Acta Univ. Sapientiae, Informatica 11, 2 (2019) 206–214

DOI: 10.2478/ausi-2019-0014

Heuristic method to determine lucky k-polynomials for k-colorable graphs

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Abstract. The existence of edges is a huge challenge with regards to determining lucky k-polynomials of simple connected graphs in general. In this paper the lucky 3-polynomials of path and cycle graphs of order, $3 \leq n \leq 8$ are presented as the basis for the heuristic method to determine the lucky k-polynomials for k-colorable graphs. The difficulty of adjacency with graphs is illustrated through these elementary graph structures . The results are also illustratively compared with the results for null graphs (edgeless graphs). The paper could serve as a basis for finding recurrence results through innovative methodology.

1 Introduction

For general notation and concepts in graphs see, [1, 2, 6]. It is assumed that the reader is familiar with the concept of graph coloring. Recall that in a proper coloring of G all edges are good i.e. $uv \Leftrightarrow c(u) \neq c(v)$. For any proper coloring $\varphi(G)$ of a graph G the addition of all good edges, if any, is called the chromatic completion of G in respect of $\varphi(G)$. The additional edges are called *chromatic completion edges*. The set of such chromatic completion edges is denoted by, $E_\varphi(G)$. The resultant graph G_φ is called a *chromatic completion graph* of G. See [3] for an introduction to chromatic completion of a graph.

Computing Classification System 1998: G.2.2
Mathematics Subject Classification 2010: 05C15, 05C38, 05C75, 05C85
Key words and phrases: chromatic completion, perfect lucky 3-coloring, lucky 3-polynomial

The *chromatic completion number* of a graph G denoted by, $\zeta(G)$ is the maximum number of good edges that can be added to G over all chromatic colorings (χ-colorings). Hence, $\zeta(G) = max\{|E_\chi(G)| : over\ all\ \varphi_\chi(G)\}$.

A χ-coloring which yields $\zeta(G)$ is called a *lucky* χ*-coloring* or simply, a lucky coloring[1] and is denoted by, $\varphi_{\mathcal{L}}(G)$. The resultant graph G_ζ is called a *minimal chromatic completion graph* of G. It is trivially true that $G \subseteq G_\zeta$. Furthermore, the graph induced by the set of completion edges, $\langle E_\chi \rangle$ is a subgraph of the complement graph, $\overline{G}$. See [4] for the notion of stability in respect of chromatic completion.

A k-coloring of a graph G which yields $max\{|E_\varphi(G)| :$ overall k-colorings$\}$ is called a lucky k-coloring.[2]

In an improper coloring an edge uv for which, $c(u) = c(v)$ is called a *bad edge*. See [5] for an introduction to defect colorings of graphs. It is observed that the number of edges of $\overline{G}$ which are omitted from E_χ is the minimum number of bad edges in a *bad chromatic completion* of a graph G.

2 Lucky 3-polynomials of paths

A path graph (or simply, a path) denoted by, P_n, is a graph on $n \geq 1$ vertices say, $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and n edges namely, $E(P_n) = \{v_i v_{i+1} : i = 1, 2, 3, \ldots, n-1\}$.

Recall that for λ distinct colors, $\lambda \geq \chi(G)$, the number of ways a graph G can be assigned a proper coloring is given by the chromatic polynomial of G and is denoted by, $\mathcal{P}_G(\lambda, n)$. For λ distinct colors, $\lambda \geq 3$, the path P_3 can be assigned a proper 3-coloring in $\mathcal{P}_{P_3}(\lambda, n) = \lambda(\lambda-1)(\lambda-2)$ ways. The aforesaid is equal to the number of ways a perfect lucky 3-coloring can be assigned to the path P_3 in accordance with lucky's theorem [3]. Since [3] has not been published as yet we recall lucky's theorem for perfect lucky k-coloring to be:

