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gr-n-ideals in graded commutative rings

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Abstract. Let G be a group with identity e and let R be a G-graded ring. In this paper, we introduce and study the concept of gr-n-ideals of R. We obtain many results concerning gr-n-ideals. Some characterizations of gr-n-ideals and their homogeneous components are given.

1 Introduction and preliminaries

Throughout this article, rings are assumed to be commutative with $1 \neq 0$. Let R be a ring, I be a proper ideal of R. By $\sqrt{I}$, we mean the radical of I which is $\{r \in R : r^n \in I$ for some positive integer $n\}$. In particular, $\sqrt{0}$ is the set of nilpotent elements in R. Recall from [11] that a proper ideal I of R is said to be an n-ideal if whenever $a, b \in R$ and $ab \in I$ with $a \notin \sqrt{0}$ implies $b \in I$. For $a \in R$, we define $Ann(a) = \{r \in R : ra = 0\}$.

The scope of this paper is devoted to the theory of graded commutative rings. One use of rings with gradings is in describing certain topics in algebraic

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geometry. Here, in particular, we are dealing with gr-n-ideals in a G-graded commutative ring.

First, we recall some basic properties of graded rings which will be used in the sequel. We refer to [6]-[8] for these basic properties and more information on graded rings.

Let G be a group with identity e. A ring R is called graded (or more precisely, G-graded) if there exists a family of subgroups $\{R_g\}$ of R such that $R = \oplus_{g\in G} R_g$ (as abelian groups) indexed by the elements $g \in G$, and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The summands R_g are called homogeneous components and elements of these summands are called homogeneous elements. If $a \in R$, then a can be written uniquely $a = \sum_{g\in G} a_g$ where a_g is the component of a in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \oplus_{g\in G}(I \cap R_g) := \oplus_{g\in G} I_g$. An ideal of a graded ring need not be graded.

If I is a graded ideal of R, then the quotient ring R/I is a G-graded ring. Indeed, $R/I = \bigoplus_{g\in G}(R/I)_g$ where $(R/I)_g = \{x + I : x \in R_g\}$. A G-graded ring R is called *a graded integral domain* (gr-*integral domain)* if whenever $r_g, s_h \in h(R)$ with $r_g s_h = 0$, then either $r_g = 0$ or $s_h = 0$.

The *graded radical* of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$, (see [10].)

Let R be a G-graded ring. A graded ideal I of R is said to be *a graded prime (gr-prime)* if $I \neq R$; and whenever $r_g, s_h \in h(R)$ with $r_g s_h \in I$, then either $r_g \in I$ or $s_h \in I$, (see [10].)

The concepts of graded primary ideals and graded weakly primary ideals of a graded ring have been introduced in [9] and [5], respectively. Let I be a proper graded ideal of a graded ring R. Then I is called *a graded primary (gr-primary) (resp. graded weakly primary) ideal* if whenever $r_g, s_h \in h(R)$ and $r_g s_h \in I$ (resp. $0 \neq r_g s_h \in I$), then either $r_g \in I$ or $s_h \in Gr(I)$.

Graded 2-absorbing and graded weakly 2-absorbing ideals of a commutative graded rings have been introduced in [2]. According to that paper, I is said to be *a graded 2-absorbing (resp. graded weakly 2-absorbing) ideal of* R if whenever $r_g, s_h, t_i \in h(R)$ with $r_g s_h t_i \in I$ (resp. $0 \neq r_g s_h t_i \in I$), then $r_g s_h \in I$ or $r_g t_i \in I$ or $s_h t_i \in I$.

Then the graded 2-absorbing primary and graded weakly 2-absorbing primary ideals defined and studied in [4]. A graded ideal I is said to be *a graded 2-absorbing primary (resp. graded weakly 2-absorbing primary) ideal of* R if whenever $r_g, s_h, t_i \in h(R)$ with $r_g s_h t_i \in I$ (resp. $0 \neq r_g s_h t_i \in I$), then $r_g s_h \in I$ or $r_g t_i \in Gr(I)$ or $s_h t_i \in Gr(I)$.

