Harmonic univalent functions defined by post quantum calculus operators

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Abstract. We study a family of harmonic univalent functions in the open unit disc defined by using post quantum calculus operators. We first obtained a coefficient characterization of these functions. Using this, co-efficients estimates, distortion and covering theorems were also obtained. The extreme points of the family and a radius result were also obtained. The results obtained include several known results as special cases.

1 Introduction

Let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ can be expressed in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}. \tag{1}$$

The theory of (p, q)-calculus (or post quantum calculus) operators are used in various areas of science and also in the geometric function theory. Let $0 < q \le p \le 1$. The (p, q)-bracket or twin-basic number $[k]_{p,q}$ is defined by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} \quad (q \ne p), \quad \text{and} \quad [k]_{p,p} = kp^{k-1}.$$

Notice that $\lim_{q \to p}[k]_{p,q} = [k]_{p,p}$. For $0 < q \le 1$, q-bracket $[k]_q$ for $k = 0, 1, 2, \cdots$ is given by

$$[k]_q = [k]_{1,q} = \frac{1 - q^k}{1 - q} \quad (q \ne 1), \quad \text{and} \quad [k]_1 = [k]_{1,1} = k.$$

The (p, q)-derivative operator $D_{p,q}$ of a function $f \in \mathcal{A}$ is given by

$$D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}. \tag{2}$$

For a function $f \in \mathcal{A}$, it can be easily seen that

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, \quad (p \ne q, z \ne 0), \tag{3}$$

$(D_{p,q}f)(0) = 1$ and $(D_{p,p}f)(z) = f'(z)$. For definitions and properties of (p, q)-calculus, one may refer to [6]. The $(1, q)$-derivative operator $D_{1,q}$ is known as the q-derivative operator and is denoted by D_q; for $z \ne 0$, it satisfies

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1 - q)z}. \tag{4}$$

For definitions and properties of q-derivative operator, one may refer to [3, 9, 10, 11, 8].

For a function h analytic in $\mathbb{D}$ and an integer $m \ge 0$, we define the (p, q)-Sălăgean differential operator $L_{p,q}^m$, using (p, q)-derivative operator, by

$$L_{p,q}^0 h(z) = h(z) \quad \text{and} \quad L_{p,q}^m h(z) = zD_{p,q}(L_{p,q}^{m-1}(h(z))).$$

For analytic function $g(z) = \sum_{k=1}^{\infty} b_k z^k$, we have

$$L_{p,q}^m g(z) = \sum_{k=1}^{\infty} [k]_{p,q}^m b_k z^k. \tag{5}$$

In particular, for $h \in \mathcal{A}$ with $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, we have

$$L_{p,q}^m h(z) = z + \sum_{k=2}^{\infty} [k]_{p,q}^m a_k z^k. \tag{6}$$

Let $\mathcal{H}$ be the family of complex-valued harmonic functions $f = h + \overline{g}$ defined in $\mathbb{D}$, where h and g has the following power series expansion

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{7}$$

Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z) \neq 0$ in $\mathbb{D}$ and the second dilatation w of f satisfies the condition $|g'(z)/h'(z)| < 1$ in $\mathbb{D}$. Let $\mathcal{S}_{\mathcal{H}}$ be a subclass of functions f in $\mathcal{H}$ that are sense-preserving and univalent in $\mathbb{D}$. Clunie and Sheil-Small studied the class $\mathcal{S}_{\mathcal{H}}$ in their remarkable paper [5]. For a survey or comprehensive study of the theory of harmonic univalent functions, one may refer to the papers [1, 2, 7]. We introduce and study a new subclass of harmonic univalent functions by using (p, q)-Sălăgean harmonic differential operator $L_{p,q}^m : \mathcal{H} \to \mathcal{H}$. For the functions in the newly introduced family, a coefficient characterization is obtained (Theorem 3). Using this, coefficients estimates (Corollary 4), distortion (Theorem 6) and covering (Corollary 7) theorems were also obtained. The extreme points of the family (Theorem 5) and a radius result (Theorem 8) were also obtained. The results obtained include several known results as special cases. Our results can be extended, for example, by using fractional q-integral operator (see Ravikumar [16]).

