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# Some relations between energy and Seidel energy of a graph

Harishchandra S. RAMANE
Department of Mathematics
Karnatak University
Dharwad, India
email: hsramane@yahoo.com

K. ASHOKA
Department of Mathematics
Christ University
Bangalore, India
email: ashokagonal@gmail.com

B. PARVATHALU
Department of Mathematics
Karnatak University's
Karnatak Arts College
Dharwad, India
email: bparvathalu@gmail.com

Daneshwari PATIL
Department of Mathematics
Karnatak University
Dharwad, India
email: daneshwarip@gmail.com

**Abstract.** The energy $E(G)$ of a graph $G$ is the sum of the absolute values of eigenvalues of $G$ and the Seidel energy $E_S(G)$ is the sum of the absolute values of eigenvalues of the Seidel matrix $S$ of $G$. In this paper, some relations between the energy and Seidel energy of a graph in terms of different graph parameters are presented. Also, the inertia relations between the graph eigenvalue and Seidel eigenvalue of a graph are given. The results in this paper generalize some of the existing results.

## 1 Introduction

Let $G$ be a simple, finite and undirected graph of order $n$ with vertex set $V = \{v_1, v_2 \ldots, v_n\}$. The adjacency matrix $A = [a_{ij}]$ of $G$ is a square matrix of

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order $n$ whose $(i, j)$-th entry $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $0$ otherwise. The complement of a graph $G$ is denoted by $\overline{G}$. The degree of a vertex $v_i$, denoted by $d(v_i)$, is the number of edges incident with $v_i$. A graph $G$ is called $r$- regular if $d(v_i) = r$ for all $v_i \in V$. Let $\Delta$ be the maximum degree of $G$. Much like adjacency matrix, in $1966$ J. H. van Lint and J. J. Seidel [24] introduced one more real symmetric $\{0, \pm 1\}$-matrix, called the Seidel matrix $S$ as a tool for studying the systems of equiangular lines in Euclidean space. Later in $1968$ J. J. Seidel studied the Seidel eigenvalues of strongly regular graphs [21]. The eigenvalues $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ of the adjacency matrix $A$ are called the eigenvalues of $G$. A graph is integral if its eigenvalues are integers. Similarly, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the Seidel matrix $S$ of $G$ are called the Seidel eigenvalues of $G$. For a given interval $\mathbf{I}$, $n_\theta(\mathbf{I})$ denotes the number of eigenvalues of $G$ which belongs to the interval $\mathbf{I}$. The number of positive, negative and zero eigenvalues of $G$ are denoted by $n^+$, $n^-$ and $n^0$ respectively, called inertia of $G$.

The graph energy defined by I. Gutman in 1978 [7] and gained its own importance in the spectral graph theory. The energy of a graph $G$ is defined as

$$E(G) = \sum_{j=1}^{n} |\theta_j|.$$

The graph energy has applications in chemistry [6, 12]. An equivalent definition to the energy of a graph $G$ is as follows:

$$E(G) = 2\sum_{j=1}^{n^+} \theta_j = -2\sum_{j=1}^{n^-} \theta_{n-j+1} = 2 \max_{1 \leq i \leq n} \sum_{j=1}^{i} \theta_j = 2 \max_{1 \leq i \leq n} \sum_{j=1}^{i} -\theta_{n-j+1}.$$