Recently, R. Abu-Dawwas and M. Bataineh in [1] introduced and studied the concepts of graded r-ideals of a commutative graded rings. A proper graded ideal I of R is said to be *a graded* r*-ideal (*gr*–*r*-ideal) of* R if whenever $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $Ann(a) = \{0\}$, then $s_h \in I$.

In this paper, we introduce the concept of graded n-ideals (gr-n-ideals) and investigate the basic properties and facts concerning gr-n-ideals.

2 Results

Definition 1 *Let* R *be a* G*-graded ring. A proper graded ideal* I *of* R *is called* a graded n-ideal of R *if whenever* $r_g, s_h \in h(R)$ *with* $r_g s_h \in I$ *and* $r_g \notin Gr(0)$, *then* $r_g \in I$. *In short, we call it a* gr-n*-ideal.*

Example 1 (i) *Suppose that* (R, M) *is a graded local ring with unique graded prime ideal. Then every graded ideal is a* gr-n*-ideal.*

(ii) *In any graded integral domain* D, *the graded zero ideal is a* gr-n*-ideal.*

(iii) *Any graded ring* R *need not have a* gr-n*-ideal. For instance, let* $G = \mathbb{Z}_2$, $R = \mathbb{Z}_6$ *be a* G*-graded ring with* $R_0 = \mathbb{Z}_6$ *and* $R_1 = \{0\}$. *Then* R *has not any* gr-n*-ideal.*

Lemma 1 *Let* R *be a* G*-graded ring and* I *be a graded ideal of* R. *If* I *is a* gr-n*-ideal of* R, *then* $I \subseteq Gr(0)$.

Proof. Assume that I is a gr-n-ideal and $I \nsubseteq Gr(0)$. Then there exists $r_g \in h(R) \cap I$ such that $r_g \notin Gr(0)$. Since $r_g 1 = r_g \in I$ and I is a gr-n-ideal, we get $1 \in I$, so $I = R$, a contradiction. Hence $I \subseteq Gr(0)$. □

Theorem 1 *Let* R *be a* G*-graded ring and* I *be a gr-prime ideal of* R. *Then* I *is a* gr-n*-ideal of* R *if and only if* $I = Gr(0)$.

Proof. Assume that I is a gr-prime ideal of R. It is easy to see $Gr(0) \subseteq Gr(I) = I$. If I is a gr-n-ideal of R, by Lemma 1, we have $I \subseteq Gr(0)$ and so $I = Gr(0)$. For the converse, assume that $I = Gr(0)$. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $r_g \notin Gr(0)$. Since I is a gr-prime ideal and $r_g \notin Gr(0) = I$, we get $s_h \in I$. □

Corollary 1 *Let* R *be a* G*-graded ring. Then* $Gr(0)$ *is a* gr-n*-ideal of* R *if and only if it is a* gr*-prime ideal of* R.

Proof. Assume that $Gr(0)$ is a gr-n-ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in Gr(0)$ and $r_g \notin Gr(0)$.Then $s_h \in Gr(0)$ as $Gr(0)$ is a gr-n-ideal of R. Hence $Gr(0)$ is a gr-prime ideal of R. Conversely, Assume that $Gr(0)$ is a gr-prime ideal of R, by Theorem 1, we conclude that $Gr(0)$ is a gr-n-ideal of R. □

The following theorem give us a characterization of gr-n-ideal of a graded rings.

Theorem 2 *Let* R *be a graded ring and* I *be a proper graded ideal of* R*. Then the following statements are equivalent:*

(i) I *is a* gr-n*-ideal of* R.

(ii) $I = (I :_R r_g)$ *for every* $r_g \in h(R) - Gr(0)$.

(iii) *For every graded ideals* J *and* K *of* R *such that* $JK \subseteq I$ *and* $J \cap (h(R) - Gr(0)) \neq \emptyset$ *implies* $K \subseteq I$.