Theorem 1 [3] *For a positive integer* $n \geq 2$ *and* $2 \leq p \leq n$ *let integers,* $1 \leq a_1, a_2, a_3, \ldots, a_{p-r}, a'_1, a'_2, a'_3, \ldots, a'_r \leq n-1$ *be such that* $n = \sum\limits_{i=1}^{p-r} a_i + \sum\limits_{j=1}^{r} a'_j$ *then, the* ℓ*-completion sum-product* $\mathcal{L} = max\{\sum\limits_{i=1}^{p-r-1}\prod\limits_{k=i+1}^{p-r} a_i a_k + \sum\limits_{i=1}^{p-r}\prod\limits_{j=1}^{r} a_i a'_j + \sum\limits_{j=1}^{r-1}\prod\limits_{k=j+1}^{r} a'_j a'_k\}$ *over all possible,* $n = \sum\limits_{i=1}^{p-r} a_i + \sum\limits_{j=1}^{r} a'_j$.

[1]Note that for many graphs a lucky coloring is equivalent an equitable χ-coloring.

[2]Note that for many graphs a lucky k-coloring is equivalent an equitable k-coloring.

Note that lucky's theorem is reliant on the notion of the ℓ-completion sum-product [3]. We recall the definition to be:

Definition 2 *Let,* $t_i = \lfloor \frac{n}{\ell} \rfloor$, $i = 1, 2, 3, \ldots, (\ell - r)$ *and* $t'_j = \lceil \frac{n}{\ell} \rceil$, $j = 1, 2, 3, \ldots, r$. *Call,* $\mathcal{L} = \sum\limits_{i=1}^{\ell-r-1} \prod\limits_{k=i+1}^{\ell-r} t_i t_k + \sum\limits_{i=1}^{\ell-r} \prod\limits_{j=1}^{r} t_i t'_j + \sum\limits_{j=1}^{r-1} \prod\limits_{k=j+1}^{r} t'_j t'_k$, *the* ℓ-completion sum-product *of* n.

Also, because of the simplicity of the graph structure of paths no figure illustrations are deemed necessary for clarity. It is assumed that the reader can easily verify the vertex set partitions obtained. For path P_3 the lucky 3-polynomial is expressed as, $\mathcal{L}_{P_3}(\lambda, 3) = \lambda(\lambda - 1)(\lambda - 2)$. Note that the lucky 3-polynomial corresponds to coloring the vertex set partition, $\{\{v_1\}, \{v_2\}, \{v_3\}\}$.

Consider the path P_4. By the definition of a path a particular convention is implicit i.e. to obtain P_n from P_{n-1} we necessarily extend from v_{n-1} to v_n with the edge $v_{n-1}v_n$. Hence, it is not permissible to insert the vertex v_4 into an existing edge of P_3. The permissible lucky partitions for a lucky 3-coloring are, $\{\{v_1, v_4\}, \{v_2\}, \{v_3\}\}$, $\{\{v_1\}, \{v_2, v_4\}, \{v_3\}\}$, $\{\{v_1, v_3\}, \{v_2\}, \{v_4\}\}$. Hence, $\mathcal{L}_{P_4}(\lambda, 3) = 3\lambda(\lambda - 1)(\lambda - 2)$. Progressing to path P_5 the permissible lucky partitions for a lucky 3-coloring are found to be,

$\{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\}$, $\{\{v_1, v_4\}, \{v_2\}, \{v_3, v_5\}\}$, $\{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}$,
$\{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}$, $\{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}\}$, $\{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}$.

Hence, $\mathcal{L}_{P_5}(\lambda, 3) = 6\lambda(\lambda - 1)(\lambda - 2)$.

Progressing to path P_6 the permissible lucky partitions for a lucky 3-coloring are found to be,

$\{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6\}\}$, $\{\{v_1, v_4\}, \{v_2, v_6\}, \{v_3, v_5\}\}$, $\{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3, v_6\}\}$,
$\{\{v_1, v_6\}, \{v_2, v_4\}, \{v_3, v_5\}\}$, $\{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4, v_6\}\}$.

Therefore, $\mathcal{L}_{P_6}(\lambda, 3) = 5\lambda(\lambda - 1)(\lambda - 2)$. Note that $\mathcal{L}_{P_6}(\lambda, 3) < \mathcal{L}_{P_5}(\lambda, 3)$.