2   Main results

We define the (p, q)-Sălăgean harmonic differential operator $L_{p,q}^m$ of a harmonic function $f = h + \overline{g} \in \mathcal{H}$ by

$$\begin{aligned} L_{p,q}^m f(z) &= L_{p,q}^m h(z) + (-1)^m \overline{L_{p,q}^m g(z)} \\ &= z + \sum_{k=2}^{\infty} [k]_{p,q}^m a_k z^k + (-1)^m \sum_{k=1}^{\infty} [k]_{p,q}^m \overline{b_k z^k}. \end{aligned} \tag{8}$$

This last expression is obtained by using (6) and (5) and is motivated by Sălăgean[17]. Recall that convolution (or the Hadamard product) of two complex-valued harmonic functions

$$f_1(z) = z + \sum_{k=2}^{\infty} a_{1k} z^k + \sum_{k=1}^{\infty} \overline{b_{1k} z^k} \quad \text{and} \quad f_2(z) = z + \sum_{k=2}^{\infty} a_{2k} z^k + \sum_{k=1}^{\infty} \overline{b_{2k} z^k}$$

is defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1k} a_{2k} z^k + \sum_{k=1}^{\infty} \overline{b_{1k} b_{2k} z^k}, \quad z \in \mathbb{D}.$$

We now introduce a family of (p,q)-Sălăgean harmonic univalent functions by using convolution and the (p,q)-Sălăgean harmonic differential operator $L_{p,q}^m$.

Definition 1 *Suppose* $i, j \in \{0,1\}$. *Let the function* Φ_i, Ψ_j *given by*

$$\Phi_i(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k + (-1)^i \sum_{k=1}^{\infty} \mu_k \bar{z}^k, \tag{9}$$

$$\Psi_j(z) = z + \sum_{k=2}^{\infty} u_k z^k + (-1)^j \sum_{k=1}^{\infty} \nu_k \bar{z}^k \tag{10}$$

be harmonic in $\mathbb{D}$ *with* $\lambda_k > u_k \geq 0$ $(k \geq 2)$ *and* $\mu_k > \nu_k \geq 0$ $(k \geq 1)$. *For* $\alpha \in [0,1)$, $0 < q \leq p \leq 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ *and* $z \in \mathbb{D}$, *let* $\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *denote the family of harmonic functions* f *in* $\mathcal{H}$ *that satisfy the condition*

$$\mathrm{Re}\left\{ \frac{(L_{p,q}^m f * \Phi_i)(z)}{(L_{p,q}^n f * \Psi_j)(z)} \right\} > \alpha, \tag{11}$$

where $L_{p,q}^m$ *is defined by* (8).

Using (8), (9) and (10), we obtain

$$(L_{p,q}^m f * \Phi_i)(z) = z + \sum_{k=2}^{\infty} \lambda_k [k]_{p,q}^m a_k z^k + (-1)^{m+i} \sum_{k=1}^{\infty} \mu_k [k]_{p,q}^m b_k \bar{z}^k, \tag{12}$$

and

$$(L_{p,q}^n f * \Psi_j)(z) = z + \sum_{k=2}^{\infty} u_k [k]_{p,q}^n a_k z^k + (-1)^{n+j} \sum_{k=1}^{\infty} \nu_k [k]_{p,q}^n b_k \bar{z}^k. \tag{13}$$

Definition 2 *Let* $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *be the family of harmonic functions* $f_m = h + \bar{g}_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *such that* h *and* g_m *are of the form*

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad \text{and} \quad g_m(z) = (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{14}$$

The families of $\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ and $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ include a variety of well-known subclasses of harmonic functions as well as many new ones. For example,

(1) $\mathcal{S}_H(m, n, \alpha) \equiv \mathcal{S}_H(m, n, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, 1, \alpha)$,
$\mathcal{TS}_H(m, n, \alpha) \equiv \mathcal{TS}_H(m, n, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, 1, \alpha)$, [18].

