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Generalized Möbius-type functions and special set of k-free numbers

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Abstract. In [3] Bege introduced the generalized Apostol's Möbius functions $\mu_{k,m}(n)$. In this paper we present new properties of these functions. By introducing the special set of k-free numbers, we have obtained some asymptotic formulas for the partial sums of these functions.

1 Introduction

Möbius function of order k, introduced by T. M. Apostol [1], is defined by the following formula:

$$\mu_k(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^{k+1} \mid n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1^k \cdots p_r^k \prod_{i>r} p_i^{\alpha_i}, \quad \text{with } 0 \le \alpha_i < k, \\ 1 & \text{otherwise.} \end{cases}$$

The generalized function is denoted by $\mu_{k,m}(n)$, where $1 < k \le m$.
If $m = k$, $\mu_{k,k}(n)$ is defined to be $\mu_k(n)$, and if $m > k$ the function is defined as follows:

$$\mu_{k,m}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if } p^k \nmid n \text{ for each prime } p, \\ (-1)^r & \text{if } n = p_1^m \cdots p_r^m \prod_{i>r} p_i^{\alpha_i}, \quad \text{with } 0 \le \alpha_i < k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

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In this paper we show some relations that hold among the functions $\mu_{k,m}(n)$. We introduce the new type of k-free integers and we make a connection between generalized Möbius function and the characteristic function $q^*_{k,m}(n)$ of these. We use these to derive an asymptotic formula for the summatory function of $q^*_{k,m}(n)$.

2 Basic lemmas

The generalization $\mu_{k,m}$, like Apostol's $\mu_k(n)$, is a multiplicative function of n, so it is determined by its values at the prime powers. We have

$$\mu_k(p^\alpha) = \begin{cases} 1 & \text{if} \quad 0 \le \alpha < k, \\ -1 & \text{if} \quad \alpha = k, \\ 0 & \text{if} \quad \alpha > k, \end{cases}$$

whereas

$$\mu_{k,m}(p^\alpha) = \begin{cases} 1 & \text{if} \quad 0 \le \alpha < k, \\ 0 & \text{if} \quad k \le \alpha < m, \\ -1 & \text{if} \quad \alpha = m, \\ 0 & \text{if} \quad \alpha > m. \end{cases} \tag{2}$$

In [1] Apostol obtained the asymptotic formula

$$\sum_{n \le x} \mu_k(n) = A_k x + O(x^{\frac{1}{k}} \log x), \tag{3}$$

where

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right).$$

Later, Suryanarayana [5] showed that, on the assumption of the Riemann hypothesis, the error term in (3) can be improved to

$$O\left(x^{\frac{4k}{4k^2+1}} \omega(x)\right), \tag{4}$$

where

$$\omega(x) = \exp\{A \log x (\log\log x)^{-1}\}$$

for some positive constant k.
In 2001 A. Bege [3] proved the following asymptotic formulas.

Lemma 1 ([3], Theorem 3.1.) *For* $x \geq 3$ *and* $m > k \geq 2$, *we have*

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \mu_{k,m}(r) = \frac{xn^2\,\alpha_{k,m}}{\zeta(k)\psi_k(n)\alpha_{k,m}(n)} + O\left(\theta(n)x^{\frac{1}{k}}\delta(x)\right) \tag{5}$$

uniformly in x, n *and* k, *where* $\theta(n)$ *is the number of square-free divisors of* n,

$$\alpha_{k,m} = \prod_p \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right),$$

$$\alpha_{k,m}(n) = n\prod_{p|n} \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right),$$

$$\psi_k(n) = n\prod_{p|n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{k-1}}\right),$$

and

$$\delta_k(x) = \ exp\,\{-A\,k^{-\frac{8}{5}}\log^{\frac{3}{5}} x\,(\log\log x)^{-\frac{1}{5}}\}, \quad A > 0.$$

Lemma 2 ([3], Theorem 3.2.) *If the Riemann hypothesis is true, then for* $x \geq 3$ *and* $m > k \geq 2$ *we have*

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \mu_{k,m}(r) = \frac{xn^2\,\alpha_{k,m}}{\zeta(k)\psi_k(n)\alpha_{k,m}(n)} + O\left(\theta(n)x^{\frac{2}{2k+1}}\omega(x)\right) \tag{6}$$

uniformly in x, n *and* k.

