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Connected certified domination edge critical and stable graphs

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Abstract. In an isolate-free graph $\mathcal{Z} = (V_\mathcal{Z}, E_\mathcal{Z})$, a set $\mathcal{C}$ of vertices is termed as a connected certified dominating set of $\mathcal{Z}$ if, $|N_\mathcal{Z}(u) \cap (V_\mathcal{Z} \backslash \mathcal{C})| = 0$ or $|N_\mathcal{Z}(u) \cap (V_\mathcal{Z} \backslash \mathcal{C})| \geq 2$ $\forall u \in \mathcal{C}$, and the subgraph $\mathcal{Z}[\mathcal{C}]$ induced by $\mathcal{C}$ is connected. The cardinality of the minimal connected certified dominating set of graph $\mathcal{Z}$ is called the connected certified domination number of $\mathcal{Z}$ denoted by $\gamma^c_{cer}(\mathcal{Z})$. In graph $\mathcal{Z}$, if the deletion of any arbitrary edge changes the connected certified domination number, then we call it a connected certified domination edge critical. If the deletion of any random edge does not affect the connected certified domination number, then we refer to it as a connected certified domination edge stable graph. In this paper, we investigate those graphs which are connected certified domination edge critical and stable upon edge removal. We then study some properties of connected certified domination edge critical and stable graphs.

1 Introduction

Detlaff et al. [8] introduced certified domination in 2020, and it is now a well-studied domination-related parameter in the domination theory of graphs

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(see, for example, [9, 15, 16, 14] for recent literature on this topic). A set $\mathcal{C} \subseteq V_{\mathcal{Z}}$ of a graph $\mathcal{Z} = (V_{\mathcal{Z}}, E_{\mathcal{Z}})$ is called certified dominating set (CFDS) if $|N_{\mathcal{Z}}(u) \cap (V_{\mathcal{Z}} \backslash \mathcal{C})| = 0$ or $|N_{\mathcal{Z}}(u) \cap (V_{\mathcal{Z}} \backslash \mathcal{C})| \geq 2$ $\forall u \in \mathcal{C}$. The cardinality of the minimal CFDS is the certified domination number (CFDN) of the graph $\mathcal{Z}$, represented by $\gamma_{cer}(\mathcal{Z})$ [8]. A $\gamma_{cer}-$ set $\mathcal{C}$ is said to be a connected certified dominating set (CCDS), if the induced subgraph $\mathcal{Z}[\mathcal{C}]$ is connected and $|N_{\mathcal{Z}}(u) \cap (V_{\mathcal{Z}} \backslash \mathcal{C})| = 0$ or $|N_{\mathcal{Z}}(u) \cap (V_{\mathcal{Z}} \backslash \mathcal{C})| \geq 2$ $\forall u \in \mathcal{C}$. The connected certified domination number (CCDN) of the graph $\mathcal{Z}$ is the cardinality of the smallest CCDS and is represented by $\gamma^{c}_{cer}(\mathcal{Z})$. An element $u \in V_{\mathcal{Z}}$ is a $\gamma^{c}_{cer}-$ good vertex if u is in some $\gamma^{c}_{cer}-$ set of the graph $\mathcal{Z}$, and set of all $\gamma^{c}_{cer}-$ good vertices of the graph $\mathcal{Z}$ will be represented by $T^{c}_{cer}(\mathcal{Z})$.
Criticality and stability are important considerations for a lot of graph parameters. It is generally essential to understand how a graphical property behaves when the graph is altered when it is relevant in an application. Much has been written on graphs where the deletion (addition) of an edge (vertex) affects a parameter (such as domination number or chromatic number). The $\gamma-$critical graphs when one edge is eliminated were examined by Walikar and Acharya [17] and in contrast, Dutton and Brigham first studied $\gamma-$ stable graphs [10]. These problems were then used to investigate critical and stable graphs with respect to different domination variations such as, “Roman Domination”, “Total Domination”, “Connected Domination”, etc. γ_c-critical graphs were first studied by [5] in 2004, while γ_c-stable graphs were first studied by [7] in 2015. In 2020, Detlaff et al. [8] studied the influence of edge addition and deletion on the CFDN of graphs.
The criticality and stability of graph upon edge addition and deletion have been studied for various domination-related parameters, for example, [6, 2, 4, 12]. In this research, we investigate those graphs where the CCDN increases when an edge is deleted. We also study those graphs where CCDN remains unchanged on the deletion of an edge. To analyse stable or critical graphs when one edge is eliminated, we state that $\gamma^{c}_{cer}(\mathcal{Z}) = \infty$ if a graph $\mathcal{Z}$ contains an isolated vertex. Consequently, $\gamma^{c}_{cer}(\mathcal{Z} - e) = \infty$ if we delete an edge $e \in E_{\mathcal{Z}}$ that is incident with a leaf vertex in $\mathcal{Z}$. In addition, $\gamma^{c}_{cer}(\mathcal{Z} - e) = \infty$ if edge $e \in E_{\mathcal{Z}}$ divides the graph $\mathcal{Z} - e$ into two components.
We state that a graph $\mathcal{Z}$ is connected certified domination edge ($ccde$) stable or $[\gamma^{c}_{cer}]^{e^-}$-stable, if $\gamma^{c}_{cer}(\mathcal{Z} - e) = \gamma^{c}_{cer}(\mathcal{Z})$ $\forall e \in E_{\mathcal{Z}}$. If $\gamma^{c}_{cer}(\mathcal{Z}) = k$, and $\mathcal{Z}$ is $[\gamma^{c}_{cer}]^{e^-}$-stable, then $\mathcal{Z}$ is $[k^{c}_{cer}]^{e^-}$-stable. A graph $\mathcal{Z}$ is $ccde$ critical or $[\gamma^{c}_{cer}]^{e^-}$-critical , if $\gamma^{c}_{cer}(\mathcal{Z} - e) \neq \gamma^{c}_{cer}(\mathcal{Z})$ $\forall e \in E_{\mathcal{Z}}$. We note that eliminating an edge of a graph $\mathcal{Z}$ cannot decrease the CCDN of the graph $\mathcal{Z}$. Hence if $\mathcal{Z}$ is $[\gamma^{c}_{cer}]^{e^-}$-