Proof. (i) ⇒ (ii) Assume that I is a gr-n-ideal of R. Let $r_g \in h(R) - Gr(0)$. Clearly, $I \subseteq (I :_R r_g)$. Now, Let $s = \sum_{h \in G} s_h \in (I :_R r_g)$. This yields that $r_g s_h \in I$ for each $h \in G$. Since I is a gr-n-ideal of R and $r_g \in h(R) - Gr(0)$, we have $s_h \in I$ for each $h \in G$ and so $s \in I$. This implies that $(I :_R r_g) \subseteq I$. Therefore, $I = (I :_R r_g)$.

(ii) ⇒ (iii) Assume that $JK \subseteq I$ with $J \cap (h(R) - Gr(0)) \neq \emptyset$ for graded ideals J and K of R. Then there exists $r_g \in J \cap h(R)$ such that $r_g \notin Gr(0)$. Hence $r_g K \subseteq I$, it follows that $K \subseteq (I :_R r_g)$. By our assumption, we obtain $K \subseteq (I :_R r_g) = I$.

(iii) ⇒ (i) Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $r_g \notin Gr(0)$. Let $J = r_g R$ and $K = s_h R$ be two graded ideals of R generated by r_g and s_h, respectively. Then $JK \subseteq I$. By our assumption, we obtain, $K \subseteq I$ and so $s_h \in I$. Thus I is a gr-n-ideal of R. □

Theorem 3 *Let* R *be a* G*-graded ring and* $\{I_\alpha\}_{\alpha \in \Lambda}$ *be a non empty set of* gr-n*-ideals of* R*. Then* $\cap_{i \in \Delta} I_i$ *is* gr-n*-ideal of* R.

Proof. Clearly, $\cap_{\alpha \in \Lambda} I_\alpha$ is a graded ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in \cap_{\alpha \in \Lambda} I_\alpha$ and $r_g \notin Gr(0)$. Then $r_g s_h \in I_\alpha$ for every $\alpha \in \Lambda$. Since I_α is a gr-n-ideal of R, we have $s_h \in I_\alpha$ for every $\alpha \in \Lambda$ thus $s_h \in \cap_{\alpha \in \Lambda} I_\alpha$. □

Theorem 4 *Let* R *be a* G*-graded ring and* I *be a graded ideal of* R*. If* I *is a* gr-n*-ideal of* R*, then* I *is a* gr-r*-ideal of* R*.*

Proof. Assume that I is a gr-n-ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $ann(r_g) = 0$. Since $ann(r_g) = 0$, $r_g \notin Gr(0)$. Then $s_h \in I$ as I is a gr-n-ideal. Thus I is a gr-r-ideal of R. □

Remark 1 *It is easy to see that every graded nilpotent element is also a graded zero divisor. So graded zero divisors and graded nilpotent elements are equal in case* $< 0 >$ *is a graded primary ideal of* R*. Thus the* gr-n*-ideals and* gr-r*-ideals are equivalent in any graded commutative ring whose graded zero ideal is graded primary.*

Recall that a G-graded ring R is called a G-graded reduced ring if $r^2 = 0$ implies $r = 0$ for any $r \in h(R)$; i.e. $Gr(0) = 0$.

Theorem 5 *Let* R *be a* G*-graded ring. Then the following hold:*

(i) *Any* G*-graded reduced ring* R*, which is not graded integral domain, has no* gr-n*-ideal.*

(ii) *If* R *is a* G*-graded reduced ring, then* R *is a graded integral domain if and only if* 0 *is a* gr-n*-ideal.*

Proof. (i) Let R be a G-graded reduced ring such that R is not graded integral domain. Assume that there exists a gr-n-ideal I of R. Since R is a G-graded reduced ring, $Gr(0) = 0$. By Lemma 1, we get, $I \subseteq Gr(0) = 0$ and so $Gr(0) = 0 = I$. Since $Gr(0) = 0$ is not gr-prime ideal of R, by Corollary 1, we get $I = Gr(0)$ is not a gr-n-ideal, a contradiction.