Lemma 3 ([2]) *If* $s > 0$, $s \neq 1$, $x \geq 1$, *then*

$$\sum_{n \leq x} \frac{1}{n^s} = \zeta(s) - \frac{1}{(s-1)x^{s-1}} + O\left(\frac{1}{x^s}\right).$$

3 Generalized k-free numbers

Let Q_k denote the set of k-free numbers and let $q_k(n)$ to be the characteristic function of this set. Cohen [4] introduced the Q_k^* set, the set of positive

integers n with the property that the multiplicity of each prime divisor of n is not a multiple of k. Let $q_k^*(n)$ be the characteristic function of these integers.

$$q_k^*(n) = \begin{cases} 1, & \text{if } n = 1 \\ 1, & \text{if } n = p_1^{\alpha_1} \dots p_k^{\alpha_k},\ \alpha_i \not\equiv 0 \pmod{k} \\ 0, & \text{otherwise.} \end{cases}$$

We introduce the following special set of integers

$$\begin{aligned} Q_{k,m} := \ & \{n \mid n = n_1 \cdot n_2,\ (n_1, n_2) = 1,\ n_1 \in Q_k, \\ & n_2 = 1 \text{ or } n_2 = (p_1 \dots p_i)^m,\ p_i \in \mathbb{P}\}, \end{aligned}$$

with the characteristic function

$$q_{k,m}(n) = \begin{cases} 1, & \text{if } n \in Q_{k,m} \\ 0, & \text{if } n \notin Q_{k,m}. \end{cases}$$

The function $q_{k,m}(n)$ is multiplicative and

$$q_{k,m}(n) = |\mu_{k,m}(n)|. \tag{7}$$

We introduce the following set $Q_{k,m}^*$ which, in the generalization of Q_k^*. The integer n is in the set $Q_{k,m}^*$, $1 < k < m$ iff the power of each prime divisor of n divided by m has the remainder between 1 and $k-1$. The characteristic functions of these numbers is

$$q_{k,m}^*(n) = \begin{cases} 1, & \text{if } n = p_1^{\alpha_1} \dots p_k^{\alpha_k},\ \exists \ell :\ \ell m < \alpha_i < \ell m + k \\ 0, & \text{otherwise.} \end{cases}$$

If we write the generating functions for this functions, we have the following result.

Theorem 1 *If $m \geq k$ and the series converges absolutely, we have*

$$\sum_{n=1}^{\infty} \frac{\mu_{k,m}(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right), \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{q_{k,m}^*(n)}{n^s} = \zeta(s)\zeta(ms) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right), \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{q_{k,m}(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{1}{p^{ks}} + \frac{1}{p^{ms}} - \frac{1}{p^{(m+1)s}}\right). \tag{10}$$

Proof. Because the function $\mu_{k,m}(n)$ is multiplicative, when the series converges absolutely ($s > 1$), we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu_{k,m}(n)}{n^s} &= \prod_p \left(1 + \frac{\mu_{k,m}(p)}{p^s} + \ldots + \frac{\mu_{k,m}(p^\alpha)}{p^{\alpha s}} + \ldots\right) = \\
&= \prod_p \left(1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{(k-1)s}} - \frac{1}{p^{ms}}\right) = \\
&= \prod_p \frac{1}{1 - \frac{1}{p^s}} \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right) = \\
&= \zeta(s) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right).
\end{aligned}$$

In a similar way, because $q^*_{k,m}(n)$ is multiplicative, we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^*_{k,m}(n)}{n^s} &= \prod_p \left(1 + \frac{q^*_{k,m}(p)}{p^s} + \ldots + \frac{q^*_{k,m}(p^\alpha)}{p^{\alpha s}} + \ldots\right) = \\
&= \prod_p \left(1 + \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{(k-1)s}}\right) + \right. \\
&+ \left. \left(\frac{1}{p^{(m+1)s}} + \frac{1}{p^{(m+2)s}} \ldots + \frac{1}{p^{(m+k-1)s}}\right) + \ldots\right) = \\
&= \prod_p \left(1 + \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{(k-1)s}}\right)\left(1 + \frac{1}{p^{ms}} + \frac{1}{p^{2ms}} + \ldots\right)\right) \\
&= \prod_p \left(1 + \frac{\frac{1}{p^s} - \frac{1}{p^{ks}}}{1 - \frac{1}{p^s}} \frac{1}{1 - \frac{1}{p^{ms}}}\right) = \\
&= \zeta(s)\zeta(ms) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right).
\end{aligned}$$

Because $q_{k,m}(n)$ is multiplicative and $q_{k,m}(n) = |\mu_{k,m}(n)|$, we have:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q_{k,m}(n)}{n^s} &= \prod_p\left(1+\frac{q_{k,m}(p)}{p^s}+\ldots+\frac{q_{k,m}(p^\alpha)}{p^{\alpha s}}+\ldots\right)= \\ &= \prod_p\left(1+\frac{1}{p^s}+\ldots+\frac{1}{p^{(k-1)s}}+\frac{1}{p^{ms}}\right)= \\ &= \prod_p\frac{1}{1-\frac{1}{p^s}}\prod_p\left(1-\frac{1}{p^{ks}}+\frac{1}{p^{ms}}-\frac{1}{p^{(m+1)s}}\right)= \\ &= \zeta(s)\prod_p\left(1-\frac{1}{p^{ks}}+\frac{1}{p^{ms}}-\frac{1}{p^{(m+1)s}}\right).\end{aligned}$$

■

In the particular case when $m = k$, we have $\mu_{k,m}(n) = \mu_k(n)$, $q_{k,m}(n) = q_{k+1}(n)$ and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s} &= \zeta(s)\prod_p\left(1-\frac{2}{p^{ks}}+\frac{1}{p^{(k+1)s}}\right), \\ \sum_{n=1}^{\infty}\frac{q_{k+1}(n)}{n^s} &= \frac{\zeta(s)}{\zeta\big((k+1)s\big)}.\end{aligned}$$

We have the following convolution type formulas.