Progressing to path P_7 the permissible lucky partitions for a lucky 3-coloring are found to be,

$\{\{v_1, v_4, v_7\}, \{v_2, v_5\}, \{v_3, v_6\}\}$, $\{\{v_1, v_4\}, \{v_2, v_5, v_7\}, \{v_3, v_6\}\}$,
$\{\{v_1, v_4, v_7\}, \{v_2, v_6\}, \{v_3, v_5\}\}$, $\{\{v_1, v_4\}, \{v_2, v_6\}, \{v_3, v_5, v_7\}\}$,
$\{\{v_1, v_5, v_7\}, \{v_2, v_4\}, \{v_3, v_6\}\}$, $\{\{v_1, v_5\}, \{v_2, v_4, v_7\}, \{v_3, v_6\}\}$,

$\{\{v_1, v_6\}, \{v_2, v_4, v_7\}, \{v_3, v_5\}\}$, $\{\{v_1, v_6\}, \{v_2, v_4\}, \{v_3, v_5, v_7\}\}$,
$\{\{v_1, v_3, v_7\}, \{v_2, v_5\}, \{v_4, v_6\}\}$, $\{\{v_1, v_3\}, \{v_2, v_5, v_7\}, \{v_4, v_6\}\}$,
$\{\{v_1, v_4, v_6\}, \{v_2, v_5\}, \{v_3, v_7\}\}$, $\{\{v_1, v_4, v_6\}, \{v_2, v_7\}, \{v_3, v_5\}\}$,
$\{\{v_1, v_5\}, \{v_2, v_4, v_6\}, \{v_3, v_7\}\}$, $\{\{v_1, v_3, v_6\}, \{v_2, v_4\}, \{v_5, v_7\}\}$,
$\{\{v_1, v_3, v_5\}, \{v_2, v_7\}, \{v_4, v_6\}\}$, $\{\{v_1, v_3, v_6\}, \{v_2, v_5\}, \{v_4, v_7\}\}$.

Therefore, $\mathcal{L}_{P_7}(\lambda, 3) = 16\lambda(\lambda - 1)(\lambda - 2)$.

For path P_8 the permissible lucky partitions for a lucky 3-coloring are found to be,