critical, then $\gamma^c_{cer}(\mathcal{Z}-e) > \gamma^c_{cer}(\mathcal{Z})$ for every edge $e \in E_{\mathcal{Z}}$. If $\gamma^c_{cer}(\mathcal{Z}) = k$, and $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical, we say that $\mathcal{Z}$ is $[k^c_{cer}]^{e^-}$-critical.
An edge $e \in E_{\mathcal{Z}}$ is a critical edge of $\mathcal{Z}$ if $\gamma^c_{cer}(\mathcal{Z}-e) > \gamma^c_{cer}(\mathcal{Z})$, whereas an edge $e \in E_{\mathcal{Z}}$ is a stable edge of $\mathcal{Z}$ if $\gamma^c_{cer}(\mathcal{Z}-e) = \gamma^c_{cer}(\mathcal{Z})$. Thus, if $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical graph, then every edge of the graph $\mathcal{Z}$ is a critical edge, while every edge in a $[\gamma^c_{cer}]^{e^-}$-stable graph is a stable edge.

1.1 Definitions and notations

We refer to [13] and [18] for general graph-theoretic definitions and notations. Throughout this paper, by a graph $\mathcal{Z}$ we mean a connected, undirected, and unweighted simple graph (i.e., graph without loops or multiple edges). A graph $\mathcal{Z} = (V_{\mathcal{Z}}, E_{\mathcal{Z}})$ with no isolated vertex is an isolate-free graph. The order of $\mathcal{Z}$ is denoted by $n(\mathcal{Z}) = |V_{\mathcal{Z}}|$ and size of $\mathcal{Z}$ by $m(\mathcal{Z}) = |E_{\mathcal{Z}}|$. For any vertex $u \in \mathcal{Z}$, $d_{\mathcal{Z}}(u)$ will denote the degree of u in $\mathcal{Z}$. The neighborhood of u, represented by $N_{\mathcal{Z}}(u)$, is the set of all nodes adjacent to u, and the degree of u in $\mathcal{Z}$ is $|N_{\mathcal{Z}}(u)|$.

Vertex $u \in \mathcal{Z}$ is called an isolated vertex if $d_{\mathcal{Z}}(u) = 0$ and is called a pendant or leaf if $d_{\mathcal{Z}}(u) = 1$. $\delta(\mathcal{Z}), (\Delta(\mathcal{Z}))$ denotes the minimal (maximal) degree among the vertices of $\mathcal{Z}$. The diameter of a graph is the largest distance between two vertices and the maximum distance between $x \in V_{\mathcal{Z}}$, and all other vertices is the eccentricity of the vertex.A universal vertex of a graph $\mathcal{Z}$ is a vertex of degree $|V_{\mathcal{Z}}| - 1$. A leaf is a degree one vertex whose only neighbor is referred to as a support vertex. A support vertex is strong if it has at least two leaves as neighbors; otherwise, it is considered weak. We will use $L_{\mathcal{Z}}$ and $S_1(\mathcal{Z})(S_2(\mathcal{Z})$, respectively) to represent the set of leaves and weak supports (strong supports, respectively) of graph $\mathcal{Z}$. For a connected graph $\mathcal{Z}$, a vertex $u \in V_{\mathcal{Z}}$ is called a cut vertex if $\mathcal{Z} - u$ is not connected. The number of cut vertices of $\mathcal{Z}$ is denoted by $\zeta(\mathcal{Z})$.
The set $N_{\mathcal{Z}}(u) \cup \{u\} = N_{\mathcal{Z}}[u]$ is a closed neighborhood of u. More specifically, the neighborhood (closed, respectively) of a subset $A \subseteq V_{\mathcal{Z}}$ of vertices, represented by $N_{\mathcal{Z}}(A)$ (resp. $N_{\mathcal{Z}}[A]$), is defined to be the set $\bigcup_{u \in A} N_{\mathcal{Z}}(u)$ (resp. $N_{\mathcal{Z}}(A) \cup A$). Let $\mathcal{C} \subseteq V_{\mathcal{Z}}$ and $u \in \mathcal{C}$. The $\mathcal{C}-$private neighborhood of u denoted by $pn(u, \mathcal{C})$, and is defined by $pn(u, \mathcal{C}) = N_{\mathcal{Z}}[u] - N_{\mathcal{Z}}[\mathcal{C} - u]$. Thus if $w \in pn(u, \mathcal{C})$, then $N_{\mathcal{Z}}(w) \cap \mathcal{C} = \{u\}$. We refer to a vertex $w \in pn(u, \mathcal{C})$ as a $\mathcal{C}-$private neighborhood of u. We construct the set $epn(u, \mathcal{C}) = pn(u, \mathcal{C}) \cap (V - \mathcal{C})$ and designate a vertex $y \in epn(u, \mathcal{C})$ an external $\mathcal{C}-$private neighbor of u. If the context makes the graph $\mathcal{Z}$ clear, we simply write $N(u)$ and $N[u]$ instead

