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Generalized operator for Alexander integral operator

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Abstract. Let T_n be the class of functions f which are defined by a power series

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots$$

for every z in the closed unit disc $\overline{\mathbb{U}}$. With m different boundary points $z_s, (s = 1, 2, \dots, m)$, we consider $\alpha_m \in e^{i\beta}A_{-j-\lambda}f(\mathbb{U})$, here $A_{-j-\lambda}$ is the generalized Alexander integral operator and $\mathbb{U}$ is the open unit disc. Applying $A_{-j-\lambda}$, a subclass $B_n(\alpha_m, \beta, \rho; j, \lambda)$ of T_n is defined with fractional integral for functions f. The object of present paper is to consider some interesting properties of f to be in $B_n(\alpha_m, \beta, \rho; j, \lambda)$.

1 Introduction

Let T_n be the class of functions

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad n \in \mathbb{N} = \{1, 2, 3, \dots\} \tag{1}$$

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that are analytic in the closed unit disc $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$. For $f \in T_n$, J.W.Alexander [2] had defined the following the Alexander integral operator $A_{-1}f(z)$ given by

$$A_{-1}f(z) = \int_0^z \frac{f(t)}{t}dt = z + \sum_{k=n+1}^{\infty} \frac{a_k}{k}z^k. \tag{2}$$

The above the Alexander integral operator was applied for some subclasses of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ by M.Acu [1] and by K. Kugita et al. [4].

For the above the Alexander integral operator $A_{-1}f(z)$, we consider

$$A_{-j}f(z) = A_{-j+1}(A_{-1}f(z)) = z + \sum_{k=n+1}^{\infty} \frac{a_k}{k^j}z^k, \quad j \in \mathbb{N} \tag{3}$$

where $A_0f(z) = f(z)$.

From the various definitions of fractional calculus of $f \in T_n$ (that is, fractional integrals and fractional derivatives) given in the literature, we would like to recall here the following definitions for fractional calculus which were used by Owa [7] and Owa and Srivastava [8].

Definition 1 *The fractional integral of order* λ *for* $f \in T_n$ *is defined by*

$$D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}dt, \quad (\lambda > 0) \tag{4}$$

where f *is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of* $(z-t)^{\lambda-1}$ *is removed by requiring* $\log(z-t)$ *to be real when* $z-t > 0$ *and* Γ *is the Gamma function.*

With the above definition, we know that

$$D_z^{-\lambda}f(z) = \frac{1}{\Gamma(2+\lambda)}z^{1+\lambda} + \sum_{k=n+1}^{\infty} \frac{k!}{\Gamma(k+1+\lambda)}a_kz^{k+\lambda} \tag{5}$$

for $\lambda > 0$ and $f \in T_n$. Further applying the fractional integral for $f \in T_n$, we define a new operator $A_{-\lambda}f(z)$ given by

$$A_{-\lambda}f(z) = \frac{\Gamma\left(\frac{3+\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)}z^{\frac{1-\lambda}{2}}D_z^{-\lambda}\left(z^{\frac{-1-\lambda}{2}}f(z)\right), \tag{6}$$

where $0 \leq \lambda \leq 1$. If $\lambda = 0$, then (6) becomes $A_0 f(z) = f(z)$ and if $\lambda = 1$, then (6) leads us that

$$A_{-1} f(z) = D_z^{-1}\left(\frac{f(z)}{z}\right) = \int_0^z \frac{f(t)}{t} dt. \tag{7}$$