Two graphs $G_1$ and $G_2$ of same order are said be equienergetic if $E(G_1) = E(G_2)$. Similarly, the Seidel energy $E_S(G)$ [8] of a graph $G$ is defined as the sum of the absolute values of eigenvalues of Seidel matrix $S$. The Seidel energy is invariant under Seidel switching and complement of a graph. The Seidel energy $E_S(G)$ of a graph $G$ introduced by W. H. Haemers in 2012 and presented a relation between energy of a complete graph and Seidel energy of $G$. However, the exact relation of Seidel energy of graph and bounds for it haven't been much studied in the literature so far. One of the interesting problem on graph energy is to characterize those graphs which are equienergetic with respect to both adjacency and other matrices like distance matrix, Seidel matrix etc. A weaker problem is to construct the families of graphs which are equienergetic with respect to both the adjacency and the other matrices related to graphs. For instance, see [11, 16]. This motivates us to study some relations between the energy and the Seidel energy in terms of different graph parameters. The research related to the Seidel energy and its variants, see [2, 13, 14, 15, 18, 22, 23]. Two graphs $G_1$ and $G_2$ of same order are said be Seidel equienergetic if $E_S(G_1) = E_S(G_2)$. Let $K_n$ and $K_{n_1,n_2}$ $(n = n_1 + n_2)$ denote the complete graph and the complete bipartite graph of order $n$ respectively. This paper is organized as follows. In section 2, basic definitions, known results on eigenvalues, energy, Seidel eigenvalues and Seidel energy of graph are presented. In section 3, the exact relations between the Seidel energy and energy of a regular graph in terms of other graph parameters are given. Also, a large class of Seidel equienergetic graphs are presented. The obtained results in this section generalize some of the existing results. Section 4 provides inertia relations between the graph eigenvalues and Seidel eigenvalues. Also, the relations between the Seidel energy and energy of a graph in terms of other graph parameters are given. As a consequence, some bounds for the Seidel energy of a graph are obtained.

## 2 Preliminaries

**Definition 1** [9] *The line graph* $L(G)$ *of a graph* $G$ *is the graph with vertex set same as the edge set of* $G$. *In the line graph* $L(G)$ *any two vertices are adjacent if the corresponding edges in* $G$ *have a common vertex. The* $k^{th}$ *iterated line graph of* $G$ *for* $k \in \{0, 1, 2, \ldots\}$ *is defined as* $L^k(G) = L(L^{k-1}(G))$, *where* $L^0(G) = G$ *and* $L^1(G) = L(G)$.

**Theorem 2** [3] *If* $G$ *is an* $r$*-regular graph of order* $n$ *with the eigenvalues* $r, \theta_2, \ldots, \theta_n$, *then the eigenvalues of* $S$ *are* $n-2r-1, -1-2\theta_2, \ldots, -1-2\theta_n$.

**Lemma 3** [10] *Let* $P$ *and* $Q$ *be two Hermitian matrices of same order* $n$ *and* $R = P + Q$. *Then*

$$\mu_{n-i-k}(R) \geq \mu_{n-i}(P) + \mu_{n-k}(Q)$$
$$\mu_{s+t+1}(R) \leq \mu_{s+1}(P) + \mu_{t+1}(Q)$$

*where* $0 \leq i, k, s, i+k+1, s+t+1 \leq n$ *and* $\mu_j(B)$ *is the* $j^{th}$ *largest eigenvalue of a Hermitian matrix* $B$.

**Theorem 4** [1] *Let* $G$ *be a graph of order* $n$. *Then*

$$E(G) + E(\overline{G}) - 2(n-1) < E_S(G) \leq E(G) + E(\overline{G}).$$

*The equality in the right side holds if* $G \cong K_n$ *or* $G \cong \overline{K_n}$.

**Proposition 5** [25] *Let* $G$ *be a graph of order* $n$*. Then*

1. $E(G) + E(\overline{G}) \leq n(\sqrt{n} + 1)$
2. $E(G) + E(\overline{G}) \leq (n-1)(\sqrt{n+1} + 1)$ *if* $G$ *is a regular graph.*

**Lemma 6** [4] *Let* $P$, $Q$ *be two real symmetric matrices of same order* $n$ *such that* $R = P + Q$*. Then*

$$E(R) \leq E(P) + E(Q),$$

*where* $E(P) = \sum_{j=1}^{n} |\mu_j|$ *is the energy of* $P$*, and* $\mu_j$ $(j = 1, 2, \dots, n)$ *are the eigenvalues of* $P$*.*

## 3 The exact relation between Seidel energy and energy of a regular graph

In this section, we study the relations between the Seidel energy and the energy of a regular graph. As a consequence, the Seidel equienergetic graphs are constructed by taking regular equienergetic graphs.

