



On domination in signed graphs

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Abstract. In this article the concept of domination in signed graphs is examined from an alternate perspective and a new definition of the same is introduced. A vertex subset D of a signed graph S is a dominating set, if for each vertex v not in D there exists a vertex $u \in D$ such that the sign of the edge uv is positive. The domination number $\gamma(S)$ of S is the minimum cardinality among all the dominating sets of S . We obtain certain bounds of $\gamma(S)$ and present a necessary and sufficient condition for a dominating set to be a minimal dominating set. Further, we characterise the signed graphs having small and large values for domination number.

1 Introduction

A *signed graph* is an ordered pair $S = (G, \sigma)$, where $G = (V, E)$ is a simple graph called the *underlying graph* of S and $\sigma : E(G) \rightarrow \{-1, 1\}$ is a function called a *signing* of G or the *signature* of S . The negative and positive edges are depicted using dashed and solid lines respectively. The set of all positive(negative) edges is denoted by $E^+(S)(E^-(S))$. The subgraph obtained by removing the negative(positive) edges is denoted by $S^+(S^-)$. A signed

Key words and phrases: signed graphs, dominating set, domination number, minimal dominating set

graph where every edge is positive(negative) is called *all-positive(all-negative)*. For any vertex u , $N^+(u) = \{v \in N(u) | \sigma(uv) = 1\}$ and $N^-(u) = \{v \in N(u) | \sigma(uv) = -1\}$. The positive and negative degree of a vertex u is defined as $d^+(u) = |N^+(u)|$ and $d^-(u) = |N^-(u)|$ respectively, while the degree of u in G is $d_G(u) = d^+(u) + d^-(u)$. The maximum(minimum) positive degree of S is denoted using $\Delta^+(S)$ ($\delta^+(S)$), whereas $\Delta^-(S)$ ($\delta^-(S)$) denote the maximum (minimum) negative degree of S . Signed graphs provide a large scope for researchers to investigate both theoretical and application problems in graph theory [1, 2, 5, 7, 13, 15, 17, 18, 19].

The concept of domination in graphs is a well established research area in graph theory [6, 8, 9, 10, 11, 12]. Although the concept of domination in graphs was introduced by Berge [4] in the year 1962, it is to be noted that the first article on domination in signed graphs appeared only in the year 2013. Another interesting fact is that the idea of domination can be viewed from various perspectives. It was Acharya[2] who made the first attempt in articulating the concept of domination in signed graphs. He defined a dominating set of a signed graph $S = (G, \sigma)$ as a set $D \subseteq V$ such that all the vertices of S are either in D or there exists a function $\mu : V \rightarrow \{-1, 1\}$ called a marking of S such that all the vertices $u \in V \setminus D$ are adjacent to at least one vertex $v \in D$ such that $\sigma(uv) = \mu(u)\mu(v)$. Later in the year 2020, Jeyalakshmi [13] proposed another definition for a dominating set of a signed graph. A subset D of the vertex set V is called a dominating set of a signed graph S if for all $v \in V \setminus D$, $|N^+(v) \cap D| > |N^-(v) \cap D|$. In this article we study the concept of domination in signed graphs from yet another point of view.

Joseph and Joseph [14] considered the fact that in any network that can be represented as a signed graph, a vertex dominates another vertex provided there exist a positive edge between them. In this sense, a set D of vertices of a signed graph S that are connected to the remaining vertices of S by positive edges can be considered as a dominating set of S . Accordingly they presented the following definition for a dominating set of signed graphs.

Definition 1 [14] *Let $S = (G, \sigma)$ be a signed graph. A set $D \subseteq V$ is said to be a dominating set of S if each vertex $v \in V \setminus D$ is adjacent to at least one vertex $u \in D$ such that $\sigma(uv) = 1$. The minimum cardinality among all the dominating sets of S is called the domination number of S , denoted by $\gamma(S)$.*

For a signed graph $S = (G, \sigma)$, by the term $\gamma(S)$ -set we mean a minimum dominating set and $\gamma(G)$ -set refers to a minimum dominating set of the underlying graph G .

Clearly, if all the edges of S are positive then the above definition reduces to that of the domination in graphs. On the other hand, if S is all negative, then the dominating set is trivially V .