With this integral operator, we know

$$A_{-j-\lambda} f(z) = A_{-j}\left(A_{-\lambda} f(z)\right) \tag{8}$$

where $j \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. This operator $A_{-j-\lambda} f(z)$ is the generalization of the Alexander integral operator $A_{-1} f(z)$. Here, we note that

$$A_{-\lambda} f(z) = z + \sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right)} a_k z^k \tag{9}$$

and

$$A_{-j-\lambda} f(z) = z + \sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right) k^j} a_k z^k \tag{10}$$

where $j \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. From the above, we know that

$$A_{-j-\lambda} f(z) = A_{-j}(A_{-\lambda} f(z)) = A_{-\lambda}(A_{-j} f(z)) \tag{11}$$

for $f \in T_n$. For m different boundary points $z_s (s = 1, 2, 3, ..., m)$ with $|z_s| = 1$, we consider

$$\alpha_m = \frac{1}{m} \sum_{s=1}^{m} \frac{A_{-j-\lambda} f(z_s)}{z_s}, \tag{12}$$

where $\alpha_m \in e^{i\beta} A_{-j-\lambda} f(\mathbb{U}), \alpha_m \neq 1$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$. For such α_m, if $f \in T_n$ satisfies

$$\left| \frac{e^{i\beta} \frac{A_{-j-\lambda} f(z)}{z} - \alpha_m}{e^{i\beta} - \alpha_m} - 1 \right| < \rho, \quad z \in \mathbb{U} \tag{13}$$

for some real $\rho > 0$, we say that the function f belongs to the subclass $B_n(\alpha_m, \beta, \rho; j, \lambda)$ of T_n. With this definition for the class $B_n(\alpha_m, \beta, \rho; j, \lambda)$, we see that the condition (13) is equivalent to

$$\left| \frac{A_{-j-\lambda} f(z)}{z} - 1 \right| < \rho \left| e^{i\beta} - \alpha_m \right|, \quad z \in \mathbb{U}. \tag{14}$$

If we consider the function $f \in T_n$ given by

$$f(z) = z + \frac{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2n+3+\lambda}{2}\right)}{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2n+3-\lambda}{2}\right)}\rho(e^{i\beta} - \alpha_m)(n+1)^j z^{n+1} \tag{15}$$

then f satisfies

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| = \rho\left|e^{i\beta} - \alpha_m\right||z|^n < \rho\left|e^{i\beta} - \alpha_m\right|, \quad z \in \mathbb{U}. \tag{16}$$

Therefore, f given by (15) belongs to the class $B_n(\alpha_m, \beta, \rho; j, \lambda)$.

Discussing our problems for $f \in B_n(\alpha_m, \beta, \rho; j, \lambda)$, we have to introduce the following lemma due to S. S. Miller and P. T. Mocanu [5, 6] (also, due to I. S. Jack [3]).

Lemma 1 *Let the function* w *given by*

$$w(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \quad (n \in \mathbb{N}) \tag{17}$$

be analytic in $\mathbb{U}$ *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r$ *at a point* $z_0, (0 < |z_0| < 1)$ *then there exists a real number* $k \geq n$ *such that*

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \tag{18}$$

and

$$\operatorname{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k. \tag{19}$$

2 Properties of functions concerning with the class $B_n(\alpha_m, \beta, \rho; j, \lambda)$

Our first property for $f \in T_n$ is as follows.

Theorem 1 *If* $f \in T_n$ *satisfies*

$$\left|\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)} - 1\right| < \frac{\left|e^{i\beta} - \alpha_m\right| n\rho}{1 + |e^{i\beta} - \alpha_m|\,\rho}, \quad z \in \mathbb{U} \tag{20}$$

for some α_m *defined by* (12) *with* $\alpha_m \neq 1$ *such that* $z_s \in \partial\mathbb{U}\,(s = 1, 2, 3, \dots, m)$, *and for some real* $\rho > 1$, *then*

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < \rho\left|e^{i\beta} - \alpha_m\right|, \quad z \in \mathbb{U} \tag{21}$$

that is, $f \in B_n(\alpha_m, \beta, \rho; j, \lambda)$.

