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Hadamard product of GCUD matrices

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Abstract. Let f be an arithmetical function. The matrix $[f(i,j)^{**}]_{n\times n}$ given by the value of f in greatest common unitary divisor of $(i,j)^{**}$, $f\big((i,j)^{**}\big)$ as its i, j entry is called the greatest common unitary divisor (GCUD) matrix. We consider the Hadamard product of these matrices and we calculate the Hadamard product and determinant of the Hadamard product of two GCUD matrices.

1 Introduction

The classical Smith determinant introduced by H. J. Smith [9] is

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdots & \cdots & \cdots & \cdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(1)\cdot\varphi(2)\cdots\varphi(n), \qquad (1)$$

where (i,j) is the greatest common divisor of i and j, and $\varphi(n)$ is Euler's totient function.
A divisor d of n is said to be a unitary divisor of n if $\left(d, \frac{n}{d}\right) = 1$ and we write $d \parallel n$. Let $(m,n)^*$ the greatest divisor of m which is unitary divisor of n (see E. Cohen [2]).

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We denote the greatest common unitary divisor of m and n as $(m,n)^{**}$.
Let $\varphi^*(n)$ be the unitary analogue of $\varphi(n)$:

$$\varphi^*(n) = \sum_{\substack{k \le n \\ (k,n)^* = 1}} 1,$$

and

$$\mu^*(n) = (-1)^{\omega(n)}$$

be the unitary analogue of the Möbius function $\mu(n)$.
The GCUD matrix with respect to f is

$$[f(i,j)^{**}]_{n\times n} = \begin{bmatrix} f((1,1)^{**}) & f((1,2)^{**}) & \cdots & f((1,n)^{**}) \\ f((2,1)^{**}) & f((2,2)^{**}) & \cdots & f((2,n)^{**}) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)^{**}) & f((n,2)^{**}) & \cdots & f((n,n)^{**}) \end{bmatrix}$$

If we consider the GCUD matrix $[f(i,j)^{**}]_{n\times n}$ where

$$f(n) = \sum_{d \| n} g(d),$$

H. Jager [5] proved that

$$\left[f\big((i,j)\big)^{**}\right]_{n\times n} = C_1 \operatorname{diag}[g(1), g(2), \ldots, g(n)] C_1^T, \tag{2}$$

where $C_1 = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & \text{ha} \quad j \,\|\, i \\ 0, & \text{ha} \quad j \,\not\|\, i \end{cases}.$$

and

$$\det[f((i,j)^{**})]_{n\times n} = g(1)\cdot g(2)\cdots g(n).$$

For $g(n) = \varphi^*(n)$

$$f\big((i,j)^{**}\big) = \sum_{d \| (i,j)^{**}} \varphi^*(d) = (i,j)^{**}.$$

and the decomposition of matrix

$$[(i,j)^{**}]_{n\times n} = C_1 \operatorname{diag}[\varphi^*(1), \varphi^*(2), \ldots, \varphi^*(n)] C_1^T,$$

$$\det[(i,j)^{**}]_{n\times n} = \varphi^*(1)\varphi^*(2)\cdots\varphi^*(n).$$

The unitary convolution of the arithmetical functions f and g is defined as

$$(f \odot g)(n) = \sum_{d \,\|\, n} f(d) g\left(\frac{n}{d}\right).$$

From this convolution we can write (2) in the following form:

$$\det\left[f\big((i,j)^{**}\big)\right]_{n\times n} = (f\odot\mu^*)(1)(f\odot\mu^*)(2)\cdots(f\odot\mu^*)(n). \tag{3}$$

Here we present some examples which are relevant in our study.

Example 1 *If*

$$g(n) = \beta^*(n) = \sum_{i=1}^{n} (i,n)^*$$

the unitary Pillai function then (see L. Tóth, [10, 11]). We have

$$\begin{aligned} \beta^*(n) &= \sum_{s\|n} d\varphi\left(\frac{n}{d})\right), \\ f(n) &= \sum_{d|n} \beta^*(n) = n\tau^*(n), \end{aligned}$$

where $\tau^*(n)$ is the number of unitary divisors. The GCUD matrix and determinant in this case have the following form:

$$[(i,j)^{**}\tau^*((i,j)^{**})]_{n\times n} = C_1 \operatorname{diag}\big(\beta^*(1),\ \beta^*(2),\ldots,\beta^*(n)\big)\ C_1^T, \tag{4}$$

$$\det[(i,j)^{**}\tau^*((i,j)^{**})]_{n\times n} = \beta^*(1)\beta^*(2)\cdots\beta^*(n). \tag{5}$$