(ii) Assume that R is a G-graded reduced ring. If R is a graded integral domain, then $Gr(0) = 0$ is a gr-prime ideal, and hence by Corollary 1, $0 = Gr(0)$ is a gr-n-ideal of R. For the converse if 0 is a gr-n-ideal of R, then by part (i) R is a graded integral domain. □

Theorem 6 *Let* R *be a* G*-graded ring,* I *be a* gr-n*-ideal of* R *and* $t_g \in h(R) - I$*. Then* $(I :_R t_g)$ *is a* gr-n*-ideal of* R*.*

Proof. By [9, Proposition 1.13], $(I :_R t_g)$ is a graded ideal. Since $t_g \notin I$, $(I :_R t_g) \neq R$. Now, let $r_h, s_\lambda \in h(R)$ such that $r_h s_\lambda \in (I :_R t_g)$ and $r_h \notin Gr((I :_R t_g))$. Then $r_h s_\lambda t_g \in I$. Since I is a gr-n-ideal of R and $r_h \notin Gr(0)$, we get $s_\lambda t_g \in I$. This yields that $s_\lambda \in (I :_R t_g)$. Therefore, $(I :_R t_g)$ is a gr-n-ideal of R. □

Theorem 7 *Let* R *be* G*-graded ring and* I *be a graded ideal of* R*. If* I *is a maximal* gr-n*-ideal of* R*, then* $I = Gr(0)$.

Proof. Assume that I is a maximal gr-n-ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $r_g \notin I$. Since I is a gr-n-ideal and $r_g \notin I$, by Theorem 6, we have $(I :_R r_g)$ is a gr-n-ideal. Thus $s_h \in (I :_R r_g) = I$ by maximality of I. This yields that I is a gr-prime ideal of R. By Theorem 1, we get $I = Gr(0)$. □

Lemma 2 *Let* R *be a* G*-graded ring and* $\{I_i : i \in \Lambda\}$ *be a directed collection of* gr-n*-ideals of* R*. Then* $I = \cup_{i\in\Lambda} I_i$ *is a* gr-n*-ideal of* R.

Proof. Suppose that $r_g s_h \in I$ and $r_g \notin Gr(0)$ for some $r_g, s_h \in h(R)$. Hence $r_g s_h \in I_k$ for some $k \in \Lambda$. Since I_k is a gr-n-ideal of R, we conclude that $s_h \in I_k \subseteq \cup_{i\in\Lambda} I_i = I$. Thus I is a gr-n-ideal. □

Theorem 8 *Let* R *be a* G*-graded ring. Then the following statements are equivalent:*

(i) $Gr(0)$ *is a* gr*-prime ideal of* R.

(ii) *There exists a* gr-n*-ideal of* R.

Proof. (i) ⇒ (ii) It is clear by Corollary 1.

(ii) ⇒ (i) First we show that R has a maximal gr-n-ideal. Let D be the set of all gr-n-ideals of R. Then by our assumption, $D \neq \emptyset$. Since D is a poset by the set inclusion, take a chain $I_1 \subseteq I_2 \subseteq \cdots$ in D. We conclude that the upper bound of this chain is $I = \cup_{i=1}^{\infty} I_i$ by Lemma 2. Then D has a maximal element which is a maximal gr-n-ideal. Thus that ideal is $Gr(0)$ by Corollary 1 and Theorem 7. □

In view of Lemma 1 and Theorem 8, we have the following result.

Theorem 9 *Let* R *be a* G*-graded ring and* I *a graded ideal of* R *such that* $I \subseteq Gr(0)$.

(i) I *is a* gr-n*-ideal if and only if* I *is a* gr*-primary ideal.*

(ii) *If* I *is a* gr-n*-ideal, then* I *is a graded weakly primary (so graded weakly 2-absorbing primary) and graded 2-absorbing primary ideal.*

(iii) *If* $Gr(0)$ *is* gr*-prime, then* I *is a graded weakly 2-absorbing primary ideal if and only if* I *is a graded 2-absorbing primary ideal of* R.