(2) $\mathcal{S}_H^*(\alpha) \equiv \mathcal{S}_H(1, 0, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, 1, \alpha)$,
$\mathcal{TS}_H^*(\alpha) \equiv \mathcal{TS}_H(1, 0, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, 1, \alpha)$, [12].

(3) $\mathcal{K}_H(\alpha) \equiv \mathcal{S}_H(2, 1, \frac{z+z^2}{(1-z)^3} + \frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3}, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, 1, 1, \alpha)$,
$\mathcal{TK}_H(\alpha) \equiv \mathcal{TS}_H(2, 1, \frac{z+z^2}{(1-z)^3} + \frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3}, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, 1, 1, \alpha)$, [13].

(4) $\mathcal{S}_{H_q}^*(\alpha) \equiv \mathcal{S}_H(1, 0, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, q, \alpha)$,
$\mathcal{TS}_{H_q}^*(\alpha) \equiv \mathcal{TS}_H(1, 0, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, q, \alpha)$, [4].

(5) $\mathcal{K}_{H_q}(\alpha) \equiv \mathcal{S}_H(2, 1, \frac{z+z^2}{(1-z)^3} + \frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3}, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, 1, q, \alpha)$,
$\mathcal{TK}_{H_q}(\alpha) \equiv \mathcal{TS}_H(2, 1, \frac{z+z^2}{(1-z)^3} + \frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3}, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, 1, q, \alpha)$.

(6) $\mathcal{S}_H(n+1, n, q, \alpha) \equiv \mathcal{S}_H(n+1, n, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, q, \alpha)$,
$\mathcal{TS}_H(n+1, n, q, \alpha) \equiv \mathcal{TS}_H(n+1, n, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, 1, q, \alpha)$, [14].

(7) $\mathcal{S}_H(\Phi_i, \Psi_j, \alpha) \equiv \mathcal{S}_H(0, 0, \Phi_i, \Psi_j, 1, 1, \alpha)$,
$\mathcal{TS}_H(\Phi_i, \Psi_j, \alpha) \equiv \mathcal{TS}_H(0, 0, \Phi_i, \Psi_j, 1, 1, \alpha)$, [15].

We first prove coefficient conditions for the functions in $\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ and $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$.

Theorem 3 *Let the function* $f = h + \overline{g}$ *be such that the functions* h *and* g *are given by* (7). *Also, let the* (p, q)-*coefficient inequality*

$$
\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n}{1-\alpha} |a_k| \\
&+ \sum_{k=1}^{\infty} \frac{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k [k]_{p,q}^n}{1-\alpha} |b_k| \leq 1,
\end{aligned}
\tag{15}
$$

be satisfied for $\alpha \in [0, 1)$, $0 < q \leq p \leq 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$, $\lambda_k > u_k \geq 0$ $(k \geq 2)$ *and* $\mu_k > v_k \geq 0$ $(k \geq 1)$. *Then*

(i) *the function* $f = h + \overline{g}$ *given by* (7) *is a sense-preserving harmonic univalent functions in* $\mathbb{D}$ *and* $f \in \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *if the inequality in* (15) *is satisfied.*

(ii) *the function* $f_m = h + \overline{g}_m$ *given by* (14) *is in the* $\mathcal{T}\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *if and only if the inequality in* (15) *is satisfied.*

Proof. (i). Using the techniques used in [14] and [15], it is a routine step to prove that $f = h + \overline{g}$ given by (7) is sense-preserving and locally univalent in $\mathbb{D}$. Using the fact $\mathsf{Re}(w) > \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$\left| 1 - \alpha + \frac{(L_{p,q}^m f * \Phi_i)(z)}{(L_{p,q}^n f * \Psi_j)(z)} \right| - \left| 1 + \alpha - \frac{(L_{p,q}^m f * \Phi_i)(z)}{(L_{p,q}^n f * \Psi_j)(z)} \right| \geq 0. \qquad (16)$$