$\{\{v_1, v_4, v_7\}, \{v_2, v_5, v_8\}, \{v_3, v_6\}\}$, $\{\{v_1, v_4, v_7\}, \{v_2, v_5\}, \{v_3, v_6, v_8\}\}$,
$\{\{v_1, v_4, v_8\}, \{v_2, v_5, v_7\}, \{v_3, v_6\}\}$, $\{\{v_1, v_4\}, \{v_2, v_5, v_7\}, \{v_3, v_6, v_8\}\}$,
$\{\{v_1, v_4, v_7\}, \{v_2, v_6, v_8\}, \{v_3, v_5\}\}$, $\{\{v_1, v_4, v_7\}, \{v_2, v_6\}, \{v_3, v_5, v_8\}\}$,
$\{\{v_1, v_4, v_8\}, \{v_2, v_6\}, \{v_3, v_5, v_7\}\}$, $\{\{v_1, v_4\}, \{v_2, v_6, v_8\}, \{v_3, v_5, v_7\}\}$,
$\{\{v_1, v_5, v_7\}, \{v_2, v_4, v_8\}, \{v_3, v_6\}\}$, $\{\{v_1, v_5, v_7\}, \{v_2, v_4\}, \{v_3, v_6, v_8\}\}$,
$\{\{v_1, v_5, v_8\}, \{v_2, v_4, v_7\}, \{v_3, v_6\}\}$, $\{\{v_1, v_5\}, \{v_2, v_4, v_7\}, \{v_3, v_6, v_8\}\}$,
$\{\{v_1, v_6, v_8\}, \{v_2, v_4, v_7\}, \{v_3, v_5\}\}$, $\{\{v_1, v_6\}, \{v_2, v_4, v_7\}, \{v_3, v_5, v_8\}\}$,
$\{\{v_1, v_6, v_8\}, \{v_2, v_4\}, \{v_3, v_5, v_7\}\}$, $\{\{v_1, v_6\}, \{v_2, v_4, v_8\}, \{v_3, v_5, v_7\}\}$,
$\{\{v_1, v_3, v_7\}, \{v_2, v_5, v_8\}, \{v_4, v_6\}\}$, $\{\{v_1, v_3, v_7\}, \{v_2, v_5\}, \{v_4, v_6, v_8\}\}$,
$\{\{v_1, v_3, v_7\}, \{v_2, v_4, v_6\}, \{v_5, v_8\}\}$, $\{\{v_1, v_3, v_8\}, \{v_2, v_5, v_7\}, \{v_4, v_6\}\}$,
$\{\{v_1, v_3\}, \{v_2, v_5, v_7\}, \{v_4, v_6, v_8\}\}$, $\{\{v_1, v_4, v_6\}, \{v_2, v_5, v_8\}, \{v_3, v_7\}\}$,
$\{\{v_1, v_4, v_6\}, \{v_2, v_7\}, \{v_3, v_5, v_8\}\}$, $\{\{v_1, v_5, v_8\}, \{v_2, v_4, v_6\}, \{v_3, v_7\}\}$,
$\{\{v_1, v_4, v_6\}, \{v_2, v_7\}, \{v_3, v_5, v_8\}\}$, $\{\{v_1, v_3, v_6\}, \{v_2, v_4, v_8\}, \{v_5, v_7\}\}$,
$\{\{v_1, v_3, v_5\}, \{v_2, v_7\}, \{v_4, v_6, v_8\}\}$, $\{\{v_1, v_3, v_6\}, \{v_2, v_5, v_8\}, \{v_4, v_7\}\}$,
$\{\{v_1, v_4, v_6\}, \{v_2, v_5, v_7\}, \{v_3, v_8\}\}$, $\{\{v_1, v_4, v_6\}, \{v_2, v_8\}, \{v_3, v_5, v_7\}\}$,
$\{\{v_1, v_5, v_7\}, \{v_2, v_4, v_6\}, \{v_3, v_8\}\}$, $\{\{v_1, v_3, v_5\}, \{v_2, v_4, v_7\}, \{v_6, v_8\}\}$,
$\{\{v_1, v_3, v_6\}, \{v_2, v_4, v_7\}, \{v_5, v_8\}\}$, $\{\{v_1, v_4, v_6\}, \{v_2, v_8\}, \{v_3, v_5, v_7\}\}$,
$\{\{v_1, v_3, v_6\}, \{v_2, v_5, v_7\}, \{v_4, v_8\}\}$, $\{\{v_1, v_4, v_6\}, \{v_2, v_5, v_7\}, \{v_3, v_8\}\}$,
$\{\{v_1, v_5, v_7\}, \{v_2, v_4, v_6\}, \{v_3, v_8\}\}$, $\{\{v_1, v_5, v_7\}, \{v_2, v_4, v_6\}, \{v_3, v_8\}\}$,
$\{\{v_1, v_3, v_6\}, \{v_2, v_4, v_7\}, \{v_5, v_8\}\}$, $\{\{v_1, v_3, v_5\}, \{v_2, v_4, v_7\}, \{v_6, v_8\}\}$,
$\{\{v_1, v_3, v_6\}, \{v_2, v_5, v_7\}, \{v_4, v_8\}\}$.

Therefore, $\mathcal{L}_{P_8}(\lambda, 3) = 41\lambda(\lambda - 1)(\lambda - 2)$.

A cycle graph (or simply, a cycle) denoted by, C_n, is a graph on $n \geq 1$ vertices say, $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and n edges namely, $E(C_n) = \{v_i v_{i+1} : i = 1, 2, 3, \ldots, n-1\} \cup \{v_n v_1\}$. The graph structural difference between P_n and C_n is the edge $v_n v_1$. It implies that to obtain the corresponding lucky

3-polynomial, the permissible lucky partitions for $V(C_n)$ are those obtained after eliminating those lucky partitions of $V(P_n)$ with vertex subsets which have both v_1, v_n as elements. The next results follows easily without further proof.