of $N_{\mathcal{Z}}(u)$ and $N_{\mathcal{Z}}[u]$, respectively.
The star graph $S_{(1,r)}$ of order $n = r+1$, "is a tree on n vertices with one vertex having degree $|V_{(S_{(1,r)})}| - 1$ and the other $n-1$ vertices having vertex degree 1". Double star graph $S_{(q,r)}$ is a graph obtained by joining center vertex of two star graphs $S_{(1,q)}$ and $S_{(1,r)}$ with an edge. A tree in which every vertex is on a central spine or is just one edge away from the spine is known as a caterpillar graph, caterpillar tree, or simply a caterpillar (in other words, deleting its endpoints results in a path graph). "The corona product of two graphs, $\mathcal{H}_1$ and $\mathcal{H}_2$, is defined as the graph attained by taking one replica of $\mathcal{H}_1$ and $|V_{\mathcal{H}_1}|$ replicas of $\mathcal{H}_2$ and linking the j^{th} vertex of $\mathcal{H}_1$ to every vertex in the j^{th} replica of $\mathcal{H}_2$"[1, 3, 11].

2 $[\gamma^c_{cer}]^{e^-}$-critical graphs

We provide the characterization of "$[\gamma^c_{cer}(\mathcal{Z})]^{e^-}$-critical graphs" in this section. Before moving on to the key findings, we first define the family of trees $\mathcal{T}$ as follows.
We define $\mathcal{T}$ as a family of trees in which each vertex is a leaf or of degree at least 3. A tree $\mathfrak{T} \in \mathcal{T}$, if $\mathfrak{T}$ is a non-trivial star $S_{(1,r)}$, $r \geq 2$, or $\mathfrak{T}$ is double star graph $S_{(q,r)}$, $q, r \geq 2$, or $\mathfrak{T}$ is a caterpillar, or if $\mathfrak{T}$ can be constructed by subdivided star $S_{(1,r)}$, $r \geq 2$ by adding zero or at least two vertices to the vertices of degree 1, or if $\mathfrak{T}$ can be constructed by subdivided double star graph $S_{(q,r)}$, $q, r \geq 2$ by adding at least two vertices to the vertices of degree 1.
We begin this section with the following observation.

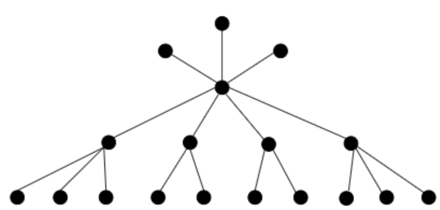

Figure 1: A graph T_{18} in the family of trees $\mathcal{T}$.

Observation 1 If $\mathcal{C}$ is the smallest γ^c_{cer}−set of a graph $\mathcal{Z}$, then for each vertex $u \in \mathcal{C}$, $|epn(u, \mathcal{C})| \geq 1$.
Note that if $\mathcal{C}$ is the smallest γ^c_{cer}−set of a connected graph $\mathcal{Z} = (V_{\mathcal{Z}}, E_{\mathcal{Z}})$ such that $\mathcal{C} = V_{\mathcal{Z}}$, then $epn(u, \mathcal{C}) = \phi$.

Proposition 1 *Let $\mathcal{Z}$ be an isolate free graph of order n with $dia(\mathcal{Z}) \leq 2$ and $\delta(\mathcal{Z}) = 1$, then $\mathcal{Z}$ is $[\gamma_{cer}^c]^{e^-}$-critical.*

Proof. Let $\mathcal{Z}$ be any connected graph of order n, with $dia(\mathcal{Z}) \leq 2$ and $\delta(\mathcal{Z}) = 1$, and let $\mathcal{C}$ be the γ_{cer}^c–set of the graph $\mathcal{Z}$. We will prove it in two cases.

Case 1. When $dia(\mathcal{Z}) = 2$.

If a graph $\mathcal{Z}$ has diameter two, then every pair of non-adjacent vertices has a common neighbor, and $\gamma_{cer}^c(\mathcal{Z}) = 1$, whenever $dia(\mathcal{Z}) = 2$ and $\delta(\mathcal{Z} = 1$. Let $e = uv \in \mathcal{Z}$ be such that either $u \in \mathcal{C}$ or $v \in \mathcal{C}$. Therefore, if we delete edge $e = uv$ from $\mathcal{Z}$, then the CCDN of the graph $\mathcal{Z} - e$ will change, and we know that deletion of an edge from any arbitrary graph cannot decrease its CCDN. Therefore $\gamma_{cer}^c(\mathcal{Z} - e) > \gamma_{cer}^c(\mathcal{Z})$, implying that $\mathcal{Z}$ is $[\gamma_{cer}^c]^{e^-}$-critical.

Case 2. When $dia(\mathcal{Z}) = 1$.

If a graph $\mathcal{Z}$ has a diameter 1, then $\mathcal{Z}$ is a path graph P_2 on two vertices u and v connected with only edge $e = uv \in \mathcal{Z}$. Now if we remove this edge e then the resultant graph will be a disconnected graph implying that $\gamma_{cer}^c(P_2 - e) = \infty$ and hence $\mathcal{Z}$ is $[\gamma_{cer}^c]^{e^-}$-critical.