Proof. We introduce the function w by

$$w(z)=\frac{e^{i\beta}\frac{A_{-j-\lambda}f(z)}{z}-\alpha_m}{e^{i\beta}-\alpha_m}-1=\frac{e^{i\beta}}{e^{i\beta}-\alpha_m}\left\{\sum_{k=n+1}^{\infty}\frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right)k^j}a_k z^{k-1}\right\}. \tag{22}$$

Then, w is analytic in $\mathbb{U}$ with $w(0)=0$ and

$$\frac{A_{-j-\lambda}f(z)}{z}=1+(1-e^{-i\beta}\alpha_m)w(z). \tag{23}$$

It follows from the above that

$$\frac{z(A_{-j-\lambda}f(z))'}{A_{-j-\lambda}f(z)}-1=\frac{(1-e^{-i\beta}\alpha_m)zw'(z)}{1+(1-e^{-i\beta}\alpha_m)w(z)}. \tag{24}$$

Note that

$$z(A_{-j-\lambda}f(z))'=A_{-j-\lambda+1}f(z). \tag{25}$$

So, (24) is the same as

$$\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1=\frac{(1-e^{-i\beta}\alpha_m)zw'(z)}{1+(1-e^{-i\beta}\alpha_m)w(z)}. \tag{26}$$

Thus, our condition (20) gives that

$$\left|\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1\right|=\left|\frac{(1-e^{-i\beta}\alpha_m)zw'(z)}{1+(1-e^{-i\beta}\alpha_m)w(z)}\right|<\frac{|e^{i\beta}-\alpha_m|n\rho}{1+|e^{i\beta}-\alpha_m|\rho}. \tag{27}$$

Now, we suppose that there exists a point $z_0, (0<|z_0|<1)$ such that

$$\max\{|w(z)|;|z|\leq|z_0|\}=|w(z_0)|=\rho>1. \tag{28}$$

Then, we can write that $w(z_0)=\rho e^{i\theta}, (0\leq\theta\leq 2\pi)$ and $z_0w'(z_0)=kw(z_0)$, $(k\geq n)$ by Lemma 1. For such a point $z_0, (0<|z_0|<1)$ we see that

$$\begin{aligned}\left|\frac{A_{-j-\lambda+1}f(z_0)}{A_{-j-\lambda}f(z_0)}-1\right|&=\left|\frac{(1-e^{-i\beta}\alpha_m)z_0w'(z_0)}{1+(1-e^{-i\beta}\alpha_m)w(z_0)}\right|\\&=\left|\frac{(1-e^{-i\beta}\alpha_m)k\rho}{1+(1-e^{-i\beta}\alpha_m)\rho e^{i\theta}}\right|\\&\geq\frac{|1-e^{-i\beta}\alpha_m|n\rho}{1+|1-e^{-i\beta}\alpha_m|\rho}\\&=\frac{|e^{i\beta}-\alpha_m|n\rho}{1+|e^{i\beta}-\alpha_m|\rho}.\end{aligned} \tag{29}$$

Since (29) contradicts our condition (20), we know that there is no z_0, $(0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. Therefore, using $|w(z)| < \rho$ for all $z \in \mathbb{U}$, we have that

$$|w(z)| = \left| \frac{e^{i\beta}\left(\frac{A_{-j-\lambda}f(z)}{z} - 1\right)}{e^{i\beta} - \alpha_m} \right| < \rho, \quad z \in \mathbb{U}, \tag{30}$$

that is, that

$$\left| \frac{A_{-j-\lambda}f(z)}{z} - 1 \right| < \rho \left| e^{i\beta} - \alpha_m \right|, \quad z \in \mathbb{U}. \tag{31}$$

This completes the proof of the theorem. □

Example 1 *We consider the function* $f \in T_n$ *given by*

$$f(z) = z + a_{n+1}z^{n+1}, \quad z \in \mathbb{U}. \tag{32}$$

Then, we see that

$$\left| \frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)} - 1 \right| = \left| \frac{P(n,j,\lambda)na_{n+1}z^n}{1 + P(n,j,\lambda)a_{n+1}z^n} \right|. \tag{33}$$

$$< \frac{nP(n,j,\lambda)|a_{n+1}|}{1 - P(n,j,\lambda)|a_{n+1}|}, \quad z \in \mathbb{U},$$

where

$$0 < |a_{n+1}| < \frac{1 - P(n,j,\lambda)}{P(n,j,\lambda)} \tag{34}$$

and

$$P(n,j,\lambda) = \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2n+3-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2n+3+\lambda}{2}\right)(n+1)^j}. \tag{35}$$