Corollary 3 *(i)* $\mathcal{L}_{C_3}(\lambda,3) = \lambda(\lambda-1)(\lambda-2)$,
(ii) $\mathcal{L}_{C_4}(\lambda,3) = 2\lambda(\lambda-1)(\lambda-2)$,
(iii) $\mathcal{L}_{C_5}(\lambda,3) = 5\lambda(\lambda-1)(\lambda-2)$,
(iv) $\mathcal{L}_{C_6}(\lambda,3) = 4\lambda(\lambda-1)(\lambda-2)$,
(v) $\mathcal{L}_{C_7}(\lambda,3) = 13\lambda(\lambda-1)(\lambda-2)$,
(vi) $\mathcal{L}_{C_8}(\lambda,3) = 34\lambda(\lambda-1)(\lambda-2)$.

Recall that a null graph, $\mathfrak{N}_n$ of order n is simply an edgeless graph with vertex set, $\{v_i : 1 \leq i \leq n\}$. Constructing a path is considered to be the simplest way to add edges to a null graph to obtain a connected simple graph with minimum maximum degree, minimum number of edges and the property of symmetry. However, to find either a closed or recurrence relation between the lucky k-polynomials of null graphs and paths and cycles remains open. The table below depicts the lucky 3-polynomials for the three families of graphs for order 3 to 8.

n	$\mathfrak{N}_n$,	P_n	C_n
3	$\lambda(\lambda-1)(\lambda-2)$	$\lambda(\lambda-1)(\lambda-2)$	$\lambda(\lambda-1)(\lambda-2)$
4	$6\lambda(\lambda-1)(\lambda-2)$	$3\lambda(\lambda-1)(\lambda-2)$	$2\lambda(\lambda-1)(\lambda-2)$
5	$15\lambda(\lambda-1)(\lambda-2)$	$6\lambda(\lambda-1)(\lambda-2)$	$5\lambda(\lambda-1)(\lambda-2)$
6	$15\lambda(\lambda-1)(\lambda-2)$	$5\lambda(\lambda-1)(\lambda-2)$	$4\lambda(\lambda-1)(\lambda-2)$
7	$51\lambda(\lambda-1)(\lambda-2)$	$16\lambda(\lambda-1)(\lambda-2)$	$13\lambda(\lambda-1)(\lambda-2)$
8	$109\lambda(\lambda-1)(\lambda-2)$	$41\lambda(\lambda-1)(\lambda-2)$	$34\lambda(\lambda-1)(\lambda-2)$

Table 1.

We recall from [3] that a graph G is perfect lucky k-colorable if and only if the graph is k-colorable in accordance with lucky's theorem hence, in accordance with the lucky partition form,

$$\{\underbrace{\{\lfloor \tfrac{n}{k} \rfloor\text{-element}\}, \{\lfloor \tfrac{n}{k} \rfloor\text{-element}\}, \ldots, \{\lfloor \tfrac{n}{k} \rfloor\text{-element}\}}_{(k-r)-\text{subsets}},$$
$$\underbrace{\{\lceil \tfrac{n}{k} \rceil\text{-element}\}, \{\lceil \tfrac{n}{k} \rceil\text{-element}\}, \ldots, \{\lceil \tfrac{n}{k} \rceil\text{-element}\}}_{(r\geq 0)-\text{subsets}}\}.$$

First we present a lemma.

Lemma 4 *If* G *of order* n *and* $\Delta(G) \neq n-1$ *is perfect lucky* k*-colorable and* H *is a graph obtained from,* G *with one pendant vertex* v_{n+1} *added to any* $v_i \in V(G)$*, then* H *is perfect lucky* k*-colorable.*