Hence from Case 1 and case 2, we conclude that $\mathcal{Z}$ is $[\gamma_{cer}^c]^{e^-}$-critical. □

Proposition 2 *If $\mathcal{Z}$ is $[\gamma_{cer}^c]^{e^-}$- critical graph and $dia(\mathcal{Z}) \leq 2$, then for every $\gamma_{cer}^c(\mathcal{Z})$–set $\mathcal{C}$ of $\mathcal{Z}$, $\mathcal{Z}[\mathcal{C}]$ is either a trivial graph or a star graph.*

The CCDN of a an isolate free graph $\mathcal{Z}$ with $dia(\mathcal{Z}) \leq 2$ in most of the cases is one and is two in the only case when $\mathcal{Z} \approxeq P_2$. So in cases when $\gamma_{cer}^c(\mathcal{Z}) = 1$, the subgraph induced by γ_{cer}^c–set of graph $\mathcal{Z}$ is a trivial graph, and when $\gamma_{cer}^c(\mathcal{Z}) = 2$ the subgraph induced by γ_{cer}^c–set is a star graph. Also, if a graph $\mathcal{Z}$ has $\delta(\mathcal{Z}) = 1$, then $\mathcal{Z}$ is always $[\gamma_{cer}^c]^{e^-}$-critical.

Corollary 3 *Let $\mathcal{T}$ be an isolate free graph such that $\mathcal{T} \in \mathcal{T}$, then γ_{cer}^c– set of $\mathcal{T}$ will be a star or double star graph.*

Proposition 4 *Let $\mathcal{Z}$ be an isolate free graph of order n and let $\mathcal{C}$ be the γ_{cer}^c–set of $\mathcal{Z}$. For any edge $e = uv \in E_{\mathcal{Z}}$, where $u \in \mathcal{C}$ and $v \notin \mathcal{C}$, if $|N(v) \cap \mathcal{C}| = 1$, then the graph $\mathcal{Z}$ is $[\gamma_{cer}^c]^{e^-}$-critical.*

Proof. Let $\mathcal{Z}$ be any isolate free graph of order n and $\mathcal{C}$ be the γ_{cer}^c–set of graph $\mathcal{Z}$. Let $e = uv \in E_{\mathcal{Z}}$ be such that $u \in \mathcal{C}, v \notin \mathcal{C}$ and $|N(v) \cap \mathcal{C}| = 1$. Since $\mathcal{C}$ is γ_{cer}^c–set of the graph $\mathcal{Z}$, therefore $|N(w) \cap (V_{\mathcal{Z}} \backslash \mathcal{C})| = 0$ or $\geq 2, \forall w \in \mathcal{C}$,

that is, every vertex in $\mathcal{C}$ has 0 or at least two neighbors in $V_{\mathcal{Z}}\backslash\mathcal{C}$, implying that $|N(u) \cap (V_{\mathcal{Z}}\backslash\mathcal{C})| \geq 2$, as v is one such neighbor of u in $V_{\mathcal{Z}}\backslash\mathcal{C}$. By assumption, the vertex u cannot be weak support of the graph $\mathcal{Z}$. We consider two cases: $u \in S_2(\mathcal{Z})$ and $u \notin S_2(\mathcal{Z})$.
Case 1. $u \in S_2(\mathcal{Z})$.
If $u \in S_2(\mathcal{Z})$, then $deg_{\mathcal{Z}}(u) \geq 2$, and $v \in L_{\mathcal{Z}}$ is one such neighbor of u in $\mathcal{Z}$. Also, every strong support vertex of a graph $\mathcal{Z}$ belongs to every $\gamma^c_{cer}-$set of the graph $\mathcal{Z}$. If we eliminate the edge $e = uv$ from the graph $\mathcal{Z}$ then the CCDN of the graph $\mathcal{Z} - e$ will change, as u is the only neibhor of v in $\mathcal{C}$, implying that $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical.
Case 2. $u \notin S_2(\mathcal{Z})$.
Since $|N(v) \cap \mathcal{C}| = 1$, so if $u \notin S_2(\mathcal{Z})$ then its neighbor v cannot be a leaf in the graph $\mathcal{Z}$, because if v is a leaf, then the vertex $u \in S_2(\mathcal{Z})$ which will be a contradiction to our assumption. As u is the only neighbor of the vertex v in $\gamma^c_{cer}-$set $\mathcal{C}$ of the graph $\mathcal{Z}$, therefore if we delete the edge $e = uv$ from the graph $\mathcal{Z}$, then the vertex $v \in \mathcal{Z} - e$ will be the only vertex in the graph $\mathcal{Z} - e$ which is not adjacent to any of the vertex in $\gamma^c_{cer}-$set $\mathcal{C}$ of the graph $\mathcal{Z}$, implying that the removal of the edge $e = uv$ from the graph $\mathcal{Z}$ changes the CCDN of $\mathcal{Z} - e$, $i.e., \gamma^c_{cer}(\mathcal{Z} - e) > \gamma^c_{cer}(\mathcal{Z})$ as removal of an edge cannot decrease the CCDN of a graph.
Hence from the above two cases, we conclude that for any edge $e = uv$ such that $u \in \mathcal{C}, v \notin \mathcal{C}$ and $|N(v) \cap \mathcal{C}| = 1$, then $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical. □

Proposition 5 *Let $\mathcal{Z}$ be an isolate free graph of order n, then $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical if there exists a vertex $u \in \mathcal{Z}$ such that $u \in S_2(\mathcal{Z})$.*

Proof. Let $\mathcal{C}$ be the $\gamma^c_{cer}-$set of the graph $\mathcal{Z}$. Suppose there exists a vertex $u \in \mathcal{Z}$ such that $u \in S_2(\mathcal{Z})$. Let $v \in L_{\mathcal{Z}}$ be a leaf of the graph $\mathcal{Z}$ adjacent to the strong support vertex u. Since u is the only neighbor of the vertex v in graph $\mathcal{Z}$ and every strong support vertex of a graph $\mathcal{Z}$ belongs to every $\gamma^c_{cer}-$set of the graph $\mathcal{Z}$. Therefore $u \in \mathcal{C}, v \in L_{\mathcal{Z}}$ $i.e., v \notin \mathcal{C}$ and $|N(v) \cap \mathcal{C}| = 1$ implies that $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical by preposition 4. □