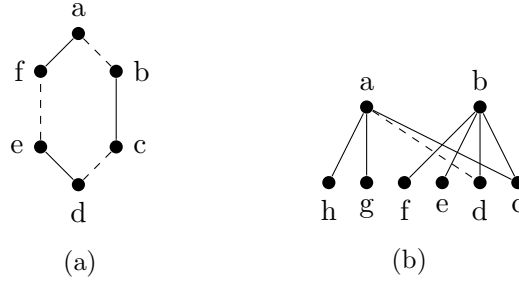


Figure 1

If $|N^+(v) \cap D| > |N^-(v) \cap D| \forall v \in V \setminus D$, then we can observe that $N^+(v) \cap D$ is non-empty whenever $v \in V \setminus D$ i.e every vertex $v \in V \setminus D$ is adjacent to a vertex $u \in D$ such that $uv \in E^+(S)$. Hence, any dominating set as given in [13] is also a dominating set as per Definition 1. For example, the signed graph in Figure 1(a) has $\{a, b, c, d\}$ as a minimum dominating set by the definition proposed by Jeyalakshmi where as $\{a, b, d\}$ is a minimum dominating set by Definition 1. But, for the signed graph given in Figure 1(b), $\{a, b\}$ is the minimum dominating set in both the cases.

2 Preliminaries

In this section we explore some of the basic results that follow from Definition 1. A characterisation of minimal dominating sets in graphs was obtained by Ore [16]. Analogously we present a characterisation for the minimal dominating sets of a signed graph. We omit the proof as it is similar to the corresponding result in [16].

Theorem 2 *Let S be a signed graph and D be a dominating set of S . Then D is a minimal dominating set of S if and only if for each vertex v in D either of the following conditions hold:*

(i) $N^+(v) \subseteq V \setminus D$,

(ii) *There exists a vertex u in $V \setminus D$ such that $N^+(u) \cap D = \{v\}$.*

For any signed graph S of order n , the vertex set is a trivial dominating set. On the other hand any dominating set of S is of cardinality at least 1 so that we have the obvious inequality $1 \leq \gamma(S) \leq n$. The upper bound is attained by all-negative signed graphs. Moreover, for any signed graph S of order n , $\gamma(S) = n$ if and only if S is all-negative.

Remark 3 *In any signed graph S , a vertex v with $N^+(v) = \emptyset$ belongs to all the dominating sets of S .*

Note that the dominating sets of signed graphs are always the dominating sets of their underlying graphs. Thus we have the following inequality.

Proposition 4 *If $S = (G, \sigma)$ is any signed graph, then $\gamma(S) \geq \gamma(G)$.*

The following theorem characterises the signed graphs having domination number equal to that of their underlying graphs.

Theorem 5 *Let $S = (G, \sigma)$ be a signed graph. Then $\gamma(S) = \gamma(G)$ if and only if there is a $\gamma(G)$ -set D such that for every vertex u in $V \setminus D$, $N^+(u) \cap D \neq \emptyset$.*

Proof. Suppose that $S = (G, \sigma)$ is a signed graph such that $\gamma(S) = \gamma(G)$. Then there is a $\gamma(S)$ -set D which is a $\gamma(G)$ -set. Also, by the definition of a dominating set of a signed graph, for every vertex u in $V \setminus D$, $N^+(u) \cap D \neq \emptyset$.

Conversely, suppose that there exists a $\gamma(G)$ -set D such that for every vertex u in $V \setminus D$, $N^+(u) \cap D \neq \emptyset$. Therefore D is a dominating set of S and $\gamma(S) \leq \gamma(G)$. Then from Proposition 4 it follows that $\gamma(S) = \gamma(G)$. \square

Now we examine the number of negative edges in a signed graph $S = (G, \sigma)$ such that $\gamma(S) \geq \gamma(G)$. Recall that the bondage number $b(G)$ of a graph G is defined as the minimum number of edges whose removal increases the domination number. Let $m^-(S)$ be the number of negative edges in S . Then we have the following proposition.

Proposition 6 *Let $S = (G, \sigma)$ be a signed graph.*

- (i) *If $\gamma(S) = \gamma(G)$, then the maximum value of $m^-(S)$ is the number of edges between the vertices of a $\gamma(G)$ -set, say D and the number of edges between the vertices of the set $V \setminus D$ and $|N(v) \cap D| - 1$ edges corresponding to each vertex v in $V \setminus D$ dominated by more than one vertex in D .*
- (ii) *If $\gamma(S) > \gamma(G)$, then $m^-(S) \geq b(G)$.*

Proposition 7 follows from the fact that joining any two non-adjacent vertices of a signed graph by a negative edge does not change the domination number.

Proposition 7 *Let S be any signed graph with a pair of non-adjacent vertices and S' be the signed graph obtained from S by joining a pair non-adjacent vertices of S by a negative edge, then $\gamma(S) = \gamma(S')$.*

Remark 8 *From Proposition 7 it follows that $\gamma(S) = \gamma(S^+)$.*

We use Ore's theorem to obtain a bound for domination number of certain class of signed graphs.