Proof. Consider any graph G of order n and $\Delta(G) \neq n-1$ which is perfect lucky k-colorable. It implies that the G permits a proper k-coloring on the vertex set partitions of the lucky partition form,

$$\{\underbrace{\{\lfloor \tfrac{n}{k} \rfloor\text{-element}\}, \{\lfloor \tfrac{n}{k} \rfloor\text{-element}\}, \ldots, \{\lfloor \tfrac{n}{k} \rfloor\text{-element}\}}_{(k-r)-\text{subsets}},$$
$$\underbrace{\{\lceil \tfrac{n}{k} \rceil\text{-element}\}, \{\lceil \tfrac{n}{k} \rceil\text{-element}\}, \ldots, \{\lceil \tfrac{n}{k} \rceil\text{-element}\}}_{(r\geq 0)-\text{subsets}}\}.$$

Let graph H be, graph G with one pendant vertex v_{n+1} added to any $v_i \in V(G)$. Assume without loss of generality that in H the pendant vertex v_{n+1} is adjacent to vertex v_j.
Case 1: Assume $r > 0$. Because $\Delta(G) \neq n-1$, there exists at least one vertex partition which contains a vertex subset say, X such that, $|X| = \lceil \frac{n}{k} \rceil$ such that $v_j \in X$ and there exists at least one vertex subset say, Y such that, $|Y| = \lfloor \frac{n}{k} \rfloor$. Therefore, with regards to a lucky partition form for $V(H)$, the vertex subset $Y \cup \{v_{n+1}\}$ is permissible. It means that, the lucky partition form,

$$\{\underbrace{\{\lfloor \tfrac{n+1}{k} \rfloor\text{-element}\}, \{\lfloor \tfrac{n+1}{k} \rfloor\text{-element}\}, \ldots, \{\lfloor \tfrac{n+1}{k} \rfloor\text{-element}\}}_{(k-r-1)-\text{subsets}},$$
$$\underbrace{\{\lceil \tfrac{n+1}{k} \rceil\text{-element}\}, \{\lceil \tfrac{n+1}{k} \rceil\text{-element}\}, \ldots, \{\lceil \tfrac{n+1}{k} \rceil\text{-element}\}}_{(r+1\geq 0)-\text{subsets}}\},$$

yielding a vertex partition having the vertex subset $Y \cup \{v_{n+1}\}$ is a permissible to assign a perfect lucky k-coloring to graph H.
Case 2: Assume $r = 0$. By similar reasoning to that, found in Case 1 the result follows conclusively. □

Theorem 5 *Let* G *of order* $n = k(t+1) - 1$*,* $t \geq 1$ *with* $\Delta(G) \neq n-1$ *be a simple connected graph. Let* G *permit a perfect lucky* k*-coloring. Let* H *be the graph,* G *with one pendant vertex* $v_{t(k+1)}$ *added to any* $v_i \in V(G)$*. Then,* $\mathcal{L}_H(\lambda, k) < \mathcal{L}_G(\lambda, k)$*.*

Proof. Clearly the perfect lucky colorings are assigned to vertex partitions in accordance to the lucky partition form,

$$\{\underbrace{\{\lfloor\tfrac{n}{k}\rfloor\text{-element}\},\{\lfloor\tfrac{n}{k}\rfloor\text{-element}\},\dots,\{\lfloor\tfrac{n}{k}\rfloor\text{-element}\}}_{1-\text{subset}},$$
$$\underbrace{\{\lceil\tfrac{n}{k}\rceil\text{-element}\},\{\lceil\tfrac{n}{k}\rceil\text{-element}\},\dots,\{\lceil\tfrac{n}{k}\rceil\text{-element}\}}_{(k-1)-\text{subsets}}\}.$$

Assume without loss of generality that in H the pendant vertex v_{n+1} is adjacent to vertex v_j. Since, $\Delta(G) \neq n-1$, there exist at least two permissible vertex partitions. If we relax adjacency (allow a bad edge) then vertex v_{n+1} can only be added to all the $\{\lfloor\frac{n}{k}\rfloor$-element$\},\{\lfloor\frac{n}{k}\rfloor$-element$\},\dots,\{\lfloor\frac{n}{k}\rfloor$-element$\}$ vertex subsets, over all permissible vertex partitions. For this relaxed case, $\mathcal{L}_H(\lambda,k) = \mathcal{L}_G(\lambda,k)$. Else, Lemma 2 above ensures a perfect lucky coloring and $\mathcal{L}_H(\lambda,k) < \mathcal{L}_G(\lambda,k)$. □

Theorem 3 above explains the observation that, $\mathcal{L}_{P_6}(\lambda,3) < \mathcal{L}_{P_5}(\lambda,3)$.

