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# Improvement of some results concerning starlikeness and convexity

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**Abstract.** We prove sharp versions of several inequalities dealing with univalent functions. We use differential subordination theory and Herglotz representations in our proofs.

## 1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane. Let $\mathcal{A}_n$ be the class of analytic functions of the form

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$$

which are defined in the unit disk $\mathbb{U}$, and let $\mathcal{A}_1 = \mathcal{A}$. Evidently $\mathcal{A}_{n+1} \subset \mathcal{A}_n$. The subclass of $\mathcal{A}$, consisting of functions $f$ for which the domain $f(\mathbb{U})$ is starlike with respect to 0, is denoted by $S^*$. It is well-known that $f \in S^* \Leftrightarrow \mathrm{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U}$. A function $f \in \mathcal{A}$ for which the domain $f(\mathbb{U})$ is convex, is called convex function. The class of convex functions is denoted by $K$. We have $f \in K \Leftrightarrow \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{U}$. Let $\mu \in [0,1)$. If for some function $f \in \mathcal{A}$ we have $\mathrm{Re}\frac{zf'(z)}{f(z)} > \mu,\ z \in U,\ \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \mu,\ z \in U,\ \left|\arg\frac{zf'(z)}{f(z)}\right| < \mu\frac{\pi}{2},\ z \in U$, then we say that the function $f$ is starlike of order $\mu$, convex of order $\mu$, and strongly starlike of order $\mu$, respectively. We introduce the notations:

$$\mathcal{V}[\lambda,\gamma;f](z) \equiv \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\gamma}, \tag{1}$$

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$$\mathcal{W}[\lambda,\gamma;f](z) \equiv \gamma \frac{\mathcal{F}(z)}{\mathcal{F}'(z)} \left( \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \right)', \tag{2}$$

where

$$\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad z \in \mathbb{U}, \ \gamma \in \mathbb{C}^*, \ \lambda \in [0,1], \ f \in \mathcal{A}_n.$$

The authors proved in a recent paper [3] the following results:

**Theorem 1** *If* $0 < \beta \leq 1, \quad f \in \mathcal{A}$, *then*

$$\left|\mathcal{W}[\lambda,\gamma;f](z)\right| < \beta, \ z \in \mathbb{U}, \Rightarrow \left| \arg \mathcal{V}[\lambda,\gamma;f](z)\right| < \frac{\pi}{2}\beta, \ z \in \mathbb{U}.$$

**Theorem 2** *If* $M \geq 1, \ z \in \mathbb{U}, \ n \in \mathbb{N}, \ f \in \mathcal{A}_n$, *then*

$$\mathcal{R}e\{\mathcal{W}[\lambda,\gamma;f](z)\} < \frac{nM}{1+nM}, \ z \in \mathbb{U} \Rightarrow \ \left|\mathcal{V}[\lambda,\gamma;f](z)\right| < M, \ z \in \mathbb{U}.$$

**Theorem 3** *If* $0 \leq \mu < 1$, $z \in \mathbb{U}$, $n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *then*

$$\mathcal{R}e\{\mathcal{W}[\lambda,\gamma;f](z)\} > \Delta_n(\mu), \ z \in \mathbb{U} \Rightarrow \ \mathcal{R}e\mathcal{V}[\lambda,\gamma;f](z) > \mu, \ z \in \mathbb{U},$$

*where*

$$\Delta(\mu) = \begin{cases} \frac{n\mu}{2(\mu-1)}, & \textit{if} \quad \nu \in [0, \frac{1}{2}] \\ \frac{n(\mu-1)}{2\mu}, & \textit{if} \quad \nu \in [\frac{1}{2}, 1). \end{cases}$$

The goal of this paper is to prove the sharp version of Theorem 1 and also the sharp version of Theorem 2 and Theorem 3 in case of $n = 1$. To do this we need some preliminary results which will be exposed in the following section.

