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Generalizations of Lindelöf spaces via hereditary classes

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Abstract. In this paper by using hereditary classes [6], we define the notion of γ-Lindelöf modulo hereditary classes called $\gamma\mathcal{H}$-Lindelöf and obtain several properties of $\gamma\mathcal{H}$-Lindelöf spaces.

1 Introduction

Let (X, τ) be a topological space and $\mathcal{P}(X)$ the power set of X. In 1991, Ogata [13] introduced the notions of γ-operations and γ-open sets and investigated the associated topology τ_γ and weak separation axioms γ-T_i $(i = 0, 1/2, 1, 2)$. In 2011, Noiri [10] defined an operation on an m-structure with property $\mathcal{B}$ (the generalized topology in the sense of Lugojan [8]). The operation is defined as a function $\gamma : m \to \mathcal{P}(X)$ such that $U \subseteq \gamma(U)$ for each $U \in m$ and is called a γ-operation on m. Then, it terns out that the operation is an unified form of several operations (for example, semi-γ-operation [7], pre-γ-operation [4]) defined on the family of generalized open sets. Moreover, he obtained some characterizations of γ-compactness and suggested some generalizations of compact spaces by using recent modifications of open sets in a topological space.

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In this paper by using hereditary classes [6], we define the notion of γ-Lindelöf modulo hereditary classes called $\gamma\mathcal{H}$-Lindelöf and obtain several properties of $\gamma\mathcal{H}$-Lindelöf spaces. Recently papers [1, 2, 3] have introduced some new classes of sets via hereditary classes.

2 Preliminaries

First we state the following: in [11], a minimal structure $\mathfrak{m}$ is defined as follows: $\mathfrak{m}$ is called a minima structure if $\emptyset, X \in \mathfrak{m}$. However, in this paper, we define as follows:

Definition 1 *Let* X *be a nonempty set and* $\mathcal{P}(X)$ *the power set of* X*. A subfamily* $\mathfrak{m}$ *of* $\mathcal{P}(X)$ *is called a minimal structure (briefly* $\mathfrak{m}$*-structure) on* X *if* $\mathfrak{m}$ *satisfies the following conditions:*

1. $\emptyset, X \in \mathfrak{m}$.
2. *The union of any family of subsets belonging to* $\mathfrak{m}$ *belongs to* $\mathfrak{m}$*.*

A set X with an $\mathfrak{m}$-structure is called an $\mathfrak{m}$-space and denoted by $(X, \mathfrak{m})$. Each member of $\mathfrak{m}$ is said to be $\mathfrak{m}$-open and the complement of an $\mathfrak{m}$-open set is said to be $\mathfrak{m}$-closed.

Definition 2 [9] *Let* X *be a nonempty set and* $\mathfrak{m}$ *an* $\mathfrak{m}$*-structure on* X*. For a subset* A *of* X*, the* $\mathfrak{m}$*-closure of* A *is defined as follows:* $\mathfrak{m}cl(A) = \cap\{F : A \subseteq F, X \setminus F \in \mathfrak{m}\}$.

Lemma 1 [9] *Let* X *be a nonempty set and* $\mathfrak{m}$ *an* $\mathfrak{m}$*-structure on* X*. For the* $\mathfrak{m}$*-closure, the following properties hold, where* A *and* B *are subsets of* X*:*

1. $A \subseteq \mathfrak{m}cl(A)$,
2. $\mathfrak{m}cl(\emptyset) = \emptyset$, $\mathfrak{m}cl(X) = X$,
3. *If* $A \subseteq B$*, then* $\mathfrak{m}cl(A) \subseteq \mathfrak{m}cl(B)$,
4. $\mathfrak{m}cl(\mathfrak{m}cl(A)) = \mathfrak{m}cl(A)$.

Lemma 2 [14] *Let* $(X, \mathfrak{m})$ *be an* $\mathfrak{m}$*-space and* A *a subset of* X*. Then* $x \in \mathfrak{m}cl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in \mathfrak{m}$ *containing* x*.*

Lemma 3 [15] *Let* (X, m) *be an* m*-space and* A *a subset of* X*. Then, the following properties hold:*

1. A *is* m*-closed if and only if* $mcl(A) = A$*,*

2. $mcl(A)$ *is* m*-closed.*

Definition 3 [10] *Let* (X, m) *be an* m*-space and* γ *an operation on* m*. A subset* A *of* X *is said to be* γ*-open if for each* $x \in A$ *there exists* $U \in m$ *such that* $x \in U \subseteq \gamma(U) \subseteq A$*. The complement of a* γ*-open set is said to be* γ*-closed. The family of all* γ*-open sets of* (X, m) *is denoted by* $\gamma(X)$*.*

3 $\gamma\mathcal{H}$-Lindelöf spaces

First, we recall the definition of a hereditary class used in the sequel. A subfamily $\mathcal{H}$ of the power set $\mathcal{P}(X)$ is called a hereditary class on X [6] if it satisfies the following property: $A \in \mathcal{H}$ and $B \subseteq A$ implies $B \in \mathcal{H}$.