Example 2 *If* $g(n) = \dfrac{\varphi^*(n)}{n}$, *then*

$$f(n) = \sum_{d\|n} \frac{\varphi^*(d)}{d} = \frac{\beta^*(n)}{n},$$

$$\left[\frac{\beta^*(i,j)}{(i,j)}\right]_{n\times n} = C_1\ \mathit{diag}\left(\frac{\varphi^*(1)}{1},\ \frac{\varphi^*(2)}{2},\ldots,\frac{\varphi^*(n)}{n}\right)\ C_1^T, \tag{6}$$

$$\det\left[\frac{\beta^*(i,j)}{(i,j)}\right]_{n\times n} = \frac{\varphi^*(1)\varphi^*(2)\cdots\varphi^*(n)}{n!}.$$

For other contributions, we mention the papers of P. Haukkanen, J. Wang and J. Sillanpää [3], A. Nalli, D. Tasci [6], P. Haukkanen, J. Sillanpää [4]. We introduce the concept of Hadamard product (see F. Zhang [12]).

Definition 1 *The Hadamard product* $C = A \circ B = [c_{ij}]_{n\times n}$ *of two matrices* $A = [a_{ij}]_{n\times n}$ *and* $B = [b_{ij}]_{n\times n}$ *is simply their elementwise product,*

$$c_{ij} = a_{ij} b_{ij}, \quad i, j \in \{1, 2, \ldots, b\}.$$

A. Ocal [8], A. Nalli [7], A. Bege [1] establishes various results concerning classical GCD matrices and least common multiple (LCM) matrices. In examples 1 and 2 appears Hadamard products of special GCD matrices:

$$\det\left[[\tau((i,j)^{**})]_{n\times n} \circ [(i,j)^{**}]_{n\times n}\right]_{n\times n} = \beta^*(1)\beta^*(2)\cdots\beta^*(n),$$

$$\det\left[[\beta((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{(i,j)^{**}}\right]_{n\times n}\right]_{n\times n} = \frac{\varphi^*(1)\varphi^*(2)\cdots\varphi^*(n)}{n!}.$$

Let f and g be two arithmetical functions. In this paper we calculate the Hadamard product and the determinant of Hadamard product of $[f((i,j)^{**})]_{n\times n}$ and $[g((i,j)^{**})]_{n\times n}$.

2 Main results

Theorem 1 *Let* h *and* g *be two arithmetical functions and* g *be multiplicative. If*

$$f(n) = \sum_{d\|n} h(d) g\left(\frac{n}{d}\right), \tag{7}$$

then

1.

$$\left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = C_1 \operatorname{diag}\left(\frac{h(1)}{g(1)}, \frac{h(2)}{g(2)}, \ldots, \frac{h(n)}{g(n)}\right) C_1^T, \tag{8}$$

where $C_1 = [c_{ij}]_{n\times n}$*,*

$$c_{ij} = \begin{cases} 1, & \text{if} \quad j \| i \\ 0, & \text{if} \quad j \not\| i \end{cases},$$

2.

$$\det\left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \frac{h(1)}{g(1)}\frac{h(2)}{g(2)}\cdots\frac{h(n)}{g(n)}, \tag{9}$$

3. *There exist* $H(n)$ *and* $G(n)$ *arithmetical functions, such that*

$$\det\left[[f((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n}=\frac{\det[H((i,j)^{**})]}{\det[G((i,j)^{**})]}.$$

Proof.
Let

$$h(n)=\big(f\odot(\mu^{*}g)\big)(n).$$

We have

$$g\odot(\mu^{*}g)=I,$$

which means that $\mu^{*}g$ is the inverse respecting to the unitary convolution. From this

$$h\odot g=f\odot\big((\mu^{*}g)\odot g\big)=f\odot I=f.$$

Because g is a multiplicative function

$$f(n)=\sum_{d\|n}h(d)g\left(\frac{n}{d}\right)=\sum_{d\|n}\frac{h(d)}{g(d)}g(d)g\left(\frac{n}{d}\right)=g(n)\sum_{d\|n}\frac{h(d)}{g(d)}.$$