Theorem 9 (Ore's Theorem[16]) *If a graph G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

The following result gives a bound similar to the bound given in Ore's Theorem on the domination number of signed graphs S with positive $\delta^+(S)$.

Theorem 10 *If S is any signed graph of order n such that $\delta^+(S) > 0$, then $\gamma(S) \leq \frac{n}{2}$.*

Proof. Given that S is any signed graph with $\delta^+(S) > 0$, then $d^+(v) > 0$ for every vertex $v \in V$. This implies that the subgraph S^+ is a graph without any isolates. Therefore by Ore's Theorem $\gamma(S^+) \leq \frac{n}{2}$. Now by applying Remark 8, $\gamma(S) \leq \frac{n}{2}$. \square

The corona of two graphs G and H , denoted by $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H such that i^{th} vertex of G is adjacent to all the vertices of i^{th} copy of H . We refer the following theorem to obtain a characterisation of the signed graphs with order n having domination number $\frac{n}{2}$, when n is even.

Theorem 11 [11] *For any graph G with even order n and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$, where H is any connected graph.*

Analogously, we have the next result that follows from Theorem 11 and Remark 8.

Theorem 12 *For any signed graph S of even order n and $\delta^+(S) > 0$, $\gamma(S) = n/2$ if and only if the components of S^+ are the cycle C_4 or the corona $H \circ K_1$, where H is any connected graph.*

Haynes et al. [3] have defined a class of graphs \mathcal{G} in order to characterise the connected graphs G with $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ and the characterisation is as follows.

Theorem 13 [3] *A connected graph G have $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G \in \mathcal{G}$.*

In this direction, using Remark 8 and Theorem 13 we have the corresponding result for signed graphs S with the property that S^+ is connected.

Theorem 14 *For any signed graph S that contains a connected S^+ , $\gamma(S) = \lfloor \frac{n}{2} \rfloor$ if and only if $S^+ \in \mathcal{G}$.*

3 Main results

Now we examine the signed graphs with small and large values for domination number. First we obtain the characterisation for signed graphs S with $\gamma(S) = 1$. Whenever S is a signed graph with $\gamma(S) = 1$ and $\{v\}$ is a $\gamma(S)$ -set, then v is a vertex with $d^+(v) = n - 1$, where n is the order of S . This implies that $\Delta^+(S) = n - 1$. Conversely, if $\Delta^+(S) = n - 1$, then $\gamma(S) = 1$. Thus we have the following characterisation for signed graphs with domination number equal to 1.

Theorem 15 *For any signed graph S of order n , $\gamma(S) = 1$ if and only if $\Delta^+(S) = n - 1$.*

From the above theorem we can see that for any signed graph S of order n with $\gamma(S) > 1$, $\Delta^+(S) \leq n - 2$. The following theorem characterises signed graphs with domination number equal to 2.

Theorem 16 *Let S be any signed graph of order n . Then $\gamma(S) = 2$ if and only if $\Delta^+(S) \leq n - 2$ and there exists a pair of vertices u and v such that $V \setminus \{u, v\} \subseteq N^+(u) \cup N^+(v)$.*

Proof. Assume that $\gamma(S) = 2$ and let $D = \{u, v\}$ be a $\gamma(S)$ -set of S . By Theorem 15, $\Delta^+(S) \leq n - 2$. Since each vertex in $V \setminus D$ is adjacent to at least one vertex in D by a positive edge, $V \setminus D \subseteq N^+(u) \cup N^+(v)$.

Suppose that $\Delta^+(S) \leq n - 2$ and there exist a pair of vertices u and v such that $V \setminus \{u, v\} \subseteq N^+(u) \cup N^+(v)$. If $D = \{u, v\}$, then by our assumption for every vertex w in $V \setminus D$ there exists a vertex in $N^+(w) \cap D$. Therefore D is a dominating set of S . Since $\Delta^+(S) \leq n - 2$, $\gamma(S) > 1$ and hence $\gamma(S) = 2$. \square

Using the above two characterisations we now characterise the signed graphs with domination number equal to 3.