Definition 4 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space and* γ *an operation on* m*, where* $\mathcal{H}$ *is a hereditary class on* X*. Then* m*-space* (X, m) *is said to be* $\gamma\mathcal{H}$*-Lindelöf (resp.* $\mathcal{H}$*-Lindelöf) if every cover* $\{U_\alpha : \alpha \in \Delta\}$ *of* X *by* m*-open sets, there exists a countable subset* Δ_0 *of* Δ *such that* $X \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ *(resp.* $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$*).*

Theorem 1 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space and* γ *an operation on* m*, where* $\mathcal{H}$ *is a hereditary class. Then the following properties are equivalent:*

1. $(X, \gamma(X))$ *is* $\mathcal{H}$*-Lindelöf;*

2. *For every family* $\{F_\alpha : \alpha \in \Delta\}$ *of* γ*-closed sets such that* $\cap\{F_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}$ *for every countable subfamily* Δ_0 *of* Δ*,* $\cap\{F_\alpha : \alpha \in \Delta\} \neq \emptyset$*.*

Proof. (1) $\Rightarrow$ (2): Let $(X, \gamma(X))$ be $\mathcal{H}$-Lindelöf. Suppose that $\cap\{F_\alpha : \alpha \in \Delta\} = \emptyset$, where F_α is γ-closed set. Then $X \setminus F_\alpha$ is γ-open for each $\alpha \in \Delta$ and $\cup_{\alpha\in\Delta}(X \setminus F_\alpha) = X \setminus \cap_{\alpha\in\Delta}(F_\alpha) = X$. By (1), there exists a countable subfamily Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0}(X \setminus F_\alpha) = \cap\{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This is a contradiction.
(2) $\Rightarrow$ (1): Suppose that $(X, \gamma(X))$ is not $\mathcal{H}$-Lindelöf. There exists a cover $\{U_\alpha : \alpha \in \Delta\}$ of X by γ-open sets such that $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}$ for

every countable subset Δ_0 of Δ. Since $X \setminus U_\alpha$ is γ-closed for each $\alpha \in \Delta$ and $\cap\{(X \setminus U_\alpha) : \alpha \in \Delta_0\} \notin \mathcal{H}$ for every countable subset Δ_0 of Δ. By (2), we have $\cap\{(X \setminus U_\alpha) : \alpha \in \Delta\} \neq \emptyset$. Therefore, $X \setminus \cup\{U_\alpha : \alpha \in \Delta\} \neq \emptyset$. This is contrary that $\{U_\alpha : \alpha \in \Delta\}$ is a γ-open cover of X.

□

Lemma 4 [10] *Let* (X, m) *be an* m*-space. For* $\gamma(X)$*, the following properties hold:*

1. $\emptyset$, $X \in \gamma(X)$,
2. *If* $A_\alpha \in \gamma(X)$ *for each* $\alpha \in \Lambda$*, then* $\cup_{\alpha\in\Lambda} A_\alpha \in \gamma(X)$,
3. $\gamma(X) \subseteq m$.

Definition 5 [10] *An* m*-space* (X, m) *is said to be* γ*-regular if for each* $x \in X$ *and each* $U \in m$ *containing* x*, there exists* $V \in m$ *such that* $x \in V \subseteq \gamma(V) \subseteq U$.

Lemma 5 [10] *For an* m*-space* (X, m)*, the following properties are equivalent:*

1. $m = \gamma(X)$;
2. (X, m) *is* γ*-regular;*
3. *For each* $x \in X$ *and each* $U \in m$ *containing* x*, there exists* $W \in \gamma(X)$ *such that* $x \in W \subseteq \gamma(W) \subseteq U$.