Theorem 6 *A connected graph $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical if and only if $\mathcal{Z} \in \mathcal{T}$.*

Proof. Suppose that $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical graph. Let $\mathcal{C}$ be the $\gamma^c_{cer}-$set of the graph $\mathcal{Z}$. If l is the leaf in the subgraph $\mathcal{Z}[\mathcal{C}]$ induced by $\gamma^c_{cer}-$ set $\mathcal{C}$, then l is adjacent to a node of degree at least 2 in $\mathcal{Z}[\mathcal{C}]$ and by observation 1 $|epn(l, \mathcal{C})| \geq 1$. Thus, l is a neighbor of at least two nodes in $V_{\mathcal{Z}}\backslash\mathcal{C}$ because l has an external

private neighbor. The fact that $\mathcal{Z}$ is $[\gamma^{c}_{cer}]^{e^-}$-critical is contradicted by the assumption that if $\mathcal{C}$ is $\gamma^{c}_{cer}-$set in $\mathcal{Z}-e$, $\forall e \in \mathcal{Z}$, then $\gamma^{c}_{cer}(\mathcal{Z}-e) \leq |\mathcal{C}| = \gamma^{c}_{cer}(\mathcal{Z})$. As a result, the set $\mathcal{C}$ is not a $\gamma^{c}_{cer}-$set in $\mathcal{Z}-e$ $\forall e \in \mathcal{Z}$. Hence $V_{\mathcal{Z}}\backslash\mathcal{C}$ is an independent set and every node in $V_{\mathcal{Z}}\backslash\mathcal{C}$ is adjacent to precisely one node of $\mathcal{C}$ and is thus, a leaf of $\mathcal{Z}$. Since $\mathcal{C}$ is the $\gamma^{c}_{cer}-$ set of the graph $\mathcal{Z}$, the subgraph $\mathcal{Z}[\mathcal{C}]$ induced by $\mathcal{C}$ in $\mathcal{Z}$ is connected. Hence $\mathcal{Z}[\mathcal{C}]$ will either be a double star or star graph, and so $\mathcal{Z} \in \mathcal{T}$. We may suppose that $\mathcal{Z}[\mathcal{C}]$ is a star $S_{(1,r)}$. As each node in the induced star $\mathcal{Z}[\mathcal{C}]$ is adjacent to at least two nodes in $V_{\mathcal{Z}}\backslash\mathcal{C}$. Let $\mathcal{J}$ denote the set of $q = 2r$, $r \geq 2$ nodes in $V_{\mathcal{Z}}\backslash\mathcal{C}$ that is adjacent to the set of r leaves in $\mathcal{Z}[\mathcal{C}]$, then $\mathcal{Z}[\mathcal{C} \cup \mathcal{J}] = S^{*}_{(1,r)}$ is a subdivided star graph and can be obtained by adding each node in $\mathcal{C}$ with at least two pendant edges. Thus $\mathcal{Z} \in \mathcal{T}$.
Now, assume that $\mathcal{Z} = (V_{\mathcal{Z}}, E_{\mathcal{Z}}) \in \mathcal{T}$ and let $e = uv \in E_{\mathcal{Z}}$ be any edge in the graph $\mathcal{Z}$. If the edge $e = uv$ is such that one of the end vertex of e is a leaf in $\mathcal{Z}$, then $\gamma^{c}_{cer}(\mathcal{Z}-e) = \infty$, and so the edge $e = uv$ is critical. Therefore, we may suppose that the edge e is not a pendant edge in $\mathcal{Z}$. More precisely, $\mathcal{Z}$ is not a star graph $S_{(1,r)}$. If $\mathcal{Z}$ is a double-star graph $S_{(q,r)}$ with central vertices u_1 and u_2, then the edge $e = u_1u_2$ joins the two vertices u_1 and u_2 of $\mathcal{Z}$. Thus, $\gamma^{c}_{cer}(\mathcal{Z}-e) = \infty$ while $\gamma^{c}_{cer}(\mathcal{Z}) = 2$, so the edge $e = uv$ is critical, and graph $\mathcal{Z}$ is $[\gamma^{c}_{cer}]^{e^-}$-critical. Therefore, let's suppose $\mathcal{Z}$ isn't a double star. Henceforth, $\mathcal{Z}$ is the graph formed from a star $S_{(1,r)}$ for some $r \geq 2$, by appending at least two pendant edges to each leaf of $S_{(1,r)}$. In the set $E_{\mathcal{Z}}\backslash E_{S_{(1,r)}}$ every edge is a pendant edge in $\mathcal{Z}$. Hence, by our previous assumption $e \in E_{S_{(1,r)}}$. But then $\gamma^{c}_{cer}(\mathcal{Z}-e) = \infty$, which again implies that the edge e is critical. Therefore, $\mathcal{Z}$ is $[\gamma^{c}_{cer}]^{e^-}$-critical. □

Corollary 7 *If $\mathcal{Z}$ is an isolate free graph of order $n \geq 4$, then $\mathcal{Z}$ is $[\gamma^{c}_{cer}]^{e^-}$-critical if $\mathcal{Z}$ has unique minimal $\gamma^{c}_{cer}-$set.*

Observation 2 If $\mathcal{Z} = \mathcal{H} \circ \mathcal{K}$ is the corona of graphs $\mathcal{H}$ and $\mathcal{K}$, then $\mathcal{Z}$ is $[\gamma^{c}_{cer}]^{e^-}$-critical.