Now, we consider five boundary points

$$z_1 = e^{-i\frac{\arg(a_{n+1})}{n}} \tag{36}$$

$$z_2 = e^{i\frac{\pi - 6\arg(a_{n+1})}{6n}} \tag{37}$$

$$z_3 = e^{i\frac{\pi - 4\arg(a_{n+1})}{4n}} \tag{38}$$

$$z_4 = e^{i\frac{\pi - 3\arg(a_{n+1})}{3n}} \tag{39}$$

and

$$z_5 = e^{i\frac{\pi - 2\arg(a_{n+1})}{2n}}. \tag{40}$$

For such $z_s (s = 1, 2, 3, 4, 5)$, *we have that*

$$\frac{A_{-j-\lambda} f(z_1)}{z_1} = 1 + P(n, j, \lambda)|a_{n+1}|, \tag{41}$$

$$\frac{A_{-j-\lambda} f(z_2)}{z_2} = 1 + P(n, j, \lambda)|a_{n+1}|\frac{\sqrt{3} + i}{2}, \tag{42}$$

$$\frac{A_{-j-\lambda} f(z_3)}{z_3} = 1 + P(n, j, \lambda)|a_{n+1}|\frac{\sqrt{2}(1 + i)}{2}, \tag{43}$$

$$\frac{A_{-j-\lambda} f(z_4)}{z_4} = 1 + P(n, j, \lambda)|a_{n+1}|\frac{1 + \sqrt{3}i}{2}, \tag{44}$$

and

$$\frac{A_{-j-\lambda} f(z_5)}{z_5} = 1 + P(n, j, \lambda)|a_{n+1}|i. \tag{45}$$

It follows from the above that

$$\alpha_5 = \frac{1}{5}\sum_{s=1}^{5} \frac{A_{-j-\lambda} f(z_s)}{z_s} = 1 + \frac{(3 + \sqrt{2} + \sqrt{3})P(n, j, \lambda)|a_{n+1}|(1 + i)}{10} \tag{46}$$

and that

$$\left|1 - e^{-i\beta}\alpha_5\right| = \frac{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})P(n, j, \lambda)|a_{n+1}|}{10} \tag{47}$$

with $\beta = 0$. *For such* α_5 *and* β, *we consider* $\rho > 1$ *with*

$$\frac{nP(n, j, \lambda)|a_{n+1}|}{1 - P(n, j, \lambda)|a_{n+1}|} \leq \frac{|e^{i\beta} - \alpha_5|n\rho}{1 + |e^{i\beta} - \alpha_5|\rho}. \tag{48}$$

This gives us that

$$\rho \geq \frac{10}{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})(1 - (1 + |a_{n+1}|)P(n, j, \lambda))} > \frac{10}{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})} > 1. \tag{49}$$

For such α_5 *and* $\rho > 1$, *the function* f *satisfies*

$$\left|\frac{A_{-j-\lambda} f(z)}{z} - 1\right| < P(n, j, \lambda)|a_{n+1}| \leq \rho|e^{i\beta} - \alpha_5|, \quad z \in \mathbb{U}.$$

Next, we derive the following theorem.