Theorem 2 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space and* γ *an operation on* m*, where* $\mathcal{H}$ *is a hereditary class. The implications* $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ *hold. If* (X, m) *is* γ*-regular, then the following properties are equivalent:*

1. (X, m) *is* $\mathcal{H}$*-Lindelöf;*
2. (X, m) *is* $\gamma\mathcal{H}$*-Lindelöf;*
3. $(X, \gamma(X))$ *is* $\mathcal{H}$*-Lindelöf;*
4. $(X, \gamma(X))$ *is* $\gamma\mathcal{H}$*-Lindelöf.*

Proof. (1) $\Rightarrow$ (2): Let (X, m) be $\mathcal{H}$-Lindelöf. For any cover $\{U_\alpha : \alpha \in \Delta\}$ of X by m-open sets, there exists a countable subset Δ_0 of Δ such that $X \setminus \cup\{\gamma(U\alpha) : \alpha \in \Delta_0\} \subseteq X \setminus \cup\{U\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore, (X, m) is $\gamma\mathcal{H}$-Lindelöf.

(2) $\Rightarrow$ (3): Let $(X, \mathfrak{m})$ be $\gamma\mathcal{H}$-Lindelöf and $\{U_\alpha : \alpha \in \Delta\}$ a cover of X by γ-open sets. For each $x \in X$ there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is γ-open, there exists $V_{\alpha(x)} \in \mathfrak{m}$ such that $x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}$. Since the family $\{V_{\alpha(x)} : x \in X\}$ is a cover of X by $\mathfrak{m}$-open sets and $(X, \mathfrak{m})$ is $\gamma\mathcal{H}$-Lindelöf, there exists a countable number of points, say, $x_1, x_2, x_3, \cdots \in X$ such that $X \setminus \cup_{i=1}^{\infty} \gamma(V_{\alpha(x_i)}) \in \mathcal{H}$ and hence $X \setminus \cup_{i=1}^{\infty} U_{\alpha(x_i)} \in \mathcal{H}$. This shows that $(X, \gamma(X))$ is $\mathcal{H}$-Lindelöf.

(3) $\Rightarrow$ (4): By Lemma 4, $\gamma(X)$ is an $\mathfrak{m}$-structure and it follows that the same argument as (1) $\Rightarrow$ (2) that $(X, \gamma(X))$ is $\gamma\mathcal{H}$-Lindelöf.

(4) $\Rightarrow$ (1): Suppose that $(X, \mathfrak{m})$ is γ-regular. Let $(X, \gamma(X))$ be $\gamma\mathcal{H}$-Lindelöf. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of X by $\mathfrak{m}$-open sets. For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since $(X, \mathfrak{m})$ is γ-regular, by Lemma 5 there exists $V_{\alpha(x)} \in \gamma(X)$ such that $x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : x \in X\}$ is a cover of X by γ-open sets and $(X, \gamma(X))$ is $\gamma\mathcal{H}$-Lindelöf, there exist a countable number of points, say, $x_1, x_2, x_3, \cdots \in X$ such that $X \setminus \cup_{i=1}^{\infty} \gamma(V_{\alpha(x_i)}) \in \mathcal{H}$; and hence $X \setminus \cup_{i=1}^{\infty} U_{\alpha(x_i)} \in \mathcal{H}$. This shows that $(X, \mathfrak{m})$ is $\mathcal{H}$-Lindelöf. $\square$

Definition 6 *Let $(X, \mathfrak{m}, \mathcal{H})$ be a hereditary $\mathfrak{m}$-space. A subset A of X is said to be $\gamma\mathcal{H}$-Lindelöf relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by $\mathfrak{m}$-open sets of X, there exists a countable subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$.*