**Theorem 7** *Let* $G$ *be an* $r$*-regular graph of order* $n$*. Then*

$$E_S(G) = |n - 2r - 1| + n - 2r - 1 - 2n^- + 2E(G) + 2 \sum_{\substack{2 \leq j \leq n \\ \theta_j \in (-1/2, 0)}} (2\theta_j + 1).$$

**Proof.** Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues and the Seidel eigenvalues of a graph $G$ respectively. By definition of the Seidel energy and Theorem 2, we have

$$\begin{aligned}
E_S(G) &= |n - 2r - 1| + \sum_{j=2}^{n} |\lambda_j| = |n - 2r - 1| + \sum_{j=2}^{n} |-1 - 2\theta_j| \\
&= |n - 2r - 1| + \sum_{\substack{2 \leq j \leq n \\ \theta_j \leq -1/2}} (-1 - 2\theta_j) + \sum_{\substack{2 \leq j \leq n \\ \theta_j > -1/2}} (1 + 2\theta_j) \\
&= |n - 2r - 1| - n_\theta[-r, -1/2] + 2 \sum_{\substack{2 \leq j \leq n \\ \theta_j \leq -1/2}} |\theta_j| + n_\theta(-1/2, r) \\
&\quad + 2 \sum_{\substack{2 \leq j \leq n \\ \theta_j \in (-1/2, 0)}} \theta_j + 2 \sum_{\substack{2 \leq j \leq n \\ \theta_j \geq 0}} |\theta_j|. \qquad (1)
\end{aligned}$$

For convenience, the energy of G can be written as

$$E(G) = r + \sum_{\substack{2\le j\le n\\ \theta_j\le -1/2}} |\theta_j| + \sum_{\substack{2\le j\le n\\ \theta_j\ge 0}} |\theta_j| + \sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} |\theta_j|.$$

With this, (1) becomes

$$\begin{aligned}
E_S(G) &= |n-2r-1| - n_\theta[-r,-1/2] + n_\theta(-1/2,r) + 2\sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} \theta_j \\
&\quad + 2\Big(E(G) - r - \sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} |\theta_j|\Big) \\
&= |n-2r-1| - n_\theta[-r,-1/2] + n_\theta(-1/2,r) + 2\sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} \theta_j \\
&\quad + 2E(G) - 2r - 2\sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} |\theta_j| \\
&= |n-2r-1| + 2E(G) - 2r + 4\sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} \theta_j - n_\theta[-r,-1/2] \\
&\quad + n - 1 - n_\theta[-r,-1/2] \\
&= |n-2r-1| + 2E(G) - 2r + 4\sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} \theta_j + n - 1 - 2n_\theta[-r,-1/2].
\end{aligned} \tag{2}$$

We have the following

$$n_\theta[-r,-1/2] = n^- - n_\theta(-1/2,0).$$

and

$$\sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} (2\theta_j + 1) = \sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2,0)}} 2\theta_j + n_\theta(-1/2,0). \tag{3}$$

Now, using (3) in (2)

$$E_S(G) = |n-2r-1| + n - 2r - 1 - 2n^- + 2E(G) + 2\sum_{\substack{2\le j\le n\\ \theta_j\in(-1/2),0)}} (2\theta_j + 1)$$

which completes the proof. □

It is easy to see that $\sum_{\substack{2\leq j\leq n \\ \theta_j\in(-1/2,0)}} (2\theta_j+1) > 0$. With this we have the following:

**Corollary 8** *Let* $G$ *be an* $r$*-regular graph of order* $n$*. Then*

1. $E_S(G) > 2(n-1-2r-n^- + E(G))$ *if* $r \leq \frac{(n-1)}{2}$

2. $E_S(G) > 2(E(G)-n^-)$ *if* $r \geq \frac{(n-1)}{2}$.