Theorem 2 *If* $f \in T_n$ *satisfies*

$$\left| \left(\frac{A_{-j-\lambda+1} f(z)}{A_{-j-\lambda} f(z)} - 1 \right) \left(\frac{A_{-j-\lambda} f(z)}{z} - 1 \right) \right| < \frac{\left| e^{i\beta} - \alpha_m \right|^2 n\rho^2}{1 + |e^{i\beta} - \alpha_m| \, \rho}, \quad z \in \mathbb{U} \tag{50}$$

for some α_m *defined by* (12) *with* $\alpha_m \neq 1$ *and for some real* $\rho > 1$, *then*

$$\left| \frac{A_{-j-\lambda} f(z)}{z} - 1 \right| < \rho \left| e^{i\beta} - \alpha_m \right|, \quad z \in \mathbb{U} \tag{51}$$

that is, $f \in B_n(\alpha_m, \beta, \rho; j, \lambda)$.

Proof. Define the function w by (22). Applying (25), our condition (50) leads us that

$$\begin{aligned} \left| \left(\frac{A_{-j-\lambda+1} f(z)}{A_{-j-\lambda} f(z)} - 1 \right) \left(\frac{A_{-j-\lambda} f(z)}{z} - 1 \right) \right| &= \left| \frac{\left(1 - e^{-i\beta} \alpha_m\right)^2 z w(z) w'(z)}{1 + \left(1 - e^{-i\beta} \alpha_m\right) w(z)} \right| \\ &\leq \frac{\left| e^{i\beta} - \alpha_m \right|^2 n\rho^2}{1 + |e^{i\beta} - \alpha_m| \, \rho}, \quad z \in \mathbb{U}. \end{aligned} \tag{52}$$

Suppose that there exists a point $z_0, (0 < |z_0| < 1)$ such that

$$\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1. \tag{53}$$

Then, applying Lemma 1, we write that $w(z_0) = \rho e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = k w(z_0), (k \geq n)$. This shows us that

$$\begin{aligned} \left| \left(\frac{A_{-j-\lambda+1} f(z)}{A_{-j-\lambda} f(z)} - 1 \right) \left(\frac{A_{-j-\lambda} f(z)}{z} - 1 \right) \right| &= \left| \frac{\left(1 - e^{-i\beta} \alpha_m\right)^2 z_0 w(z_0) w'(z_0)}{1 + \left(1 - e^{-i\beta} \alpha_m\right) w(z_0)} \right| \\ &= \frac{\left| e^{i\beta} - \alpha_m \right|^2 \rho^2 k}{|1 + (1 - e^{-i\beta} \alpha_m) \rho e^{i\theta}|} \\ &\geq \frac{\left| e^{i\beta} - \alpha_m \right|^2 n\rho^2}{1 + |e^{i\beta} - \alpha_m| \, \rho} \end{aligned} \tag{54}$$

which contradicts our condition (50). Thus there is no $z_0, (0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. This shows us that

$$\left| \left(\frac{A_{-j-\lambda} f(z)}{z} - 1 \right) \right| < \rho |e^{i\beta} - \alpha_m|, \quad z \in \mathbb{U}. \tag{55}$$

□

Example 2 *Consider a function* $f \in T_n$ *given by*

$$f(z) = z + a_{n+1}z^{n+1}\ , z \in \mathbb{U} \tag{56}$$

with $0 < |a_{n+1}| < \frac{1}{P(n,j,\lambda)}$, *where* $P(n, j, \lambda)$ *is given by* (35). *It follows that*

$$\begin{aligned}\left|\left(\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1\right)\left(\frac{A_{-j-\lambda}f(z)}{z}-1\right)\right| &= \left|\frac{nP(n,j,\lambda)^2 a_{n+1}^2 z^{2n}}{1+P(n,j,\lambda)a_{n+1}z^n}\right| \\ &< \frac{nP(n,j,\lambda)^2|a_{n+1}|^2}{1-P(n,j,\lambda)|a_{n+1}|}, \quad z \in \mathbb{U}.\end{aligned} \tag{57}$$