(iv) *If* R *has at least one* gr-n-*ideal, then* I *is a graded weakly 2-absorbing primary ideal if and only if* I *is a graded 2-absorbing primary ideal of* R.

Proof. Straightforward. □

Theorem 10 *Let* R *be a* G-*graded ring. Then* R *is a graded integral domain if and only if* 0 *is the only* gr-n-*ideal of* R.

Proof. Let R be a graded integral domain. Assume that I is a nonzero gr-n-ideal of R. Then we have $I \subseteq Gr(0) = 0$ by Lemma 1, a contradiction. Hence 0 is a gr-n-ideal by Example 1 (ii). Conversely, if 0 is the only gr-n-ideal, we get $Gr(0)$ is a gr-prime ideal and also a gr-n-ideal by Corollary 1 and Theorem 8. Hence $Gr(0) = 0$ is a gr-prime ideal. Thus R is a graded integral domain. □

Theorem 11 *Let* R *be a* G-*graded ring and* J *be a graded ideal of* R *with* $J \cap (h(R) - Gr(0)) \neq \emptyset$. *Then the following statements hold:*

(i) *If* I_1 *and* I_2 *are* gr-n-*ideals of* R *such that* $I_1J = I_2J$, *then* $I_1 = I_2$.

(ii) *If* IJ *is a* gr-n-*ideal of* R, *then* $IJ = I$.

Proof.
(i) Suppose that $I_1J = I_2J$. Since $I_2J \subseteq I_1$, $J \cap (h(R) - Gr(0)) \neq \emptyset$, and I_1 is a gr-n-ideal, by Theorem 2, we conclude that $I_2 \subseteq I_1$. Similarly, since I_2 is a gr-n-ideal, we have the inverse inclusion.
(ii) It is clear from (i). □

For G-graded rings R and R', a G-graded ring homomorphism $f : R \to R'$ is a ring homomorphism such that $f(R_g) \subseteq R'_g$ for every $g \in G$.

The following result studies the behavior of gr-n-ideals under graded homomorphism.

Theorem 12 *Let* R_1 *and* R_2 *be two* G-*graded rings and* $f : R_1 \to R_2$ *a graded ring homomorphism. Then the following statements hold:*

(i) *If* f *is a graded epimorphism and* I_1 *is a* gr-n-*ideal of* R_1 *containing* $\ker f$, *then* $f(I_1)$ *is a* gr-n-*ideal of* R_2.

(ii) *If* f *is a graded monomorphism and* I_2 *is a* gr-n-*ideal of* R_2, *then* $f^{-1}(I_2)$ *is a* gr-n-*ideal of* R_1.

Proof. (i) Suppose that $r_g s_h \in f(I_1)$ and $r_g \notin Gr(0_{R_2})$ for some $r_g, s_h \in h(R_2)$. Since f is onto, $f(x_g) = r_g$, $f(y_h) = s_h$ for some $x_g, y_h \in h(R_1)$. Hence $f(x_g y_h) \in f(I_1)$ implies that $x_g y_h \in I_1$ as $\mathrm{Ker} f \subseteq I_1$. It is clear that $x_g \notin Gr(0_{R_1})$. Since I_1 is a gr-n-ideal of R_1, we conclude that $y_h \in I_1$; and so $s_h = f(y_h) \in f(I_1)$. Thus $f(I_1)$ is a gr-n-ideal of R_2.

(ii) Suppose that $r_g s_h \in f^{-1}(I_2)$ and $r_g \notin Gr(0_{R_1})$ for some $r_g, s_h \in h(R_1)$. Since $\ker f = \{0\}$, we have $f(r_g) \notin Gr(0_{R_2})$. Since $f(r_g s_h) = f(r_g) f(s_h) \in I_2$ and I_2 is a gr-n-ideal of R_2, we conclude that $f(s_h) \in I_2$. It means $s_h \in f^{-1}(I_2)$, we are done. □

Corollary 2 *Let I_1 and I_2 be two graded ideals of a G-graded ring R with $I_1 \subseteq I_2$. Then the following statements hold:*