**Corollary 9** *Let* $G$ *be an* $r$*-regular graph of order* $n$ *and* $\theta_j \notin (-1/2, 0)$*. Then*

$$E_S(G) = \begin{cases} 2(n-1-2r-n^- + E(G)) & \text{if } r \leq \frac{(n-1)}{2} \\ 2(E(G)-n^-) & \text{if } r \geq \frac{(n-1)}{2}. \end{cases}$$

**Proof.** If $\theta_j \notin (-1/2, 0)$ then $\sum_{\substack{2\leq j\leq n \\ \theta_j\in(-1/2,0)}} (2\theta_j+1) = 0$. Now $r \leq \frac{(n-1)}{2}$ implies $n-2r-1 \geq 0$ and $r \geq \frac{(n-1)}{2}$ implies $n-2r-1 \leq 0$. With these, Theorem 7 gives the desired results. □

**Remark 10** *In Theorem 3.11 of* [1] *it is proved that* $E_S(G) = 2(n-1-2r-n^- + E(G))$ *if* $\theta_j \notin (-1, 0)$ *and* $r \leq \frac{n-1}{2}$*. The Corollary 9 gives the same even if* $\theta_j \notin (-1/2, 0)$*, which shows that Theorem 3.11 of* [1] *is enriched.*

**Corollary 11** *Let* $G$ *be an* $r$*-regular integral graph of order* $n$*. Then*

$$E_S(G) = \begin{cases} 2(n-1-2r-n^- + E(G)) & \text{if } r \leq \frac{(n-1)}{2} \\ 2(E(G)-n^-) & \text{if } r \geq \frac{(n-1)}{2}. \end{cases}$$

**Corollary 12** *Let* $G$ *be an* $r$*-regular graph of order* $n$*. Then* $E_S(G) = E(G)$ *if and only if*

$$E(G) = \begin{cases} -2(n-2r-1-n^- + \sum_{\substack{2\leq j\leq n \\ \theta_j\in(-1/2,0)}} (2\theta_j+1)) & \text{if } r \leq \frac{(n-1)}{2} \\ -2(-n^- + \sum_{\substack{2\leq j\leq n \\ \theta_j\in(-1/2,0)}} (2\theta_j+1)) & \text{if } r \geq \frac{(n-1)}{2}. \end{cases}$$

In a particular case if $\theta_j \notin (-1/2, 0)$, then $E_S(G) = E(G)$ if and only if

$$E(G) = \begin{cases} -2(n - 2r - 1 - n^-) & \text{if } r \leq \frac{(n-1)}{2} \\ 2n^- & \text{if } r \geq \frac{(n-1)}{2}. \end{cases}$$

From the first case of above results and the fact that $E(G) \geq 0$, one can easily observe that if $n - n^- > 2r + 1$ then there is no graph with $E_S(G) = E(G)$. The one of interesting problem in studying the Seidel energy of graphs is to find the Seidel equienergetic graphs. In this direction, we have the following:

**Theorem 13** *Let $G_1$ and $G_2$ be two equienergetic, $r$- regular graphs of same order $n$ with no eigenvalue in the interval $(-1/2, 0)$. If $G_1$ and $G_2$ both have same number of negative eigenvalues, then $G_1$ and $G_2$ are Seidel equienergetic.*

**Proof.** Proof follows directly from Corollary 9. □

If $G$ is an $r$-regular graph of order $n$ with $r \geq 3$, then the iterated line graphs $L^k(G)$, $k \geq 2$ have all negative eigenvalues equal to $-2$ [17]. If $G_1$ and $G_2$ are two $r$-regular graph of order $n$, where $r \geq 3$ then $L^k(G_1)$, $L^k(G_2)$ have same number of negative eigenvalues and $E(L^k(G_1)) = E(L^k(G_2))$, $k \geq 2$ [20].

**Remark 14** *In [19] Ramane et al. studied the Seidel energy of iterated line graphs $L^k(G)$, $k \geq 2$ of a $r$-regular graph, $r \geq 3$ and constructed a large class of Seidel equienergetic graphs by taking two $r$-regular graphs of same order. It is noted that the results in [19] become the particular case of Theorem 13.*

## 4 Seidel energy and energy of a graph

In this section, we study the connection between the Seidel energy and the energy of a graph $G$ in terms of different graph parameters. Also, obtained the inertia relations between the graph eigenvalues and the Seidel eigenvalues of a graph $G$.