In view of (12) and (13), left side of (16) yields

$$\left| (L_{p,q}^m f * \Phi_i)(z) + (1 - \alpha)(L_{p,q}^n f * \Psi_j)(z) \right|$$
$$- \left| (L_{p,q}^m f * \Phi_i)(z) - (1 + \alpha)(L_{p,q}^n f * \Psi_j)(z) \right|$$
$$= \left| (2 - \alpha)z + \sum_{k=2}^{\infty} (\lambda_k[k]_{p,q}^m + (1 - \alpha)u_k[k]_{p,q}^n)a_k z^k \right.$$
$$\left. + (-1)^{m+i} \sum_{k=1}^{\infty} (\mu_k[k]_{p,q}^m + (-1)^{n+j-(m+i)}(1 - \alpha)v_k[k]_{p,q}^n)b_k \overline{z}^k \right|$$
$$- \left| - \alpha z + \sum_{k=2}^{\infty} (\lambda_k[k]_{p,q}^m - (1 + \alpha)u_k[k]_{p,q}^n)a_k z^k \right.$$
$$\left. + (-1)^{m+i} \sum_{k=1}^{\infty} (\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}(1 + \alpha)v_k[k]_{p,q}^n)b_k \overline{z}^k \right|$$
$$\geq (2 - 2\alpha)|z| - 2\sum_{k=2}^{\infty} (\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n)|a_k||z|^k$$
$$- \sum_{k=1}^{\infty} (\mu_k[k]_{p,q}^m + (-1)^{n+j-(m+i)}(1 - \alpha)v_k[k]_{p,q}^n)|b|_k|z|^k$$
$$- \sum_{k=1}^{\infty} (\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}(1 + \alpha)v_k[k]_{p,q}^n)|b|_k|z|^k$$
$$\geq (1 - \alpha)|z| \left[1 - \sum_{k=2}^{\infty} \frac{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}{1 - \alpha}|a_k||z|^{k-1} \right.$$

$$-\sum_{k=1}^{\infty} \frac{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}{1-\alpha}|b_k||z|^{k-1}\Bigg]$$

$$> (1-\alpha)|z|\Bigg[1 - \Bigg(\sum_{k=2}^{\infty} \frac{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}{1-\alpha}|a_k|$$

$$+ \sum_{k=1}^{\infty} \frac{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}{1-\alpha}|b_k|\Bigg)\Bigg].$$

This last expression is non-negative because of the condition given in (15). This completes the proof of part (i) of theorem.

(ii). Since

$$\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha) \subset \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha),$$

the sufficient part of part (ii) follows from part (i). In order to prove the necessary part of part (ii), we assume that $f_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$. We notice that

$$\mathrm{Re}\left\{\frac{(L_{p,q}^m f * \Phi_i)(z)}{(L_{p,q}^n f * \Psi_j)(z)} - \alpha\right\}$$

$$= \mathrm{Re}\left\{\frac{(1-\alpha)z - \sum_{k=2}^{\infty}(\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n)a_k z^k}{z - \sum_{k=2}^{\infty} u_k[k]_{p,q}^n a_k z^k + (-1)^{m+i+n+j-1}\sum_{k=1}^{\infty} v_k[k]_{p,q}^n b_k \overline{z}^k}\right.$$

$$\left. + \frac{(-1)^{2m+2i-1}\sum_{k=1}^{\infty}(\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n)b_k \overline{z}^k}{z - \sum_{k=2}^{\infty} u_k[k]_{p,q}^n a_k z^k + (-1)^{m+i+n+j-1}\sum_{k=1}^{\infty} v_k[k]_{p,q}^n b_k \overline{z}^k}\right\}$$

$$\geq \frac{\begin{aligned}&(1-\alpha) - \sum_{k=2}^{\infty}(\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n)a_k r^{k-1}\\ &\quad - \sum_{k=1}^{\infty}(\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n)b_k r^{k-1}\end{aligned}}{1 - \sum_{k=2}^{\infty} u_k[k]_{p,q}^n a_k r^{k-1} - (-1)^{m+i+n+j}\sum_{k=1}^{\infty} v_k[k]_{p,q}^n b_k r^{k-1}}$$

$$\geq 0,$$

by (11). The above inequality must hold for all $z \in \mathbb{D}$. In particular, choosing the values of z on the positive real axis and $z \to 1^-$, we obtain the required condition (15). This completes the proof of part (ii) of theorem.