## 2 Preliminaries

**Lemma 1** [2, p.24] *Let* $f$ *and* $g$ *be two analytic functions in* $\mathbb{U}$ *such that* $f(0) = g(0)$, *and* $g$ *is univalent. If* $f \not\prec g$ *then there are two points,* $z_0 \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U}$, *and a real number* $m \in [1,\infty)$ *such that:*

$$1. \quad f(z_0) = g(\zeta_0), \qquad 2. \quad z_0 f'(z_0) = m\zeta_0 g'(\zeta_0).$$

**Lemma 2** [1, p.27] *If* $p$ *is an analytic function in* $\mathbb{U}$, *with* $p(0) = 1$ *and* $\mathcal{R}ep(z) \geq 0, \ z \in \mathbb{U}$, *then there is a probability measure* $\nu$ *on the interval* $[0, 2\pi]$, *such that* $f(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\nu(t)$.

**Lemma 3** *If* $\alpha \in [1,2)$, *then the following inequality holds:* $\left|(1+z)^{\alpha}-1\right| \leq 2^{\alpha}-1, \quad z \in \mathbb{U}$.

**Proof.** According to the maximum modulus principle for analytic functions, we have to prove the inequality only in case if $z = e^{i\theta}, \ \theta \in [-\pi, \pi]$. The inequality $\left|(1+e^{i\theta})^{\alpha}-1\right| \leq 2^{\alpha}-1, \quad \theta \in [-\pi,\pi], \ \alpha \in [1,2)$ is equivalent to

$$2^{\alpha-1}\left(1-\cos^{2\alpha}\frac{\theta}{2}\right)-1+\cos^{\alpha}\frac{\theta}{2}\cos\frac{\alpha\theta}{2} \geq 0, \quad \theta \in [-\pi,\pi], \quad \alpha \in [1,2). \qquad (3)$$

Let $f : [-\pi,\pi] \to \mathbb{R}$ be the function defined by $f(\theta) = 2^{\alpha-1}\left(1-\cos^{2\alpha}\frac{\theta}{2}\right)-1+\cos^{\alpha}\frac{\theta}{2}\cos\frac{\alpha\theta}{2}$. We have: $f'(\theta) = \alpha\cos^{\alpha-1}\frac{\theta}{2}\sin\frac{\theta}{2}\left[2^{\alpha-1}\cos^{\alpha}\frac{\theta}{2}-\frac{1}{2}\frac{\sin\frac{(\alpha+1)\theta}{2}}{\sin\frac{\theta}{2}}\right]$.
Let $\alpha \in (1,2)$, and $\theta \in [0, \frac{2\pi}{\alpha+1})$ be two fixed real numbers. We define the functions $g_1, g_2 : [0,\alpha] \to \mathbb{R}$ by $\quad g_1(x) = \left(2\cos\frac{\theta}{2}\right)^{x}, \quad g_2(x) = \frac{\sin\frac{(x+1)\theta}{2}}{\sin\frac{\theta}{2}}$.
It is simple to prove that $g_1$ is a convex and $g_2$ is a concave function. Thus the graphs of the two functions have at most two common points . Since $g_1(0) = g_2(0)$ and $g_1(1) = g_2(1)$, it follows that the two graphs have exactly two common points, and $g_2(x) > g_1(x), \quad x \in (0,1)$, and $g_1(x) > g_2(x), \quad x \in (1,\alpha]$. Thus we have $g_1(\alpha) > g_2(\alpha)$ in case of $\alpha \in (1,2)$, and $\theta \in [0, \frac{2\pi}{\alpha+1})$. The inequality $g_1(\alpha) > g_2(\alpha)$ holds in case of $\alpha \in (1,2)$ and $\theta \in [\frac{2\pi}{\alpha+1}, \pi]$ too, because in this case we have: $g_1(\alpha) > 0 \geq g_2(\alpha)$. This means that the inequality $g_1(\alpha) > g_2(\alpha)$ holds for $\alpha \in (1,2)$ and $\theta \in [0,\pi]$. It is easily seen that the inequality $g_1(\alpha) > g_2(\alpha)$ can be extended to $\alpha \in (1,2)$ and $\theta \in [-\pi,\pi]$.
Consequently, $2^{\alpha-1}\cos^{\alpha}\frac{\theta}{2}-\frac{1}{2}\frac{\sin\frac{(\alpha+1)\theta}{2}}{\sin\frac{\theta}{2}} \geq 0, \quad (\forall)\ \alpha \in (1,2), \quad (\forall)\ \theta \in [-\pi,\pi]$;
$f'(\theta) < 0, \ \theta \in (-\pi, 0)$ and $f'(\theta) > 0, \ \theta \in (0,\pi)$.
Thus it follows that $\min_{\theta \in [-\pi,\pi]} f(\theta) = f(0) = 0$, and the inequality (3) is proved. □