Theorem 2 *If* $m \geq k$

$$q^*_{k,m}(n) = \sum_{d^m\delta=n}\mu_{k,m}(\delta), \tag{11}$$

$$\mu_{k,m}(n) = \sum_{d^m\delta=n}\mu(d)q^*_{k,m}(\delta). \tag{12}$$

Proof. Because $q_{k,m}(n)$ and $\mu_{k,m}(n)$ are multiplicative, it results that both sides of (11) are multiplicative functions. Hence it is enough if we verify the identity for $n = p^\alpha$, a prime power.
If $\alpha = \ell m + i$ and $0 < i < k$

$$\begin{aligned}\sum_{d^m\delta=p^\alpha}\mu_{k,m}(\delta) &= \mu_{k,m}(p^{\ell m+i})+\mu_{k,m}(p^{(\ell-1)m+i})+\ldots+\mu_{k,m}(p^{m+i})+ \\ &+ \mu_{k,m}(p^i)=1=q_{k,m}(p^\alpha).\end{aligned}$$

If $\alpha = \ell m + i$ and $k < i < m$, then

$$\begin{aligned}\sum_{d^m\,\delta=p^\alpha} \mu_{k,m}(\delta) &= \mu_{k,m}(p^{\ell m+i}) + \mu_{k,m}(p^{(\ell-1)m+i}) + \ldots + \mu_{k,m}(p^{m+i}) + \\ &+ \mu_{k,m}(p^i) = 0 = q_{k,m}(p^\alpha).\end{aligned}$$

If $\alpha = \ell m$

$$\begin{aligned}\sum_{d^m\,\delta=p^\alpha} \mu_{k,m}(\delta) &= \mu_{k,m}(p^{\ell m}) + \mu_{k,m}(p^{(\ell-1)m}) + \ldots + \mu_{k,m}(p^m) + \mu_{k,m}(1) = \\ &= -1 + 1 = 0 = q_{k,m}(p^\alpha).\end{aligned}$$

(12) results from the Möbius inversion formula.
■

4 Asymptotic formulas

Theorem 3 *For $x \geq 3$ and $m > k \geq 2$, we have*

$$\sum_{r \leq x} q^*_{k,m}(r) = \frac{x\alpha_{k,m}\zeta(m)}{\zeta(k))} + 0\left(x^{\frac{1}{k}}\delta(x)\right) \tag{13}$$

uniformly in x, n and k, where

$$\alpha_{k,m} = \prod_p \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right)$$

$$\delta(x) = \ exp\,\{-A\ \log^{\frac{3}{5}} x\ (\log\log x)^{-\frac{1}{5}}\},$$

for some absolute constant $A > 0$.

Proof. Based on (11) and (5) with $n = 1$, we have

$$\sum_{r \leq x} q^*_{k,m}(n) = \sum_{\delta d^m \leq x} \mu_{k,m}(\delta) = \sum_{d \leq x^{\frac{1}{m}}} \sum_{\delta \leq \frac{x}{d^m}} \mu_{k,m}(\delta) =$$

$$= \sum_{d \leq x^{\frac{1}{m}}} \left\{\frac{\left(\frac{x}{d^m}\right)\alpha_{k,m}}{\zeta(k)} + 0\left(\frac{x^{\frac{1}{k}}}{d^{\frac{m}{k}}}\delta\left(\frac{x}{d^m}\right)\right)\right\} =$$

$$= \frac{x\alpha_{k,m}}{\zeta(k)} \sum_{d \le x^{\frac{1}{m}}} \frac{1}{d^m} + O\left(\delta(x) x^{\epsilon} x^{\frac{1}{k}-\epsilon} \sum_{d \le x^{\frac{1}{m}}} \frac{1}{d^{\frac{m}{k}-\epsilon m}} \right).$$

Now we use (3), and the fact that $\delta(x)x^{\epsilon}$ is increasing for all $\epsilon > 0$, we choose $\epsilon > 0$, so that $\frac{m}{k} - \epsilon m > 1 + \epsilon'$ and we obtain (13). ■

Applying the method used to prove Theorem 1, and making use of Lemma 2, we get

Theorem 4 *If the Riemann hypothesis is true, then for* $x \ge 3$ *and* $m > k \ge 2$ *we have*

$$\sum_{r \le x} q^*_{k,m}(r) = \frac{x\alpha_{k,m}\zeta(m)}{\zeta(k)} + O\left(x^{\frac{2}{2k+1}} \omega(x)\right) \tag{14}$$

uniformly in x, n *and* k.

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