2.1 Heuristic method to determine lucky k-polynomials.

It is observed from Table 1 that $\mathcal{L}_{C_n}(\lambda,3) < \mathcal{L}_{P_n}(\lambda,3)$, $4 \leq n \leq 8$. The next theorem follows from this observation.

Theorem 6 *Let graph G be k-colourable and let $H = G - e$, $e \in E(G)$. Then, $\mathcal{L}_H(\lambda,k) > \mathcal{L}_G(\lambda,k)$.*

Proof. Because G is k-colourable it follows trivially that H is k-colourable. The lucky partitions of $V(H)$ serves as a basis to determine the permissible lucky partitions of $V(G)$ because the only graph structural difference between G and H is the edge e. Hence, with regards to G the lucky partitions of $V(H)$ which have vertex subsets which have the end-points of e as elements must be eliminated. Since, at least one such lucky partition exists, the result $\mathcal{L}_H(\lambda,k) > \mathcal{L}_G(\lambda,k)$ follows immediately. □

Let G be a graph of order n. Note that loops in G, if any, are irrelevant and may be deleted. For application of the heuristic method G is considered to be free of loops. Assume G is k-colourable.

Heuristic method:

Step 1: Since the null graph $\mathfrak{N}_n$ is k-colourable, let the set $\mathfrak{P}_0 = \{$lucky partitions of $V(\mathfrak{N}_n)\}$. Let $E(G) = \{e_i : 1 \leq i \leq \varepsilon(G)\}$. Also let $j = 0$.
Step 2: Let $i = j+1$. Let $\mathfrak{P}_i = \mathfrak{P}_{i-1} \backslash \{$lucky partitions of $\mathfrak{P}_{i-1}$ which have vertex subsets which have the endpoints of e_i as elements$\}$.
Step 3: If $i = \varepsilon(G)$ then go to Step 4. Else, let $j = i$ and go to Step 2.

Step 4: Let $\mathcal{L}_G(\lambda, k) = |\mathfrak{P}_{\varepsilon(G)}|\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-k+1)$ and exit.

Claim 2.5. The heuristic method converges and yields a unique and correct result.

Motivation. Since G is finite it implies that $\varepsilon(G)$ is finite. Hence, the iterative looping between Step 2 and Step 3 will reach go to Step 4 after $\varepsilon(G)$ iterations.
Furthermore, the lucky partitions of $V(G)$ are finite and due to the combinatorial properties of the lucky partitions all vertex subsets which have endpoints of an edge as elements are unique and finite in number. Therefore, the elimination of the corresponding lucky partitions yields a unique result. Finally, it is obvious that after exhaustive iterations, $i = 1, 2, 3, \dots, \varepsilon(G)$, the unique maximum number i.e. $|\mathfrak{P}_{\varepsilon(G)}|$, of lucky partitions remain to ensure a proper lucky k-colouring of G.

3 Conclusion

No step function or recurrence formula is known to determine $L_{P_n}(\lambda, 3)$ and $L_{C_n}(\lambda, 3)$, $n \geq 9$. Finding recurrence formula to determine lucky numbers where-after, finding a combinatorial formula to determine the number of lucky partitions which have vertex subsets without the endpoints of edges are needed to resolve these open questions.

For perfect lucky 3-colorings of paths and cycles the lucky 3-polynomial's coefficient decreases by 1 when P_{kt-1} (or C_{kt-1}), $t \geq 2$ extends to P_{kt} (or to C_{kt}). It is clear from Theorem 3 that for sufficiently large n and for $k \geq 4$ the decrease values will be greater than 1. Finding the decreases is considered a worthy avenue for research.

It is deemed worthy to have an algorithm coded to obtain the Lucky partitions of $V(\mathfrak{N}_n)$ in respect of a given lucky k-colouring. Such is needed to advance research.

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Received: October 18, 2019 • Revised: December 9, 2019