3 $[\gamma^{c}_{cer}]^{e^-}$-stable graphs

We present the characterization of $[\gamma^{c}_{cer}]^{e^-}$-stable graphs in this section. Note that $\gamma^{c}_{cer}(\mathcal{Z}) = \infty$ if $\mathcal{Z}$ is a graph containing atleast one isolated vertex. As a result, if we eliminate any pendant edge e from the graph $\mathcal{Z}$, then $\gamma^{c}_{cer}(\mathcal{Z}-e) = \infty$.

We have observed that if $\delta(\mathcal{Z})=1$, then the graph $\mathcal{Z}$ is always $[\gamma^c_{cer}]^{e^-}$ −critical. For a graph $\mathcal{Z}$ to be a $[\gamma^c_{cer}]^{e^-}$-stable graph if $\delta(\mathcal{Z})\geq 2$. For this purpose, we need the following proposition.

Proposition 8 *Let $\mathcal{Z}$ be an isolate-free graph, then the following two conditions hold:*

(1) If $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable, then $\delta(\mathcal{Z})\geq 2$.

(2) If the edge $e\in E_{\mathcal{Z}}$ is stable, then every $\gamma^c_{cer}(\mathcal{Z}-e)-$set is a $\gamma^c_{cer}(\mathcal{Z})-$set.

Proof.

(1) Suppose, on the contrary, that $\delta(\mathcal{Z})=1$. Then graph $\mathcal{Z}$ has at least one pendant edge e incident on a leaf vertex. Since $\mathcal{Z}$ is an isolate-free graph, the removal of the edge e from the graph $\mathcal{Z}$ will result in a graph $\mathcal{Z}-e$ containing an isolated vertex and, therefore $\gamma^c_{cer}(\mathcal{Z}-e)=\infty$, implying that $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical, a contradiction. Hence, $\delta(\mathcal{Z})\geq 2$ if $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable graph.

(2) Suppose $e\in E_{\mathcal{Z}}$ is a stable edge of the graph $\mathcal{Z}$; it means that the removal of the edge $e\in E_{\mathcal{Z}}$ from the graph $\mathcal{Z}$ does not change its CCDN; that is $\gamma^c_{cer}(\mathcal{Z}-e)=\gamma^c_{cer}(\mathcal{Z})=|\gamma^c_{cer}(\mathcal{Z})-\mathsf{set}|$, which implies that every $\gamma^c_{cer}-$ set of the graph $\mathcal{Z}-e$ is a $\gamma^c_{cer}-$set of the graph $\mathcal{Z}$

□

Proposition 9 *A graph $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable if and only if for each $e=uv\in E_{\mathcal{Z}}$ and $\delta(\mathcal{Z})\geq 2$, $\exists$ a $\gamma^c_{cer}(\mathcal{Z})$-set $\mathcal{C}$ such that:*

(1) $u,v\notin C$.

(2) If $u,v\in\mathcal{Z}$, then $|N_{\mathcal{Z}}(u)\cap\mathcal{C}|\geq 2$ and $|N_{\mathcal{Z}}(v)\cap\mathcal{C}|\geq 2$.

(3) If $u\in\mathcal{C}$ and $v\notin\mathcal{C}$, then $|N_{\mathcal{Z}}(v)\cap\mathcal{C}|\geq 2$.

Proof. Suppose that $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable. By preposition 8, $\delta(\mathcal{Z})\geq 2$. Let $e=uv$ be any edge of the graph $\mathcal{Z}$. Let $\mathcal{Z}'=\mathcal{Z}-uv$ and let $\mathcal{C}$ be any $\gamma^c_{cer}-$ set of the graph $\mathcal{Z}'$. By proposition 8, the set $\mathcal{C}$ is a $\gamma^c_{cer}-$set of the graph $\mathcal{Z}$. Now, if $u,v\in\mathcal{Z}$, then condition (1) holds. Assume that $u,v\in\mathcal{C}$, then since $\mathcal{C}$ is $\gamma^c_{cer}-$ set of the graph $\mathcal{Z}'$, $|N_{\mathcal{Z}'}(u)\cap\mathcal{C}|\geq 1$ and $|N_{\mathcal{Z}'}(v)\cap\mathcal{C}|\geq 1$, and so $|N_{\mathcal{Z}}(u)\cap\mathcal{C}|\geq 2$ and $|N_{\mathcal{Z}}(v)\cap\mathcal{C}|\geq 2$, thus condition (2) holds. If $u\in\mathcal{C}$

and $v \notin \mathcal{C}$, then since $\mathcal{C}$ is $\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}'$, $|N_{\mathcal{Z}'}(v) \cap \mathcal{C}| \geq 1$, and so $|N_{\mathcal{Z}}(v) \cap \mathcal{C}| \geq 2$. Thus (3) holds. Henceforth, the set $\mathcal{C}$ is a $\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}$ such that one of the three conditions (1), (2), and (3) is satisfied.
To prove the sufficiency, assume that $\delta(\mathcal{Z}) \geq 2$, and for every edge, $e = uv \in E_{\mathcal{Z}}$ there exists a $\gamma_{cer}^{c}-$set $\mathcal{C}$ of the graph $\mathcal{Z}$ satisfying the three conditions $(1), (2)$, and (3). Note that in all three conditions, the set $\mathcal{C}$ is also a $\gamma_{cer}^{c}-$set for $\mathcal{Z} - uv$. Hence, $\gamma_{cer}^{c}(\mathcal{Z}) \leq \gamma_{cer}^{c}(\mathcal{Z} - uv) \leq |\mathcal{C}| = \gamma_{cer}^{c}(\mathcal{Z})$, implying that $\gamma_{cer}^{c}(\mathcal{Z}) = \gamma_{cer}^{c}(\mathcal{Z} - uv)$. Therefore, the graph $\mathcal{Z}$ is $[\gamma_{cer}^{c}]^{e^-}$-stable.