Theorem 17 *For any signed graph S of order n , $\gamma(S) = 3$ if and only if*

- (i) $\Delta^+(S) \leq n - 3$,
- (ii) *for no pair of vertices u and v , $V \setminus \{u, v\} \subseteq N^+(u) \cup N^+(v)$ and*
- (iii) *there exist three vertices u , v and w such that $V \setminus \{u, v, w\} \subseteq N^+(u) \cup N^+(v) \cup N^+(w)$.*

Proof. To prove the necessary part suppose that $\gamma(S) = 3$. By Theorem 15 we get $\Delta^+(S) < n - 1$. Now assume that $\Delta^+(S) = n - 2$ and let u be a vertex with $d^+(u) = n - 2$. Then the set $\{u, v\}$ is a dominating set of S , where $v \notin N^+(u)$. This contradicts the fact that $\gamma(S) = 3$. Therefore, $\Delta^+(S) \leq n - 3$. If there exists a pair of vertices u and v such that $V \setminus \{u, v\} \subseteq N^+(u) \cup N^+(v)$, then $\{u, v\}$ is a dominating set of S contradicting $\gamma(S) = 3$. Now, let $D = \{u, v, w\}$ be a $\gamma(S)$ -set. Then by the definition of a dominating set we find that $V \setminus \{u, v, w\} \subseteq N^+(u) \cup N^+(v) \cup N^+(w)$.

Conversely, assume that S is a signed graph satisfying the conditions (i), (ii) and (iii). Then by our assumption and Theorem 16, $\gamma(S) > 2$. Now observe that by the condition (iii) and definition of a dominating set, $\{u, v, w\}$ is a dominating set of S and hence $\gamma(S) = 3$. \square

As observed earlier the signed graphs with domination number equal to n are all-negative. Now we present a necessary and sufficient conditions for a signed graph S to have $\gamma(S) = n - 1$.

Theorem 18 *Let S be a signed graph of order n . Then $\gamma(S) = n - 1$ if and only if S has exactly one positive edge.*

Proof. Assume that S has exactly one positive edge. Then by Remark 8, $\gamma(S) = n - 1$.

Conversely, suppose that S is a signed graph with $\gamma(S) = n - 1$. Let D be a $\gamma(S)$ -set and $V \setminus D = \{v\}$. Then there exists a vertex u in D such that $vu \in E^+(S)$. We prove that v is incident with exactly one positive edge. On the contrary let there be another edge $vw \in E^+(S)$. Then the set $D \setminus \{u, w\} \cup \{v\}$ is a dominating set with cardinality less than that of D , which is a contradiction. Hence there exists no positive edge between the sets D and $V \setminus D$ except the edge uv . Now we claim that there is no positive edge between any two vertices x, y in D where $y \neq u$. Suppose that $xy \in E^+(S)$, then the set $D \setminus \{y\}$ is a dominating set having cardinality less than that of D , which is a contradiction. Therefore there are no positive edges between the vertices in D . Hence S has only one positive edge which is the edge uv . \square

Now we characterise the signed graphs having domination number equal to $n - 2$.

Theorem 19 *For any signed graph S , $\gamma(S) = n - 2$ if and only if the subgraph induced by $E^+(S)$ belongs to $\{2K_2, P_3, C_3, P_4, C_4\}$.*

Proof.

To prove the sufficiency, suppose that the graph induced by the positive edges belongs to the set $\{2K_2, P_3, C_3, P_4, C_4\}$. Then by using Remark 8 $\gamma(S) = \gamma(S^+) = n - 2$.

Conversely, suppose that S is a signed graph of order n and $\gamma(S) = n - 2$. Let D be a $\gamma(S)$ -set and $V \setminus D = \{p, q\}$. First we claim that p and q cannot be dominated by more than one vertex in D . If possible assume that the vertex p is dominated by two vertices u and v belonging to D . Then the set $D \setminus \{u, v\} \cup \{p\}$ is a dominating set with cardinality less than that of D , which is a contradiction. Similarly, we can prove that q cannot be dominated by more than one vertex in D . This shows that there are only two positive edges between the sets D and $V \setminus D$. Further, p and q are dominated by the same vertex or by two distinct vertices.

Case 1: The vertices p and q are dominated by same vertex r in D . In this case, the edges $pr, qr \in E^+(S)$. We claim that there is no positive edge between the vertices in D . On the contrary, suppose that there exists a positive edge between the vertices u and v in D , where $v \neq r$. Then observe that the set $D \setminus \{v\}$ is a dominating set having lesser cardinality than D , which is a contradiction. Now observe that there is no positive edge between the sets D and $V \setminus D$ other than pr and qr and between the vertices in D . Therefore the graph induced by the positive edges is either a P_3 or a C_3 .

Case 2: The vertices p and q are dominated by two distinct vertices r and t belonging to D , respectively. Then the edges pr and qt belongs to $E^+(S)$. Using similar arguments as in **Case 1** we conclude that there are no positive edges between any two vertices u and v of D , where $v \neq r, t$. Thus the graph induced by the positive edges is either $2K_2, P_4$ or C_4 .

Therefore the graph induced by $E^+(S)$ belongs to $\{2K_2, P_3, C_3, P_4, C_4\}$. \square

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Received: February 20, 2023 • Revised: March 10, 2023