The harmonic mappings

$$
\begin{aligned}
f(z) = z &+ \sum_{k=2}^{\infty} \frac{1-\alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n} x_k z^k \\
&+ \sum_{k=1}^{\infty} \frac{1-\alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n} y_k \overline{z}^k,
\end{aligned}
\tag{17}
$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (15) is sharp. □

Theorem 3 also yields the following corollary.

Corollary 4 *For the function* $f_m = h + \overline{g}_m$ *given by* (14), *we have*

$$
|a_k| \le \frac{1-\alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}, \quad k \ge 2
$$

and

$$
|b_k| \le \frac{1-\alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}, \quad k \ge 1.
$$

The result is sharp for each k.

Using Theorem 3 (part ii), it is seen that the class $\mathcal{T}\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ is convex and closed with respect to the topology of locally uniform convergence so that the closed convex hulls of $\mathcal{T}\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ equals itself. The next theorem determines the extreme points of $\mathcal{T}\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$.

Theorem 5 *Let the function* $f_m = h + \overline{g}_m$ *be given by* (14). *Then the function* $f_m \in \mathrm{clco}\, \mathcal{T}\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *if and only if* $f_m(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_{m_k}(z))$, *where*

$$
h_1(z) = z, \quad h_k(z) = z - \frac{1-\alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n} z^k, \quad k \ge 2,
$$

$$
g_{m_k}(z) = z + (-1)^{m+i-1}\frac{1-\alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}\overline{z}^k, \quad k \ge 1,
$$

and $\sum_{k=1}^{\infty}(x_k + y_k) = 1$ *where* $x_k \ge 0$ *and* $y_k \ge 0$. *In particular, the extreme points of* $\mathcal{T}\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *are* $\{h_k\}$ *and* $\{g_{m_k}\}$.

Proof. For a function f_m of the form $f_m(z) = \sum\limits_{k=1}^{\infty} (x_k h_k(z) + y_k g_{m_k}(z))$, where $\sum\limits_{k=1}^{\infty} (x_k + y_k) = 1$, we have

$$f_m(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n} x_k z^k$$
$$+ \sum_{k=1}^{\infty} (-1)^{m+i-1} \frac{1-\alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n} y_k \bar{z}^k.$$

Then $f_m \in \mathrm{clco}\, \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ because

$$\sum_{k=2}^{\infty} \frac{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}{1-\alpha} \left(\frac{1-\alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n} x_k \right) +$$

$$\sum_{k=1}^{\infty} \frac{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}{1-\alpha} \left(\frac{1-\alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n} y_k \right)$$

$$= \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1.$$

Conversely, suppose $f_m \in \mathrm{clco}\, \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$. Then

$$|a_k| \leq \frac{1-\alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n} \quad \text{and} \quad |b_k| \leq \frac{1-\alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}.$$

Set

$$x_k = \frac{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}{1-\alpha}|a_k| \quad \text{and} \quad y_k = \frac{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}{1-\alpha}|b_k|.$$

By Theorem 3 (ii), $\sum\limits_{k=2}^{\infty} x_k + \sum\limits_{k=1}^{\infty} y_k \leq 1$. Therefore we define $x_1 = 1 - \sum\limits_{k=2}^{\infty} x_k - \sum\limits_{k=1}^{\infty} y_k \geq 0$. Consequently, we obtain $f_m(z) = \sum\limits_{k=1}^{\infty} (x_k h_k(z) + y_k g_{m_k}(z))$ as required. $\square$

For functions in the class $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$, the following theorem gives distortion bounds which in turns yields the covering result for this class.