(i) *If I_2 is a gr-n-ideal of R, then I_2/I_1 is a gr-n-ideal of R/I_1.*

(ii) *If I_2/I_1 is a gr-n-ideal of R/I_1 and $I_1 \subseteq Gr(0)$, then I_2 is a gr-n-ideal of R.*

(iii) *If I_2/I_1 is a gr-n-ideal of R/I_1 and I_1 is a gr-n-ideal of R, then I_2 is a gr-n-ideal of R.*

Proof. (i) Considering the natural graded epimorphism $\Pi : R \to R/I_1$, the result is clear by Theorem 12.

(ii) Suppose that $r_g s_h \in I_2$ and $r_g \notin Gr(0)$ for some $r_g, s_h \in h(R)$. Hence $(r_g + I_1)(s_h + I_1) = r_g s_h + I_1 \in I_2/I_1$ and $r_g \notin Gr(0_{R/I_1})$. It implies that $s_h + I_2 \in I_1/I_2$. Thus $s_h \in I_1$, we are done.

(iii) Let I_2/I_1 be a gr-n-ideal of R/I_1 and I_1 a gr-n-ideal of R. Assume that I_2 is not gr-n-ideal. Then $I_1 \nsubseteq Gr(0)$ by (ii). From Lemma 1, we conclude that I_1 is not a gr-n-ideal, a contradiction. Thus I_2 is a gr-n-ideal of R. □

Corollary 3 *Let R be a G-graded ring, I be a gr-n-ideal of R and S a subring of R with $S \nsubseteq I$. Then $I \cap S$ is a gr-n-ideal of S.*

Proof. Consider the injection $i : S \to R$. Then i is a graded homomorphism. Since I is a gr-n-ideal of R, $i^{-1}(I) = I \cap S$ is a gr-n-ideal of S by Theorem 12 (ii). □

Let R be a G-graded ring and $S \subseteq h(R)$ a multiplicatively closed subset of R. Then graded ring of fractions is denoted by $S^{-1}R$ which defined by $S^{-1}R = \oplus_{g \in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{\frac{a}{s} : a \in R,\ s \in S,\ g = (\deg s)^{-1}(\deg a)\}$. A homogeneous element $r_g \in h(R)$ is said to be gr-regular if $ann(r_g) = 0$.

Observe that the set of all gr-regular elements of R is a multiplicatively closed subset of R.

The following result studies the behaviour of gr-$\mathfrak{n}$-ideal under localization.

Theorem 13 *Let* R *be a* G*-graded ring,* $\mathsf{S} \subseteq \mathsf{h}(\mathsf{R})$ *a multiplicatively closed subset of* R*. Then the following statements hold:*

(i) *If* I *is a* gr-$\mathfrak{n}$*-ideal of* R*, then* $\mathsf{S}^{-1}I$ *is a* gr-$\mathfrak{n}$*-ideal of* $\mathsf{S}^{-1}\mathsf{R}$.

(ii) *Let* S *be the set of all* gr*-regular elements of* R*. If* J *is a* gr-$\mathfrak{n}$*-ideal of* $\mathsf{S}^{-1}\mathsf{R}$*, then* J^c *is a* gr-$\mathfrak{n}$*-ideal of* R.

Proof. (i) Suppose that $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ with $\frac{a}{s} \notin Gr(0_{S^{-1}R})$ for some $\frac{a}{s}, \frac{b}{t} \in h(S^{-1}R)$. Hence there exists $u \in h(S)$ such that $uab \in I$. Clearly, we have $a \notin Gr(0)$. It implies that $ub \in I$; so $\frac{b}{t} = \frac{ub}{ut} \in S^{-1}I$. Thus $S^{-1}I$ is a gr-$\mathfrak{n}$-ideal of $S^{-1}R$.