Theorem 3 *Let $(X, \mathfrak{m}, \mathcal{H})$ be a hereditary $\mathfrak{m}$-space. If A is γ-closed and B is $\gamma\mathcal{H}$-Lindelöf relative to X, then $A \cap B$ is $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by $\mathfrak{m}$-open subsets of X. Then $\{V_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ is a cover of B by $\mathfrak{m}$-open sets. Since $X \setminus A$ is γ-open, for each $x \in X \setminus A$, there exists an $\mathfrak{m}$-open set V_x such that $x \in V_x \subseteq \gamma(V_x) \subseteq X \setminus A$. Thus $\{V_\alpha : \alpha \in \Delta\} \cup \{V_x : x \in X \setminus A\}$ is a cover of B by $\mathfrak{m}$-open sets of X. Since B is $\gamma\mathcal{H}$-Lindelöf relative to X, there exist a countable subset Δ_0 of Δ and a countable points, says $x_1, x_2, \cdots \in X \setminus A$ such that $B \subseteq [(\cup_{\alpha \in \Delta_0} \gamma(V_\alpha)) \cup (\cup_{i=1}^{\infty} \gamma(V_{x_i}))] \cup H_0 \in \mathcal{H}$, where $H_0 \in \mathcal{H}$. Hence $A \cap B \subseteq [(\cup_{\alpha \in \Delta_0} \gamma(V_\alpha) \cap A) \cup (\cup_{i=1}^{\infty} \gamma(V_{x_i}) \cap A)] \cup (A \cap H_0) \subseteq \cup_{\alpha \in \Delta_0} \gamma(V_\alpha) \cup H_0$. Therefore, $A \cap B \setminus (\cup_{\alpha \in \Delta_0} \gamma(V_\alpha)) \subseteq H_0 \in \mathcal{H}$. Hence $A \cap B$ is $\gamma\mathcal{H}$-Lindelöf relative to X. $\square$

Corollary 1 *If a hereditary $\mathfrak{m}$-space $(X, \mathfrak{m}, \mathcal{H})$ is $\gamma\mathcal{H}$-Lindelöf space, then every γ-closed subset of X is $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. The proof is obvious by Theorem 3. □

Lemma 6 [12] *For a hereditary* $\mathfrak{m}$*-space* $(X, \mathfrak{m}, \mathcal{H})$*, the following properties hold:*

1. $\mathfrak{m}^*_H$ *is an* $\mathfrak{m}$*-structure on* X *such that* $\mathfrak{m}^*_H$ *has property* $\mathcal{B}$ *and* $\mathfrak{m} \subseteq \mathfrak{m}^*_H$.
2. $\beta(\mathfrak{m}, \mathcal{H}) = \{U \setminus H : U \in \mathfrak{m}, H \in \mathcal{H}\}$ *is a basis for* $\mathfrak{m}^*_H$*. such that* $\mathfrak{m} \subseteq \beta(\mathfrak{m}, \mathcal{H})$.

Theorem 4 *Let* $(X, \mathfrak{m}, \mathcal{H})$ *be a hereditary* $\mathfrak{m}$*-space. Then the following properties hold:*

1. *If* $(X, \mathfrak{m}^*_H, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf, then* $(X, \mathfrak{m}, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf.*
2. *If* $(X, \mathfrak{m}, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf and* $\mathcal{H}$ *is closed under countable union, then* $(X, \mathfrak{m}^*_H, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf.*

Proof. (1): The proof follows directly from the fact that every $\mathfrak{m}$-closed set is $\mathfrak{m}^*_H$-closed.

(2): Suppose that $\mathcal{H}$ is closed under countable union and X is $\mathcal{H}$-Lindelöf. Let $\{U_\alpha : \alpha \in \Delta\}$ be an $\mathfrak{m}^*_H$-open cover of X, then for each $x \in X$, $x \in U_{\alpha(x)}$ for some $\alpha(x) \in \Delta$. By Lemma 6 there exist $V_{\alpha(x)} \in \mathfrak{m}$ and $H_{\alpha(x)} \in \mathcal{H}$ such that $x \in V_{\alpha(x)} \setminus H_{\alpha(x)} \subseteq U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : \alpha(x) \in \Delta\}$ is an $\mathfrak{m}$-open cover of X, there exists a countable subset Δ_0 of Δ such that $X \setminus \cup\{V_{\alpha(x)} : \alpha(x) \in \Delta_0\} = H \in \mathcal{H}$. Since $\mathcal{H}$ is closed under countable union, then $\cup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\} \in \mathcal{H}$. Hence, $H \cup \left[\cup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\}\right] \in \mathcal{H}$. Observe that $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subseteq H \cup \left[\cup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\}\right] \in \mathcal{H}$. By the heredity property of $\mathcal{H}$ we have $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$ and therefore, $(X, \mathfrak{m}^*_H, \mathcal{H})$ is $\mathcal{H}$-Lindelöf. □

4 Strongly $\mathcal{H}$-Lindelöf spaces

Definition 7 *A subset* A *of a hereditary* $\mathfrak{m}$*-space* $(X, \mathfrak{m}, \mathcal{H})$ *is said to be:*