□

We have the following observation as a result of proposition 9.
Observation 3. Let $\mathcal{Z}$ be a $[\gamma_{cer}^{c}]^{e^-}$- stable graph, then $\mathcal{Z}$ has at least two distinct $\gamma_{cer}^{c}-$sets.
Observation 4.

(a) Every cycle graph C_n is a $[\gamma_{cer}^{c}]^{e^-}$-stable graph $\forall n \geq 4$.

(b) For every integer $s \geq 4$, $\exists$ s_{cer}^{c}-stable graph.

Theorem 10 *Let $\mathcal{Z}$ be a complete bipartite graph $K_{(m,n)}, m, n \geq 3$, then $\mathcal{Z}$ is $[\gamma_{cer}^{c}]^{e^-}$-stable if and only if $|N_{\mathcal{Z}}(x) \cap T_{cer}^{c}(\mathcal{Z})| \geq 2$, $\forall x \in \mathcal{Z}$.*

Proof. Assume that $\mathcal{Z}$ is a complete bipartite $[\gamma_{cer}^{c}]^{e^-}$-stable graph and $x \in V_{\mathcal{Z}}$. Let $\mathcal{C}$ be a $\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}$. Then, there exists a vertex $y \in \mathcal{C}$ that is adjacent to x. Now, by definition of the set $T_{cer}^{c}(\mathcal{Z})$, we note that $\mathcal{C} \subseteq T_{cer}^{c}(\mathcal{Z})$ and so $y \in N_{\mathcal{Z}}(x) \cap T_{cer}^{c}(\mathcal{Z})$. Let C' be the $\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}' = \mathcal{Z} - uv$, and let z be a vertex in C' adjacent to x. By preposition 8, C' is a$\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}$, and so $C' \subseteq T_{cer}^{c}(\mathcal{Z})$. Thus, $z \in N_{\mathcal{Z}}(x) \in T_{cer}^{c}(\mathcal{Z})$. Since $y \notin z$, we have $|N_{\mathcal{Z}}(x) \cap T_{cer}^{c}(\mathcal{Z})| \geq |\{y, z\}| = 2$, as claimed.
For sufficiency, let $\mathcal{Z}$ be a complete bipartite graph $K_{(m,n)}, m, n \geq 3$, and $\mathcal{C}$ be the $\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}$. Suppose that $|N_{\mathcal{Z}}(x) \cap T_{cer}^{c}(\mathcal{Z})| \geq 2, \forall x \in \mathcal{Z}$. Let $y \in \mathcal{C}$ be a vertex in the $\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}$ adjacent to the vertex x. Then in the graph $\mathcal{Z}' = \mathcal{Z} - uv$, $|N_{\mathcal{Z}'}(x) \cap T_{cer}^{c}(\mathcal{Z})| \geq 1$, and so $|N_(\mathcal{Z}')(x) \cap \mathcal{C}| \geq 1$, implying that $|N_{\mathcal{Z}}(x) \cap \mathcal{C}| \geq 2$. Hence by proposition 9, $\mathcal{Z}$ is $[\gamma_{cer}^{c}]^{e^-}$-stable.

□

We have observed that the above theorem is not true for bipartite graphs, and its converse does not hold for graphs in general. See the examples below for demonstration.
Example 1. Let $\mathcal{Z}$ be a bipartite graph shown in figure 2 below. The colored vertex set u, v form the $\gamma_{cer}^{c}-$ set of the graph $\mathcal{Z}$, implying that $\gamma_{cer}^{c}(\mathcal{Z}) = 2$.

However, $\gamma^c_{cer}(\mathcal{Z}' = \mathcal{Z} - uv) = 6$, which implies that $\gamma^c_{cer}(\mathcal{Z}) \neq \gamma^c_{cer}(\mathcal{Z}')$ and hence $\mathcal{Z}$ is not $[\gamma^c_{cer}]^{e^-}$-stable graph.

Example 2. If $\mathcal{Z}$ is a graph obtained from a $6-$cycle $x_1x_2..., x_6x_1$ by adding the chord x_1x_4 and x_3x_6 then $\mathcal{Z}$ has exactly two $\gamma^c_{cer}-$sets, namely the sets x_1, x_4 and x_3, x_6. Thus $T^c_{cer}(\mathcal{Z}) = \{x_1, x_4, x_3, x_6\}$ and $|N_{\mathcal{Z}}(x) \cap T^c_{cer}(\mathcal{Z})| \geq 2, \forall x \in \mathcal{Z}$. However the edges x_1x_4 and x_3x_6, are both critical in $\mathcal{Z}$, and so $\mathcal{Z}$ is not a $[\gamma^c_{cer}]^{e^-}$-stable graph.

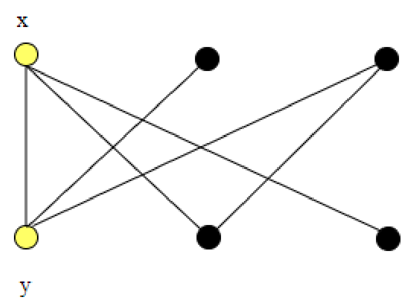

Figure 2: Bipartite graph $K_{3,3}$.

Consequently, as a direct conclusion of Theorem 10, we have the following result.