Considering five boundary points z_1, z_2, z_3, z_4 *and* z_5 *in Example* 1, *we see that*

$$\left|e^{i\beta} - \alpha_5\right| = \frac{\sqrt{2}(3+\sqrt{2}+\sqrt{3})P(n,j,\lambda)|a_{n+1}|}{10} \tag{58}$$

with $\beta = 0$. *If we consider* $\rho > 1$ *such that*

$$\frac{nP(n,j,\lambda)^2|a_{n+1}|^2|z|}{1-P(n,j,\lambda)|a_{n+1}|} \leq \frac{\left|e^{i\beta}-\alpha_5\right|^2 n\rho^2}{1+|e^{i\beta}-\alpha_5|\,\rho}, \tag{59}$$

then ρ *satisfies*

$$\rho \geq \frac{10}{\sqrt{2}(3+\sqrt{2}+\sqrt{3})P(n,j,\lambda)|a_{n+1}|} > 1. \tag{60}$$

For such α_5 *and* ρ, f *satisfies*

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < P(n,j,\lambda)|a_{n+1}| \leq \rho|e^{i\beta} - \alpha_5|, \quad z \in \mathbb{U}. \tag{61}$$

Our next result reads as follows.

Theorem 3 *If* $f \in T_n$ *satisfies*

$$\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|(n+1), \quad z \in \mathbb{U}. \tag{62}$$

for some α_m *defined by* (12) *with* $\alpha_m \neq 1$ *and for some real* $\rho > 1$, *then*

$$\left|\frac{A_{-j-\lambda+p-1}f(z)}{z} - 1\right| < \rho\left|e^{i\beta} - \alpha_m\right|, \quad z \in \mathbb{U} \tag{63}$$

where $p = 0, 1, 2, \ldots, j$.

Proof. We consider the function w defined by

$$w(z) = \frac{e^{i\beta}\frac{A_{-j-\lambda+p-1}f(z)}{z} - \alpha_m}{e^{i\beta} - \alpha_m} - 1$$
$$= \frac{e^{i\beta}}{e^{i\beta} - \alpha_m}\left\{\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right)k^{j-p+1}} a_k z^{k-1}\right\}. \tag{64}$$

Thus w is analytic in $\mathbb{U}$, $w(0) = 0$, and

$$A_{-j-\lambda+p-1}f(z) = z + (1 - e^{-i\beta}\alpha_m)zw(z). \tag{65}$$

Noting that

$$A_{-j-\lambda+p}f(z) = z\left(A_{-j-\lambda+p-1}f(z)\right)'$$
$$= z\left\{1 + (1 - e^{-i\beta}\alpha_m)w(z)\left(1 + \frac{zw'(z)}{w(z)}\right)\right\}, \tag{66}$$

we have that

$$\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| = \left|1 - e^{-i\beta}\alpha_m\right| |w(z)| \left|1 + \frac{zw'(z)}{w(z)}\right|$$
$$< \rho\left|e^{i\beta} - \alpha_m\right|(n+1), \quad z \in \mathbb{U} \tag{67}$$

by the condition (62). Suppose that there exists a point $z_0, (0 < |z_0| < 1)$ such that

$$\max\{|w(z)|; |z| \le |z_0|\} = |w(z_0)| = \rho > 1. \tag{68}$$

Then, letting $w(z_0) = \rho e^{i\theta}, (0 \le \theta \le 2\pi)$ and $z_0 w'(z_0) = kw(z_0), (k \ge n)$ with Lemma 1, we see that

$$\left|\frac{A_{-j-\lambda+p}f(z_0)}{z_0} - 1\right| = \rho\left|e^{i\beta} - \alpha_m\right|(k+1) \ge \rho\left|e^{i\beta} - \alpha_m\right|(n+1). \tag{69}$$

This contradicts the inequality (67). Therefore, we don't have any $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho > 1$. This shows us that

$$|w(z)| = \left|\frac{\alpha_m}{e^{i\beta} - \alpha_m}\left(\frac{A_{-j-\lambda+p-1}f(z)}{z} - 1\right)\right| < \rho, \quad z \in \mathbb{U}, \tag{70}$$

that is, that

$$\left|\frac{A_{-j-\lambda+p-1}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|, \quad z \in \mathbb{U}. \tag{71}$$