1. *Strongly* $\mathcal{H}$*-Lindelöf relative to* X *if for every family* $\{V_\alpha : \alpha \in \Delta\}$ *of* $\mathfrak{m}$*-open sets such that* $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$*, there exists a countable subset* Δ_0 *of* Δ *such that* $A \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$*. If* $A = X$*, then* $(X, \mathfrak{m}, \mathcal{H})$ *is said to be Strongly* $\mathcal{H}$*-Lindelöf.*

2. *Strongly $\gamma\mathcal{H}$-Lindelöf relative to* X *if for every family* $\{V_\alpha : \alpha \in \Delta\}$ *of* $\mathfrak{m}$*-open sets such that* $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$*, there exists a countable subset* Δ_0 *of* Δ *such that* $A \setminus \cup_{\alpha\in\Delta_0}\gamma(V_\alpha) \in \mathcal{H}$*. If* $A = X$*, then* $(X, \mathfrak{m}, \mathcal{H})$ *is said to be Strongly $\gamma\mathcal{H}$-Lindelöf.*

Theorem 5 *Let* $(X, \mathfrak{m}, \mathcal{H})$ *be a hereditary* $\mathfrak{m}$*-space. Then the following properties hold:*

1. *If* $(X, \mathfrak{m}^*_H, \mathcal{H})$ *is Strongly $\mathcal{H}$-Lindelöf, then* $(X, \mathfrak{m}, \mathcal{H})$ *is Strongly $\mathcal{H}$-Lindelöf.*
2. *If* $(X, \mathfrak{m}, \mathcal{H})$ *is Strongly $\mathcal{H}$-Lindelöf and $\mathcal{H}$ is closed under countable union, then* $(X, \mathfrak{m}^*_H, \mathcal{H})$ *is Strongly $\mathcal{H}$-Lindelöf.*

Theorem 6 *Let* $(X, \mathfrak{m}, \mathcal{H})$ *be a hereditary* $\mathfrak{m}$*-space. Then the following properties are equivalent:*

1. $(X, \mathfrak{m}, \mathcal{H})$ *is Strongly $\mathcal{H}$-Lindelöf;*
2. *If* $\{F_\alpha : \alpha \in \Delta\}$ *is a family of* $\mathfrak{m}$*-closed sets such that* $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$*, then there exists a countable subfamily* Δ_0 *of* Δ *such that* $\cap\{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$*.*

Proof. Suppose that $(X, \mathfrak{m}, \mathcal{H})$ is Strongly $\mathcal{H}$-Lindelöf. Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of $\mathfrak{m}$-closed sets such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{X \setminus F_\alpha : \alpha \in \Delta\}$ is a family of $\mathfrak{m}$-open sets of X. Let $H = \cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Let $X \setminus H = X \setminus \cap\{F_\alpha : \alpha \in \Delta\} = \cup\{X \setminus F_\alpha : \alpha \in \Delta\}$. Since $(X, \mathfrak{m}, \mathcal{H})$ is Strongly $\mathcal{H}$-Lindelöf, there exists a countable Δ_0 of Δ such that $X \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$.

Conversely, let $\{V_\alpha : \alpha \in \Delta\}$ be any family of $\mathfrak{m}$-open sets of X such that $X \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Then $\{X \setminus V_\alpha : \alpha \in \Delta\}$ is a family of $\mathfrak{m}$-closed sets of X. By assumption we have $\cap\{X \setminus V_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and there exists a countable subset Δ_0 of Δ such that $\cap\{X \setminus V_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that $X \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This shows that $(X, \mathfrak{m}, \mathcal{H})$ is Strongly $\mathcal{H}$-Lindelöf.

□

Definition 8 *A subset* A *of a hereditary* $\mathfrak{m}$*-space* $(X, \mathfrak{m}, \mathcal{H})$ *is said to be* $\mathfrak{m}\mathcal{H}_g$*-closed if for every* $U \in \mathfrak{m}$ *with* $A \setminus U \in \mathcal{H}$*,* $\mathfrak{m}cl(A) \subseteq U$*.*