**Lemma 15** *Let $G$ be a graph of order $n$, $n \geq 2$. Then for $j \geq 2$,*

$$2\theta_j + \lambda_{n-j+2} \leq -1 \leq 2\theta_j + \lambda_{n-j+1}. \tag{4}$$

**Proof.** By definition of Seidel matrix $S = J - I - 2A$ or $2A + S = J - I$. In Lemma 3, by taking $P = 2A$, $Q = S$ and $R = J - I$ we have $\mu_{n-i-k}(J - I) \geq 2\mu_{n-i}(A) + \mu_{n-k}(S)$, now letting $i = n - j, k = j - 2,$ we get $\mu_2(J - I) \geq 2\theta_j(A) + \lambda_{n-j+2}(S)$. But $J - I$ has only two different eigenvalues $n - 1$ and $-$

$1$ which implies $2\theta_j + \lambda_{n-j+2} \leq -1$. Similarly, by letting $s = j-1$ and $t = n-j$ in Lemma 3, we get the right side inequality of (4). □

The inertia of $S$ is the number of positive, negative and $0$ eigenvalues of $S$ and denoted by $n_S^+$, $n_S^-$ and $n_S^0$ respectively.

**Theorem 16** *Let* $G$ *be a graph of order* $n \geq 2$. *Then*

*(a)* $1 \leq n_S^+ + n^+ \leq n+1$

*(b)* $0 \leq n_S^0 + n^0 \leq n$

*(c)* $n-1 \leq n_S^- + n^-$.

**Proof.** The $n_S^+ = 1$ if and only if $G \cong K_{n_1,n_2}(n_1+n_2 = n)$ or $\overline{K_n}$ [5] which implies $\lambda_2 \geq 0$ for remaining graphs. This shows that lower bound in $(a)$ is clear. The lower bound in $(b)$ follows from the fact that many graphs are not Seidel integral. Let us rewrite the eigenvalues of $G$ as

$$\theta_1 \geq \cdots \geq \theta_{n^+} > 0 = \theta_{n^++1} = \cdots = \theta_{n^++n^0} = 0 > \theta_{n^++n^0+1} \geq \cdots$$
$$\cdots \geq \theta_{n^++n^0+j} > -\frac{1}{2} \geq \theta_{n^++n^0+j+1} \geq \cdots \geq \theta_n.$$

From the left side inequality of (4), we have $2\theta_j + \lambda_{n-j+2} \leq -1$ for $j \geq 2$. And if $j \geq 2$ and $\theta_j > -\frac{1}{2}$ then $\lambda_{n-j+2} \leq -1 - 2\theta_j < 0$.
Therefore,

$$n_S^- \geq n^+ + n^0 + j - 1 \tag{5}$$

which implies

$$n_S^- + n^- \geq n^+ + n^0 + j - 1 + n^- = n + j - 1 \geq n - 1.$$

Now by using (5),

$$n_S^+ + n^+ = n - n_S^- - n_S^0 + n^+ \leq n - (n^+ + n^0 + j - 1) - n_S^0 + n^+ \leq n + 1.$$

Next, by using above bounds

$$n_S^0 + n^0 = 2n - (n_S^+ + n^+) - (n_S^- + n^-) \leq 2n - 1 - (n-1) = n.$$

□

In the following result we give necessary and sufficient condition to hold the equality in left side of (4).

**Lemma 17** *Let* $G$ *be a graph of order* $n$*. Then* $2\theta_j + \lambda_{n-j+2} = -1$ *for all* $j \in \{2, 3, \dots, n\}$ *if and only if* $G$ *is an* $r$*-regular graph with largest Seidel eigenvalue* $n - 2r - 1$*.*

**Proof.** If $G$ is an $r$-regular graph. Then by Theorem 2 it is clear that $2\theta_j + \lambda_{n-j+2} = -1$ for all $j \in \{2, 3, \dots, n\}$. Conversely, assume that $2\theta_j + \lambda_{n-j+2} = -1$ for all $j \in \{2, 3, \dots, n\}$. Let the number of edges in $G$ and its complement $\overline{G}$ be $m$ and $\overline{m}$ respectively. By Rayleigh quotient with all one's vector $\mathbf{1}$, we have

$$\lambda_1 = \max_{x \neq 0}\{\frac{x^T S x}{x^T x}\} \geq \frac{\mathbf{1^T}(\overline{A} - A)\mathbf{1}}{\mathbf{1^T 1}} \geq \frac{2\overline{m} - 2m}{n}.$$