Corollary 11 *Let $\mathcal{Z}$ be a bipartite graph such that $\mathcal{Z}$ has two disjoint $\gamma^c_{cer}-$sets then $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable.*

Next, we will show that if a graph $\mathcal{Z}$ has at least two disjoint $\gamma^c_{cer}-$ sets then $\mathcal{Z}$ cannot have critical edges more then $\gamma^c_{cer}(\mathcal{Z})$

Proposition 12 *If $\mathcal{Z}$ is connected graph of order n such that $\mathcal{Z}$ has two disjoint $\gamma^c_{cer}-$sets, then $\mathcal{Z}$ has a maximum of $\gamma^c_{cer}(\mathcal{Z})$ critical edges.*

Proof. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ be two disjoint $\gamma^c_{cer}-$sets of an isolate free graph $\mathcal{Z}$ of order n. If the graph $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable, then every edge of $\mathcal{Z}$ stable, and the result is thus straightforward. Assume that $\mathcal{Z}$ contains at least one $[\gamma^c_{cer}]^{e^-}$-critical edge and define $e = uv$ as such an edge. If e has no end in the set $\mathcal{X}$, then the set $\mathcal{X}$ is a $\gamma^c_{cer}-$ set in $\mathcal{Z} - e$, which implies that $\gamma^c_{cer}(\mathcal{Z} - e) = \gamma^c_{cer}(\mathcal{Z})$, a contradiction. Therefore, e has at least one end in $\mathcal{X}$. Likewise, e has at least one end in $\mathcal{Y}$. Thus, $e = uv$ where $u \in \mathcal{X}$ and $v \in \mathcal{Y}$. If $|N(v) \cap \mathcal{X}| \geq 2$, then $\mathcal{X}$ is a $\gamma^c_{cer}-$ set in $\mathcal{Z} - e$, a contradiction. Hence, $N(v) \cap \mathcal{X} = \{u\}$, and similarly, $N(u) \cap \mathcal{Y} = \{v\}$. This means that $\mathcal{Z}$ has critical edges that are at most the CCDN of the graph $\mathcal{Z}$. □

Corollary 13 *For any integer* $t \geq 1, \exists$ *a graph* $\mathcal{Z}$ *with precisely two disjoint* $\gamma^c_{cer}-$ *sets such that* $\gamma^c_{cer}(\mathcal{Z}) = t$ *and* $\mathcal{Z}$ *has precisely* t *critical edges.*

Observation 5. The following graphs are $[\gamma^c_{cer}]^{e^-}$-stable.

(1) Complete graph K_n is $[\gamma^c_{cer}]^{e^-}$-stable for all $n \geq 3$.

(2) Complete bipartite graph $K_{(m,n)}$ is $[\gamma^c_{cer}]^{e^-}$-stable for all $m, n \geq 3$.

(3) Bipartite graphs satisfying the corollary 11 are $[\gamma^c_{cer}]^{e^-}$-stable.

(4) Cycle graph C_n is $[\gamma^c_{cer}]^{e^-}$-stable for all $n \geq 4$.

4 Conclusion

The study of criticality and stability of graphs upon edge removal or addition on any graph domination parameter has exciting applications in networking. In this article, we have initiated the study of connected certified domination criticality and stability upon edge removal.

For connected certified domination edge critical graphs, we have proved that every graph with $dia(\mathcal{Z}) \leq 2$ and $\delta(\mathcal{Z} = 1$,is $[\gamma^c_{cer}]^{e^-}$-critical, and if $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical graph with $dia(\mathcal{Z}) \leq 2$, then for every $\gamma^c_{cer}(\mathcal{Z})-$ set $\mathcal{C}$ of $\mathcal{Z}$, $\mathcal{Z}[\mathcal{C}]$ is either a trivial graph or a star graph. Also, if $\mathcal{C}$ is the $\gamma^c_{cer}-$ set of a graph $\mathcal{Z}$ and for any edge $e = uv \in \mathcal{Z}$, where $u \in \mathcal{C}$ and $v \notin C$, if $|N(v) \cap \mathcal{C}| = 1$, then the graph $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical, and $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical if the graph $\mathcal{Z}$ contains a support vertex. We have proved a necessary and sufficient condition that a graph $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-critical iff $\mathcal{Z} \in \mathcal{T}$.

Similarly, for connected certified domination edge stable graphs, we have proved the following results:

If a graph $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable, then $\delta(\mathcal{Z}) \geq 2$, and if the edge $e = uv \in E_{\mathcal{Z}}$ is stable, then every $\gamma^c_{cer}(\mathcal{Z}-e)-$ set is a $\gamma^c_{cer}(\mathcal{Z})-$set. Also, a graph $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable if and only if for each $e = uv \in E_{\mathcal{Z}}$ and $\delta(\mathcal{Z}) \geq 2, \exists$ a $\gamma^c_{cer}(\mathcal{Z})-$set $\mathcal{C}$ such that: (1). $u, v \notin \mathcal{C}$. (2). If $u, v \in \mathcal{C}$, then $|N_{\mathcal{Z}}(u) \cap \mathcal{C}| \geq 2$ and $|N_{\mathcal{Z}}(v) \cap \mathcal{C}| \geq 2$. (3). If $u \in \mathcal{C}$ and $v \notin \mathcal{C}$, then $|N_{\mathcal{Z}}(v) \cap \mathcal{C}| \geq 2$. We have shown that if $\mathcal{Z}$ is a complete bipartite graph $K_{(m,n)}, m, n \geq 3$, then $\mathcal{Z}$ is $[\gamma^c_{cer}]^{e^-}$-stable if and only if $|N_{\mathcal{Z}}(x) \cap T^c_{cer}(\mathcal{Z})| \geq 2, \forall x \in \mathcal{Z}$, and we have justified that this result is not true for bipartite graphs, and its converse is not valid for graphs in general. And finally, we have shown that if $\mathcal{Z}$ is a connected graph of order n such that G has precisely two disjoint $\gamma^c_{cer}-$sets, then $\mathcal{Z}$ has a maximum of $\gamma^c_{cer}(\mathcal{Z})$ critical edges.

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