\ldots,\binom{m+1}{2}+2m+2\right\}$ where

$f(E) = \left\{1, 2, \ldots, \binom{m+1}{2} + 1\right\}$.

Also

$$f^*(E(H_i^{(1)}(m,m))) = \left\{\binom{m+1}{2} + 2m + 3, \binom{m+1}{2} + 2m + 5, \ldots, 2\left[\binom{m+1}{2} + m\right] + 3\right\}$$

Therefore, $H_i^{(1)}(m,m)$ for $m \geq 4$ is E-super arithmetic graceful. □

Example 7 *E-super arithmetic graceful labelling of* $H_i^{(1)}(5,5)$.

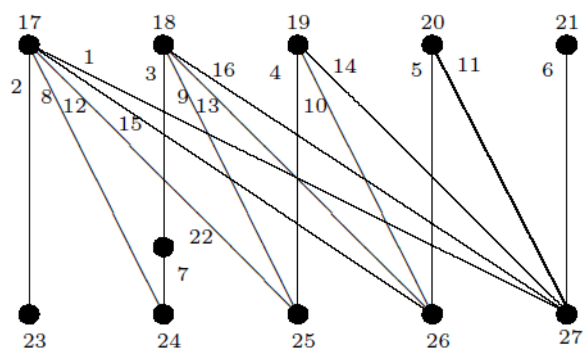

Figure 5: $H_i^{(1)}(5,5)$

Theorem 8 $C_{4,n}$ *are E-super arithmetic graceful.*

Proof. $C_{4,n}$ has $3n+1$ vertices and $4n$ edges.
Let $u_1, u_2, \ldots, u_n$ be the upper nodes, $w_1, w_2, \ldots, w_n$ be the lower nodes and $v_1, v_2, \ldots, v_{n+1}$ be the middle nodes.
Define $f(u_i) = 4n + i, \quad i = 1, 2, \ldots, n.$

$f(v_i) = 5n + i, \quad i = 1, 2, \ldots, n+1.$

$f(w_i) = 6n + 1 + i, \quad i = 1, 2, \ldots, n.$

$f(u_1v_1) = n + 1$

$f(u_iv_i) = 2n + i, \quad i = 2, \ldots, n$

$f(u_iv_{i+1}) = i, \quad i = 1, 2, \ldots, n$

$f(v_iw_i) = 3n + i, \quad i = 1, 2, \ldots, n$

$f(v_{i+1}w_i) = n+1+i, \quad i = 1, 2, \ldots, n$
$f^*(u_1v_1) = 8n+1$

$$\{f^*(u_iv_i) \mid i = 1, 2, 3, \ldots, n\} = \{7n+2, 7n+3, \ldots, 8n\}$$

$$\{f^*(u_iv_{i+1}) \mid i = 1, 2, 3, \ldots, n\} = \{9n+2, 9n+3, \ldots, 10n+1\}$$

$$\{f^*(v_iw_i) \mid i = 1, 2, 3, \ldots, n\} = \{8n+2, 8n+3, \ldots, 9n+1\}$$

$$\{f^*(v_{i+1}w_i) \mid i = 1, 2, 3, \ldots, n\} = \{10n+2, 10n+3, \ldots, 11n+1\}$$

Thus $f^*(E(C_{4,n})) = \{7n+2, 7n+3, \ldots, 11n+1\}$.
Therefore, $C_{4,n}$ is E-super arithmetic graceful. □

Example 9 *E-super arithmetic graceful labelling of* $C_{4,5}$.

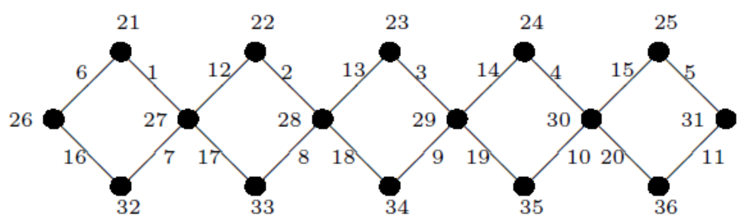

Figure 6: $C_{4,5}$

Theorem 10 $C_{6,n}$ *is E-super arithmetic graceful for all* n.

Proof. Let $G = C_{6,n}$. Let $u_1^{(1)}, u_1^{(2)}, u_2^{(1)}, u_2^{(2)}, \ldots, u_n^{(1)}, u_n^{(2)}$ be the upper level vertices of G.
Let $w_1, w_2, \ldots, w_{n+1}$ be the middle level vertices of G.
Let $v_1^{(1)}, v_1^{(2)}, v_2^{(1)}, v_2^{(2)}, \ldots, v_n^{(1)}, v_n^{(2)}$ be the upper level vertices of G.