By the definition of the Hadamard product we have

$$\frac{f((i,j)^{**})}{g((i,j)^{**})}.$$

Thus

$$\frac{f(n)}{g(n)}=\sum_{d\|n}\frac{h(d)}{g(d)},$$

and by using (2), we have (8).
If we calculate the determinant of both parts we have (9).
Let

$$H(n)=\sum_{d\|n}h(d)$$

and

$$G(n)=\sum_{d\|n}g(d).$$

Using (2) we have

$$\det[H((i,j)^{**})]_{n\times n}=h(1)h(2)\cdots h(n)$$

and

$$\det[G((i,j)^{**})]_{n\times n} = g(1)g(2)\cdots g(n).$$

which means that

$$\det\left[[f((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \frac{\det[H((i,j)^{**})]}{\det[G((i,j)^{**})]}.$$

□

Example 3 *If* $g(n) = n$ *then*

$$f(n) = \sum_{d\|n} h(d)\frac{n}{d}.$$

and

$$\begin{aligned}\left[\frac{f((i,j)^{**})}{(i,j)^{**}}\right]_{n\times n} &= \left[[f((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{(i,j)^{**}}\right]_{n\times n}\right]_{n\times n} = \\ &= C_1 \operatorname{diag}\left(\frac{h(1)}{1},\frac{h(2)}{2},\ldots,\frac{h(n)}{n}\right)\ C_1^T\end{aligned}$$

$$\begin{aligned}\det\left[\frac{f((i,j)^{**})}{(i,j)^{**}}\right]_{n\times n} &= \det\left[[f((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{(i,j)^{**}}\right]_{n\times n}\right]_{n\times n} = \\ &= \frac{h(1)h(2)\cdots h(n)}{n!}.\end{aligned}$$

Example 4 *If* $g(n) = \frac{1}{n}$ *then*

$$f(n) = \sum_{d\|n} h(d)\frac{d}{n}$$

and

$$[f((i,j)^{**}(i,j)^{**}]_{n\times n} = C_1 \operatorname{diag}(h(1)1, h(2)2,\ldots,h(n)n)\ C_1^T,$$

$$\begin{aligned}\det[f((i,j)^{**})(i,j)^{**}]_{n\times n} &= \det\left[[f((i,j)^{**}]_{n\times n}\circ[(i,j)^{**}]_{n\times n}\right]_{n\times n} = \\ &= h(1)\cdots h(n)n!.\end{aligned}$$

If we want to apply this theorem to a given f and g, using the unitary Möbius inversion formula we have

$$h(n) = \sum_{d\|n} \mu^*(d) g(d) f\left(\frac{n}{d}\right)$$

where $\mu^*(n)$ is the usual unitary Möbius function and we can formulate the following result.

Theorem 2 *Let f and g be two arithmetical functions and g be multiplicative. We have*

$$\begin{aligned}\left[\frac{f((i,j)^{**})}{g((i,j)^{**})}\right]_{n\times n} &= \left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \\ &= C_1 \operatorname{diag}\left(\frac{f(1)}{g(1)}, \ldots, \frac{\sum\limits_{d\|n} \mu^*(d) g(d) f\left(\frac{n}{d}\right)}{g(n)}\right) C_1^T,\end{aligned}$$

and

$$\det\left[[f((i,j)^{**}]_{n\times n} \circ \left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \frac{f(1)}{g(1)} \cdots \frac{\sum_{d\|n} \mu^*(d) g(d) f\left(\frac{n}{d}\right)}{g(n)}.$$

Example 5 *If f is a multiplicative arithmetical function and $g(n) = \frac{1}{n}$*

$$\det\left[f((i,j)^{**})(i,j)^{**}\right]_{n\times n} = 1 \cdots \left(n \prod_{p^\alpha\|n} \left(f(p^\alpha) - \frac{1}{p^\alpha}\right)\right).$$

In particular if $f(n) = 1$

$$\det\left[(i,j)^{**}\right]_{n\times n} = \prod_{k=1}^{n} \varphi^*(k).$$

Example 6 *For a power GCUD matrix and determinant we have*

$$[((i,j)^{**})^s]_{n\times n} = C_1 \ \mathit{diag}\big(J_s(1),\ J_s(2), \ldots, J_s(n)\big)\ C_1^T,$$

$$\det[((i,j)^{**})^s]_{n\times n} = J_s(1) J_s(2) \cdots J_s(n).$$

where $J_s(n)$ the Jordan totient function.

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