(ii) Suppose that $a, b \in h(R)$ with $ab \in J^c$ and $b \notin J^c$. Then $\frac{b}{1} \notin J$. Since J is a gr-$\mathfrak{n}$-ideal, we have $\frac{a}{1} \in Gr(0_{S^{-1}R})$. Hence $ua^k = 0$ for some $u \in S$ and $k \geq 1$. Since u is gr-regular, $a^k = 0$; i.e. $a \in Gr(0)$. Thus J^c is a gr-$\mathfrak{n}$-ideal of R. □

Definition 2 *Let* S *be a nonempty subset of a* G*-graded ring* R *with* $h(R) - Gr(0) \subseteq S \subseteq h(R)$*. Then we call* S gr-$\mathfrak{n}$*-multiplicatively closed subset of* R *if whenever* $r_g \in h(R) - Gr(0)$ *and* $s_h \in S$*, then* $r_g s_h \in S$.

Theorem 14 *Let* I *be a graded ideal of a* G*-graded ring* R*. Then the following statements are equivalent:*

(i) I *is a* gr-$\mathfrak{n}$*-ideal of* R.

(ii) $h(R) - I$ *is a* gr-$\mathfrak{n}$*-multiplicatively closed subset of* R.

Proof. (i) ⇒ (ii) Let I be a gr-$\mathfrak{n}$-ideal of R. Suppose that $r_g \in h(R) - Gr(0)$ and $s_h \in h(R) - I$. Since $r_g \notin Gr(0)$, $s_h \notin I$, and I is a gr-$\mathfrak{n}$-ideal of R, we conclude that $r_g s_h \notin I$. Therefore $r_g s_h \in h(R) - I$. Since I is a gr-$\mathfrak{n}$-ideal of R, we have $I \subseteq Gr(0)$ by Lemma 1. Then $h(R) - Gr(0) \subseteq h(R) - I$.

(ii) ⇒ (i) Suppose that $r_g, s_h \in h(R)$ with $r_g s_h \in I$ and $r_g \notin Gr(0)$. If $s_h \in h(R) - I$, then from our assumption (ii), we have $r_g s_h \in h(R) - I$, a contradiction. Thus $s_h \in I$ which means that I is a gr-$\mathfrak{n}$-ideal of R. □

Theorem 15 *Let* I *be a graded ideal of a* G*-graded ring* R *and* S *a* gr-$\mathfrak{n}$*-multiplicatively closed subset of* R *with* $I \cap S = \emptyset$*. Then there exists a* gr-$\mathfrak{n}$*-ideal* K *of* R *such that* $I \subseteq K$ *and* $K \cap S = \emptyset$.

Proof. Let $D = \{J : J \text{ is a graded ideal of } R \text{ with } I \subseteq J \text{ and } J \cap S = \emptyset\}$. Observe that $D \neq \emptyset$ as $I \in D$. Suppose $J_1 \subseteq J_2 \subseteq \cdots$ is a chain in D. Then $\cup_{i=1}^{\infty} J_i$ is a gr-n-ideal of R by Lemma 2. Since $I \subseteq \cup_{i=1}^{\infty} J_i$ and $(\cup_{i=1}^{\infty} J_i) \cap S = \cup_{i=1}^{\infty}(J_i \cap S) = \emptyset$, we get $\cup_{i=1}^{\infty} J_i$ is the upper bound of this chain. From Zorn's Lemma, there is a maximal element K of D. We show that this maximal element K is a gr-n-ideal of R. Suppose that $r_g s_h \in K$ and $s_h \notin K$ for some $r_g, s_h \in h(R)$. Then $K \subsetneq (K :_R r_g)$. Since K is maximal, it implies that $(K :_R r_g) \cap S \neq \emptyset$. Hence there is an element $t_\lambda \in (K :_R r_g) \cap S$. Then $r_g t_\lambda \in K$. If $r_g \in Gr(0)$, then we are done. So assume that $r_g \notin Gr(0)$. Since S is gr-n-multiplicatively closed, we conclude that $r_g t_\lambda \in S$. Thus $r_g t_\lambda \in S \cap K$, a contradiction. Therefore K is a gr-n-ideal of R. □

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