# 3 Main result

The following theorem is the sharp version of Theorem 1.

**Theorem 4** *If* $0 < \beta \leq 1, \quad f \in \mathcal{A}$ *then we have:*

$$\left|\mathcal{W}[\lambda,\gamma;f](z)\right| < \beta, \ z \in \mathbb{U} \Rightarrow \left|\arg\mathcal{V}[\lambda,\gamma;f](z)\right| < \beta, \ z \in \mathbb{U}.$$

**Proof.** Let $p(z) = \mathcal{V}[\lambda, \gamma; f](z)$. We have $\frac{zp'(z)}{p(z)} = \mathcal{W}[\lambda, \gamma; f](z)$, and consequently $\left|\frac{zp'(z)}{p(z)}\right| < \beta, \ z \in \mathbb{U}$. This inequality is equivalent to

$$\frac{zp'(z)}{p(z)} \prec h(z) = \beta z, \ z \in \mathbb{U}. \tag{4}$$

We prove the subordination $p(z) \prec q(z) = e^{\beta z}, \ \ z \in \mathbb{U}$. If this subordination does not hold, then according to Lemma 1, there are two points $z_0 \in \mathbb{U}$, $\zeta_0 \in \partial\mathbb{U}$ and a real number $m \in [1, \infty)$, such that $p(z_0) = q(\zeta_0)$, and $z_0 p'(z_0) = m\zeta_0 q(\zeta_0)$. Thus $\frac{z_0 p'(z_0)}{p(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = mh(\zeta_0) \notin h(\mathbb{U})$.
This contradicts (4) and the contradiction implies $p(z) \prec q(z), \ \ z \in \mathbb{U}$. The proved subordination implies $\left|\arg p(z)\right| \le \max_{z\in\mathbb{U}}\{\arg(e^{\beta z})\} = \beta, \ \ z \in \mathbb{U}$, and the proof is done. □

We present in the followings the sharp version of Theorem 2 and Theorem 3 in case of $n = 1$.

**Theorem 5** *If* $M \ge 1, \ z \in \mathbb{U}, , \ f \in \mathcal{A}$, *then*

$$\mathcal{R}e\{\mathcal{W}[\lambda, \gamma; f](z)\} < \frac{M}{1+M}, \ z \in \mathbb{U} \Rightarrow \ \left|\mathcal{V}[\lambda, \gamma; f](z)\right| < 2^{\frac{2M}{M+1}} - 1, \ z \in \mathbb{U}.$$

**Proof.** The condition of the theorem can be rewritten in the following way $\mathcal{R}e\left\{1 - \frac{M+1}{M}\mathcal{W}[\lambda, \gamma; f](z)\right\} > 0, \ \ z \in \mathbb{U}$. The Herglotz formula implies that there is a probability measure $\nu$ on $[0, 2\pi]$ such that $1 - \frac{M+1}{M}\mathcal{W}[\lambda, \gamma; f](z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\nu(t)$. This is equivalent to $\mathcal{W}[\lambda, \gamma; f](z) = -\frac{M}{M+1}\int_0^{2\pi} \frac{2ze^{-it}}{1-ze^{-it}} d\nu(t)$.
On the other hand, if we denote $1 + p(z) = \mathcal{V}[\lambda, \gamma; f](z)$, we get $\frac{zp'(z)}{1+p(z)} = \mathcal{W}[\lambda, \gamma; f](z)$ and $\frac{p'(z)}{1+p(z)} = -\frac{M}{M+1}\int_0^{2\pi} \frac{2e^{-it}}{1-ze^{-it}} d\nu(t)$. This implies

$$\log(1 + p(z)) = \frac{2M}{M+1}\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\nu(t).$$