This completes the proof of our theorem. □

Corollary 1 *If* $f \in T_n$ *satisfies*

$$\left| \frac{A_{-j-\lambda+p} f(z)}{z} - 1 \right| < \rho |e^{i\beta} - \alpha_m| (n+1)^p, \quad z \in \mathbb{U} \tag{72}$$

for some α_m *given by* (12) *with* $\alpha_m \neq 1$, *and for some real* $\rho > 1$, *then*

$$\left| \frac{A_{-j-\lambda} f(z)}{z} - 1 \right| < \rho |e^{i\beta} - \alpha_m|, \quad z \in \mathbb{U} \tag{73}$$

where $p = 0, 1, 2, ..., j$.

Proof. With Theorem 3, we say that if $f \in T_n$ satisfies

$$\left| \frac{A_{-j-\lambda+p} f(z)}{z} - 1 \right| < \rho |e^{i\beta} - \alpha_m| (n+1)^p, \quad z \in \mathbb{U}, \tag{74}$$

then

$$\left| \frac{A_{-j-\lambda+p-1} f(z)}{z} - 1 \right| < \rho |e^{i\beta} - \alpha_m| (n+1)^{p-1}, \quad z \in \mathbb{U}. \tag{75}$$

Further, we have that

$$\left| \frac{A_{-j-\lambda+p-2} f(z)}{z} - 1 \right| < \rho |e^{i\beta} - \alpha_m| (n+1)^{p-2}, \quad z \in \mathbb{U}, \tag{76}$$

from (75). Finally, we obtain that

$$\left| \frac{A_{-j-\lambda} f(z)}{z} - 1 \right| < \rho |e^{i\beta} - \alpha_m|, \quad z \in \mathbb{U}. \tag{77}$$

□

Example 3 *Consider the function* $f \in T_n$ *given by*

$$f(z) = z + a_{n+1} z^{n+1}, z \in \mathbb{U}. \tag{78}$$

Since

$$A_{-j-\lambda+p} f(z) = z + \frac{\Gamma\left(\frac{3+\lambda}{2}\right) \Gamma\left(\frac{2n+3-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right) \Gamma\left(\frac{2n+3+\lambda}{2}\right) (n+1)^{j-p+2}} a_{n+1} z^{n+1}, \tag{79}$$

we have

$$\left| \frac{A_{-j-\lambda+p} f(z)}{z} - 1 \right| = \left| P(n,j,\lambda)(n+1)^{p-2} a_{n+1} z^n \right| < P(n,j,\lambda)(n+1)^{p-2} |a_{n+1}| \tag{80}$$

where

$$0 < |a_{n+1}| < \frac{1}{P(n,j,\lambda)} \tag{81}$$

and $P(n,j,\lambda)$ *is given by* (35).

Consider five boundary points z_1, z_2, z_3, z_4 *and* z_5 *in Example* 1. *Then* α_5 *satisfies* (46) *and* $|1-e^{-i\beta}\alpha_5|$ *satisfies* (47) *for* $\beta = 0$. *For such* α_5 *and* β, *we consider* $\rho > 1$ *by*

$$\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| < P(n,j,\lambda)(n+1)^{p-2}|a_{n+1}| \leq \rho\left|e^{i\beta} - \alpha_5\right|(n+1)^{p-2}, \quad z \in \mathbb{U}, \tag{82}$$

Then ρ *satisfies*

$$\rho \geq \frac{P(n,j,\lambda)|a_{n+1}|}{|e^{i\beta} - \alpha_5|} = \frac{10}{\sqrt{2}(3+\sqrt{2}+\sqrt{3})} > 1. \tag{83}$$

With the above α_5 *and* ρ, *we have*

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < P(n,j,\lambda)|a_{n+1}| \leq \rho\left|e^{i\beta} - \alpha_5\right|, \quad z \in \mathbb{U}. \tag{84}$$

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