Illustration: $C_{6,4}$

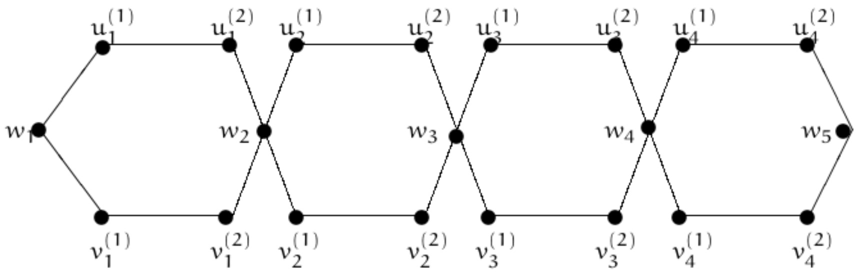

Figure 7: $C_{6,4}$

$C_{6,n}$ has 5n+1 vertices and 6n edges.
Define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, 11n + 1\}$ as follows:
$f(u_i^{(1)}) = 6n + i, \quad i = 1, 2, \ldots, n$
$f(u_i^{(2)}) = 7n + i, \quad i = 1, 2, \ldots, n$
$f(w_i) = 8n + i, \quad i = 1, 2, \ldots, n + 1$
$f(v_i^{(1)}) = 9n + 1 + i, \quad i = 1, 2, \ldots, n$
$f(v_i^{(2)}) = 10n + 1 + i, \quad i = 1, 2, \ldots, n$
$f(u_i^{(1)}u_i^{(2)}) = i, \quad i = 1, 2, \ldots, n$
$f(u_i^{(2)}w_{i+1}) = n + i, \quad i = 1, 2, \ldots, n$
$f(w_iu_i^{(1)}) = 3n - 1i, \quad i = 1, 2, \ldots, n$
$f(v_i^{(2)}w_{i+1}) = 2n + i, \quad i = 1, 2, \ldots, n - 1$
$f(v_i^{(1)}v_i^{(2)}) = 4n - 1 + i, \quad i = 1, 2, \ldots, n - 1$
$f(v_n^{(1)}v_n^{(2)}) = 6n$
$f(w_iv_i^{(1)}) = 5n + i, \quad i = 1, 2, \ldots, n - 1$
$f(w_nv_n^{(1)}) = 5n$
$f(v_n^{(2)}w_{n+1}) = 5n - 1$
Clearly f is a bijection and $f(E(G)) = \{1, 2, \ldots, 6n\}$
$\left\{f^*(u_i^{(1)}u_i^{(2)}) \mid i = 1, 2, \ldots, n\right\} = \{13n + 1, 13n + 2, \ldots, 14n\}$
$\left\{f^*(u_i^{(2)}w_{i+1}) \mid i = 1, 2, \ldots, n\right\} = \{14n + 2, 14n + 3, \ldots, 15n + 1\}$
$\left\{f^*(w_iu_i^{(1)}) \mid i = 1, 2, \ldots, n\right\} = \{11n + 2, 11n + 3, \ldots, 12n + 1\}$
$\left\{f^*(w_iv_i^{(1)}) \mid i = 1, 2, \ldots, n - 1\right\} = \{12n + 2, 12n + 3, \ldots, 13n\}$

$\left\{f^*(v_i^{(2)}w_{i+1}) \mid i = 1, 2, \ldots, n-1\right\} = \{16n+3, 16n+4, \ldots, 17n+1\}$
$\left\{f^*(v_i^{(1)}v_i^{(2)}) \mid i = 1, 2, \ldots, n-1\right\} = \{15n+4, 15n+5, \ldots, 16n+2\}$
$\left\{f^*(w_n v_n^{(1)})\right\} = 14n+1$
$\left\{f^*(v_n^{(1)}v_n^{(2)})\right\} = 15n+2$
$\left\{f^*(v_n^{(2)}w_{n+1})\right\} = 15n+3$
Combining all the above we have $f^*(E(G)) = \{11n+2, 11n+3, \ldots, 17n+1\}$
Therefore, G is E-super arithmetic graceful. □

Example 11 *E-super arithmetic graceful labelling of* $C_{6,6}$.

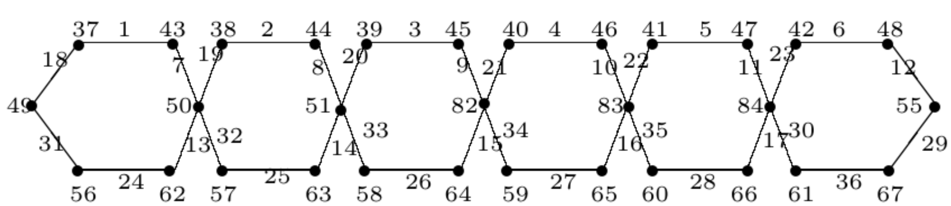

Figure 8: $C_{6,6}$

Conjecture: *Chains of all even cycles* $C_{2m,k}$ *are E-super arithmetic graceful.*

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