It is easily seen that $g(z) = \log(1+z) \in K$. Thus it follows $\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\nu(t) \in g(\mathbb{U}), \ \ \forall\, z \in \mathbb{U}$, and this leads to the subordination $\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\nu(t) \prec g(z), \ z \in \mathbb{U}$. Consequently we have $p(z) \prec \exp\left\{\frac{2M}{M+1}\log(1+z)\right\} - 1 = (1+z)^{\frac{2M}{M+1}} - 1, \ z \in \mathbb{U}$. This subordination implies

$$\left|p(z)\right| \le \max_{z\in\mathbb{U}}\left|(1+z)^{\frac{2M}{M+1}} - 1\right|, \ \ z \in \mathbb{U}.$$

Now from Lemma 3 we obtain the inequality $\left|p(z)\right| \le 2^{\frac{2M}{M+1}} - 1,\ \ z \in \mathbb{U}$. This inequality is equivalent to $\left|\mathcal{V}[\lambda,\gamma;f](z)\right| \le 2^{\frac{2M}{M+1}} - 1,\ \ z \in \mathbb{U}$. It is easy to show that if $M \ge 1$, then $2^{\frac{2M}{M+1}} - 1 \le M$, so the proved result is an improvement of Theorem 2 in case $n = 1$. Moreover the proof shows that this is the best possible result in this particular case. □

**Theorem 6** *Let* $0 \le \mu < 1$, $z \in \mathbb{U}$, $f \in \mathcal{A}$. *Then:*

$$\mathcal{R}e\{\mathcal{W}[\lambda,\gamma;f](z)\} > \Delta(\mu),\ z \in \mathbb{U} \Rightarrow\ \mathcal{R}e\mathcal{V}[\lambda,\gamma;f](z) > 2^{2\Delta(\mu)},\ z \in \mathbb{U},$$

*where*

$$\Delta(\mu) = \begin{cases} \frac{\mu}{2(\mu-1)}, & \text{if } \mu \in [0, \frac{1}{2}] \\ \frac{\mu-1}{2\mu}, & \text{if } \mu \in [\frac{1}{2}, 1). \end{cases}$$

**Proof.** We rewrite the condition $\mathcal{R}e\{\mathcal{W}[\lambda,\gamma;f](z)\} > \Delta(\mu),\ z \in \mathbb{U}$ in the following form: $\mathcal{R}e\frac{\Delta(\mu)-\mathcal{W}[\lambda,\gamma;f](z)}{\Delta(\mu)} > 0,\ z \in \mathbb{U}$. We use the Herglotz formula again and we get:

$$\frac{\Delta(\mu) - \mathcal{W}[\lambda,\gamma;f](z)}{\Delta(\mu)} = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\nu(t),$$