From the fact that $\sum_{j=1}^{n} \theta_j = 0$ and $\sum_{j=1}^{n} \lambda_j = 0$, we have

$$\sum_{j=1}^{n}(2\theta_j + \lambda_j) = 0 \text{ which implies } 2\theta_1 + \lambda_1 = n - 1.$$

On the other hand, from the fact $\theta_1 \geq \dfrac{2m}{n}$, we have

$$2\theta_1 + \lambda_1 \geq \frac{4m}{n} + \frac{2\overline{m} - 2m}{n} = n - 1.$$

Now we arrive at $\theta_1 = \dfrac{2m}{n}$ and $\lambda_1 = \dfrac{2\overline{m} - 2m}{n}$, which shows that $G$ is an $r$-regular graph with $r = \dfrac{2m}{n}$ and $\lambda_1 = n - 2r - 1$.
This completes the proof. □

**Proposition 18** *Let* $G$ *be a graph of order* $n$*. Then*

$$|E_S(G) - 2E(G)| \leq 2n - 2.$$

*Equality holds if and only if* $G$ *is a complete graph.*

**Proof.** By definition of Seidel matrix, $S = J - I - 2A$. Now using Lemma 6, we have, $E(S) \leq E(J - I) + 2E(A)$ and $2E(A) \leq E(J - I) + E(S)$ which implies $|E_S(G) - 2E(G)| \leq 2n - 2$. □

**Theorem 19** *Let* $G$ *be a graph of order* $n$*. Then*

*(a)* $2E(G) - E_S(G) \leq 4\theta_1$

*(b)* $E_S(G) - 2E(G) \leq 2\lambda_1$.

**Proof.** Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ are the Seidel eigenvalues and the eigenvalues of a graph $G$ respectively. From the definition of the Seidel energy and the energy of a graph $G$, we have

$$E_S(G) = \lambda_1 + \sum_{j=2}^{n} |\lambda_j|$$

$$2E(G) = 2\theta_1 + \sum_{j=2}^{n} |2\theta_{n-j+2}|.$$

By combining the above two equalities we get

$$\begin{aligned} 2E(G) - 2\theta_1 - E_S(G) + \lambda_1 &= \sum_{j=2}^{n} (|2\theta_{n-j+2}| - |\lambda_j|) \\ &\leq \sum_{j=2}^{n} |2\theta_{n-j+2} + \lambda_j|. \end{aligned}$$

Now from the left side inequality of (4), we have $2\theta_j + \lambda_{n-j+2} \leq -1$ for $j \geq 2$. Therefore

$$\sum_{j=2}^{n} |2\theta_{n-j+2} + \lambda_j| = -\sum_{j=2}^{n} 2\theta_{n-j+2} + \lambda_j = 2\theta_1 + \lambda_1.$$

With this fact we arrive at $2E(G) - E_S(G) \leq 4\theta_1$.
Similarly one can prove $(b)$ easily. □

**Remark 20** *The case* $(b)$ *of Theorem 19 with the fact* $\lambda_1 \leq n-1$ *gives*

$$E_S(G) - 2E(G) \leq 2\lambda_1 \leq 2n - 2.$$

*And Proposition 18 gives that* $E_S(G) - 2E(G) \leq 2n - 2$ *which implies the result* $(b)$ *of Theorem 19 is better than this inequality.*

**Theorem 21** *Let* $G$ *be a graph of order* $n$ *with Seidel eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. *Then*

$$2E(G) - E_S(G) \leq 4\theta_1 - 2n + 2n^- + 2. \tag{6}$$

*Equality holds if and only if* $G$ *is a graph with* $2\theta_j + \lambda_{n-j+2} = -1$ *for all* $j \in \{2, 3, \ldots, z\}$ *and* $n^- + n_S^- = n - 1$, *where* $z = n^+ + n^0$.