Theorem 6 *Let the function* $f_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$, $\gamma_k = \lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n$, $k \geq 2$ *and* $\phi_k = \mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n$, $k \geq 1$. *If* $\{\gamma_k\}$ *and* $\{\phi_k\}$ *are non-decreasing sequences, then we have*

$$|f_m(z)| \leq (1 + |b_1|)|z| + \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{\beta}|b_1|\right)|z|^2 \quad (18)$$

and

$$|f_m(z)| \geq (1 - |b_1|)|z| - \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{\beta}|b_1|\right)|z|^2, \quad (19)$$

for all $z \in \mathbb{D}$, *where* $b_1 = f_{\bar{z}}(0)$ *and*

$\beta = \min\{\gamma_2, \phi_2\} = \min\{\lambda_2[2]_{p,q}^m - \alpha u_2[2]_{p,q}^n, \mu_2[2]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_2[2]_{p,q}^n\}$.

Proof. Let the function $f_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$. Taking the absolute value of f_m, we obtain

$$|f_m(z)| \leq (1 + |b_1|)|z| + \sum_{k=2}^{\infty}(|a_k| + |b_k|)|z|^k$$

$$\leq (1 + |b_1|)|z| + \sum_{k=2}^{\infty}(|a_k| + |b_k|)|z|^2$$

$$\leq (1 + |b_1|)|z| + \frac{1-\alpha}{\beta}\sum_{n=2}^{\infty}\left(\frac{\beta}{1-\alpha}|a_k| + \frac{\beta}{1-\alpha}|b_k|\right)|z|^2$$

$$\leq (1 + |b_1|)|z| + \frac{1-\alpha}{\beta}\sum_{k=2}^{\infty}\left(\frac{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}{1-\alpha}|a_k|\right.$$

$$\left. + \frac{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}{1-\alpha}|b_k|\right)|z|^2$$

$$\leq (1 + |b_1|)|z| + \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{1-\alpha}|b_1|\right)|z|^2.$$

This proves (18). The proof of (19) is omitted as it is similar to the proof of (18). $\square$

The following covering result follows from the inequality (19).

Corollary 7 *Under the hypothesis of Theorem 6, we have*

$$\left\{w : |w| < \frac{1}{\beta}\left(\beta - 1 + \alpha + (\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1 - \beta)|b_1|\right)\right\} \subset f(\mathbb{D}).$$

Theorem 8 *If the function* $f_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$, *then the function* f_m *is convex in the disc*

$$|z| \leq \min_k \left\{ \frac{1 - b_1}{k[1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{1 - \alpha} b_1]} \right\}^{\frac{1}{k-1}}, \quad k \geq 2.$$

Proof. Let $f_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ and let $r, 0 < r < 1$, be fixed. Then $r^{-1}f_m(rz) \in \mathcal{TS}_H(m, n, \Phi_i, p, q, \alpha)$ and we have

$$\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) = \sum_{k=2}^{\infty} k(|a_k| + |b_k|)kr^{k-1}$$

$$\leq \sum_{k=2}^{\infty} \left(\frac{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}{1 - \alpha}|a_k| + \frac{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}^n}{1 - \alpha}|b_k| \right) kr^{k-1}$$

$$\leq \sum_{k=2}^{\infty} \left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{1 - \alpha}|b_1| \right) kr^{k-1}$$

$$\leq 1 - b_1$$

provided

$$kr^{k-1} \leq \frac{1 - b_1}{1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{1 - \alpha} b_1}$$

which is true if

$$r \leq \min_k \left\{ \frac{1 - b_1}{k[1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{1 - \alpha} b_1]} \right\}^{\frac{1}{k-1}}, \quad k \geq 2.$$

$\square$

Remark 9 *Our results naturally includes several results known for those sub-classes of harmonic functions listed after Definition 2.*

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