where $\nu$ is a probability measure on $[0, 2\pi]$. If we denote $p(z) = \mathcal{V}[\lambda,\gamma;f](z)$ then: $\frac{zp'(z)}{p(z)} = \mathcal{W}[\lambda,\gamma;f](z)$ and $\frac{p'(z)}{p(z)} = -\Delta(\mu)\int_0^{2\pi} \frac{2e^{-it}}{1-ze^{-it}} d\nu(t)$.
This leads to: $p(z) = \exp\left\{2\Delta(\mu)\int_0^{2\pi} \log(1-ze^{-it})d\nu(t)\right\}$.
Since $g(z) = \log(1+z) \in K$, it follows the inclusion: $\int_0^{2\pi} \log(1-ze^{-it})d\nu(t) \in g(\mathbb{U}),\ \ z \in \mathbb{U}$, and this implies the subordination:
$\int_0^{2\pi} \log(1-ze^{-it})d\nu(t) \prec g(z),\ z \in \mathbb{U}$. Thus we obtain: $p(z) \prec q(z) = (1+z)^{2\Delta(\mu)},\ z \in \mathbb{U}$, and consequently: $\mathcal{R}ep(z) \ge \mathcal{R}e(1+z)^{2\Delta(\mu)},\ z \in \mathbb{U}$. According to the definition of $\Delta(\mu)$, we have $-2\Delta(\mu) \in (0,1)$. This implies $q \in K$. The equivalency $f(z) \in \mathbb{R} \Leftrightarrow z \in \mathbb{R}$, and the fact that the domain $q(\mathbb{U})$ is convex and symmetric with respect to the real axis, imply the inequality: $\mathcal{R}eq(z) \ge \min\{q(-1), q(1)\} = 2^{2\Delta(\mu)},\ z \in \overline{\mathbb{U}}$. Thus it follows:

$$\mathcal{R}ep(z) \ge 2^{2\Delta(\mu)},\ \ z \in \overline{\mathbb{U}}.$$

It is easily seen that $2^{2\Delta(\mu)} \ge \mu$, for every $0 \le \mu < 1$, and $2^{2\Delta(\mu)}$ is the biggest value, for which the inequality

$$\mathcal{R}e\mathcal{V}[\lambda,\gamma;f](z) \ge 2^{2\Delta(\mu)},\ \ z \in \overline{\mathbb{U}}$$

holds. According to the minimum principle, inside the unit disk we have the strict inequality: $\mathcal{R}e\mathcal{V}[\lambda,\gamma;f](z) > 2^{2\Delta(\mu)}, \quad z \in \mathbb{U}.$ □

By choosing suitable values of the parameters, we obtain sharp results concerning starlikeness. Theorem 4 implies in case of $\gamma = 1,\ \lambda = 0$ the following result:

**Corollary 1** *If* $\beta \in (0,1]$, $f \in \mathcal{A}$, *then:*

$$\left|1 + z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right| < \beta, \quad z \in \mathbb{U} \Rightarrow \left|\arg \frac{zf'(z)}{f(z)}\right| < \beta.$$

*The result is sharp, the extremal function is:* $f(z) = z\exp\left(\int_0^z \frac{e^{\beta t}-1}{t}dt\right)$.

If we take $\gamma = 1,\ M = \alpha + 1,\ \lambda = 0$ then Theorem 2 implies:

**Corollary 2** *If* $\alpha \in [0,1)$, $f \in \mathcal{A}$, *then:*

$$\mathcal{R}e\left[z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right] < \frac{-1}{\alpha+2},\ z \in \mathbb{U} \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 2^{\frac{2\alpha+2}{\alpha+3}} - 1,\ z \in \mathbb{U},$$

*and the result is sharp. The extremal function is:*
$f(z) = z\exp\left(\int_0^z \frac{(1+t)^{\frac{2\alpha+2}{\alpha+3}}-1}{t}dt\right)$. *Since* $2^{\frac{2\alpha+2}{\alpha+3}} - 1 < 1$, *it follows that* $f$ *is a starlike function.*

Finally, for $\gamma = \lambda = 1$ Theorem 6 implies:

**Corollary 3** *If* $\mu \in [0,1)$, $f \in \mathcal{A}$, *then:*

$$\mathcal{R}e\left\{z\left[\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right]\right\} > \Delta(\mu) - 1 \Rightarrow \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 2^{2\Delta(\mu)},\ z \in \mathbb{U}.$$

*The result is sharp. The extremal function is:*

$$f(z) = \int_0^z \exp\left(\int_0^v \frac{(1+t)^{2\Delta(\mu)} - 1}{t}dt\right)dv.$$

*Since* $2^{2\Delta(\mu)} > \mu,\ \mu \in [0,1)$, *it follows that* $f$ *is a convex function of order* $\mu$.

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