**Proof.** Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ be the eigenvalues of $G$. From the definition of energy of a graph $G$, we have

$$\begin{aligned} E(G) &= \sum_{j=1}^{n} |\theta_j| = 2\sum_{j=1}^{z} \theta_j = 2\theta_1 + 2\sum_{j=2}^{z} \theta_j \\ &\leq 2\theta_1 + 2\sum_{j=2}^{z} \frac{1}{2}(-1 - \lambda_{n-j+2}) \quad \text{by left side of (4)} \qquad (7) \\ &= 2\theta_1 - z + 1 - \sum_{j=1}^{z-1} \lambda_{n-j+1} \\ &\leq 2\theta_1 - z + 1 + \max_{1\leq i\leq n} \sum_{j=1}^{i} -\lambda_{n-i+1} \qquad (8) \\ &= 2\theta_1 - n + n^- + 1 + \frac{1}{2}E_S(G) \\ 2E(G) - E_S(G) &\leq 4\theta_1 - 2n + 2n^- + 2. \end{aligned}$$

For equality we have the following. Let $G$ be a complete graph of order $n$, then $\theta_1 = n - 1$ and $n^- = n - 1$. Hence equality holds in (6). Suppose $G$ is not a complete graph then $\theta_2 \geq 0$, which gives $z = n^+ + n^0 = n - n^- \geq 2$.
Now, to have equality in (6) the inequalities (7) and (8) must be equalities. Equality in (7) holds if and only if $2\theta_j + \lambda_{n-j+2} = -1$ for all $j \in \{2, 3, \ldots, z\}$. From the energy of a graph and equality in (8) holds if and only if

$$n_S^- \leq z - 1 \leq n_S^- + n_S^0.$$

Since $z = n - n^-$ we get $n_S^- + n^- \leq n - 1$, and $n_S^- + n^- \geq n - 1 - n_S^0$. Now, from $(c)$ of the Theorem 16, the above right side inequality is obvious. Again using $(c)$ of the Theorem 16 with left side inequality of above, we have $n^- + n_S^- = n - 1$. This completes the proof. □

**Corollary 22** *Let $G$ be a graph of order $n$. Then*

$$2E(G) - E_S(G) \leq 4\Delta - 2n + 2n^- + 2.$$

*Equality holds if and only if $G$ is regular graph with $n^- + n_S^- = n - 1$.*

**Proof.** Let $\Delta$ be the maximum degree of $G$. It is well-known fact that $\theta_1 \leq \Delta$. And $\theta_1 = \Delta$ holds if and only if $G$ is a regular. Using these in Theorem 21, we arrive at required results. □

**Remark 23** *From the fact that* $n^- \leq n-1$*, Theorem 21 gives*

$$2E(G) - E_S(G) \leq 4\theta_1 - 2n + 2n^- + 2 \leq 4\theta_1.$$

*And the case* $(a)$ *of Theorem 19 shows that* $2E(G) - E_S(G) \leq 4\theta_1$ *which implies the result in Theorem 21 is better than this inequality.*

**Corollary 24** *Let* $G$ *be a graph of order* $n$*. Then*

$$2E(G) - E_S(G) \leq 2(n-1+n^-) \leq 2(2n-1-\alpha). \tag{9}$$

**Proof.** Let $\alpha$ be the independence number of $G$. Using the fact that $\theta_1 \leq n-1$ in Theorem 21 the left side inequality in (9) is clear. Now using the well known inequality $n^- \leq n - \alpha$, we get the right side inequality in (9). □

**Theorem 25** *Let* $G$ *be a graph of order* $n$*. Then*

*(a)* $2(n^+ + n^0 - 1) \leq E_S(G) \leq n(\sqrt{n}+1)$

*(b)* $2(n^+ + n^0 - 1) \leq E_S(G) \leq (n-1)(\sqrt{n+1}+1)$ *if* $G$ *is a regular graph.*

**Proof.** Left side inequality in $(a)$ and $(b)$ follows by using the fact $E(G) \geq 2\theta_1$ in (6). Now by using the right side inequality of Theorem 4 and (1) of Proposition 5, we have the right side inequality of $(a)$. Now by using the right side inequality of Theorem 4 and (2) of Proposition 5, we have the right side inequality of $(b)$. □

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