Proposition 1 *Let* $(X, \mathfrak{m}, \mathcal{H})$ *be a hereditary* $\mathfrak{m}$*-space. If* $(X, \mathfrak{m}, \mathcal{H})$ *is Strongly $\mathcal{H}$-Lindelöf and* $A \subseteq X$ *is* $\mathfrak{m}\mathcal{H}_g$*-closed, then* A *is Strongly $\mathcal{H}$-Lindelöf relative to* X*.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of m-open subsets of X such that $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Since A is $m\mathcal{H}_g$-closed, $mcl(A) \subseteq \cup_{\alpha\in\Delta} V_\alpha$. Then $(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta} V_\alpha]$ is an m-open cover of X and so $X \setminus [(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta} V_\alpha]] \in \mathcal{H}$. Since X is Strongly $\mathcal{H}$-Lindelöf, there exists a countable subset Δ_0 of Δ such that $X \setminus [(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta_0} V_\alpha]] \in \mathcal{H}$. $X \setminus [(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta_0} V_\alpha]] = mcl(A) \cap (X \setminus \cup_{\alpha\in\Delta_0} V_\alpha) \supseteq A \setminus \cup_{\alpha\in\Delta_0} V_\alpha$. Therefore, $A \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. Thus, A is Strongly $\mathcal{H}$-Lindelöf relative to X. □

Theorem 7 *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Let A be an $m\mathcal{H}_g$-closed set such that $A \subseteq B \subseteq mcl(A)$. Then A is Strongly $\mathcal{H}$-Lindelöf elative to X if and only if B is Strongly $\mathcal{H}$-Lindelöf relative to X.*

Proof.
Suppose that A is Strongly $\mathcal{H}$-Lindelöf elative to X and $\{V_\alpha : \alpha \in \Delta\}$ is a family of m-open sets of X such that $B \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. By the heredity property, $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$ and A is Strongly $\mathcal{H}$-Lindelöf elative to X and hence there exists a countable subset Δ_0 of Δ such that $A \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. Since A is $m\mathcal{H}_g$-closed, $mcl(A) \subseteq \cup_{\alpha\in\Delta_0} V_\alpha$ and so $mcl(A) \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. This implies that $B \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$.

Conversely, suppose that B is Strongly $\mathcal{H}$-Lindelöf elative to X and $\{V_\alpha : \alpha \in \Delta\}$ is a family of m-open subsets of X such that $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Given that A is $m\mathcal{H}_g$-closed, $mcl(A) \setminus \cup_{\alpha\in\Delta} V_\alpha = \emptyset \in \mathcal{H}$ and this implies $B \subseteq \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Since B is Strongly $\mathcal{H}$-Lindelöf elative to X, there exists a countable subset Δ_0 of Δ such that $B \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. Hence $A \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. □

5 (γ, δ)-continuous functions

Definition 9 *Let (X, m) and (Y, n) be minimal spaces and γ (resp. δ) be an operation on m (resp. n). Then a function $f : (X, m) \to (Y, n)$ is said to be (γ, δ)-continuous if for each $x \in X$ and each $V \in n$ containing $f(x)$, there exists $U \in m$ containing x such that $f(\gamma(U)) \subseteq \delta(V)$.*

Lemma 7 *Let $f : X \to Y$ be a function.*

1. *If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H})$ is a hereditary class on Y.*
2. *If $\mathcal{H}$ is a hereditary class on Y, then $f^{-1}(\mathcal{H})$ is a hereditary class on X.*

Proof. (1): This is due to Lemma 3.8 of [5].
(2): Let $A \subseteq f^{-1}(H)$, where $H \in \mathcal{H}$. Then $f(A) \subseteq f(f^{-1}(H)) \subseteq H$. Hence $f(A) \in \mathcal{H}$ and $A \subseteq f^{-1}(f(A)) \in f^{-1}(\mathcal{H})$ and hence $A \in f^{-1}(\mathcal{H})$. □

Theorem 8 *Let* (X, m) *and* (Y, n) *be minimal spaces and* γ *(resp.* δ*) be an operation on* m *(resp.* n*) and* $\mathcal{H}$ *be a hereditary class on* X*. If* $(X, m, \mathcal{H})$ *is* $\gamma\mathcal{H}$*-Lindelöf and* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a* (γ, δ)*-continuous surjection, then* $(Y, n, f(\mathcal{H}))$ *is* $\delta f(\mathcal{H})$*-Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of Y by n-open sets. For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is (γ, δ)-continuous, there exists $U_{\alpha(x)} \in m$ containing x such that $f(\gamma(U_{\alpha(x)})) \subseteq \delta(V_{\alpha(x)})$. Since $\{U_{\alpha(x)} : x \in X\}$ is a cover of X by m-open sets and $(X, m, \mathcal{H})$ is $\gamma\mathcal{H}$-Lindelöf, there exist a countable points $x_1, x_2, x_3, \cdots \in X$ such that $X \setminus \cup_{i=1}^{\infty} \gamma(U_{\alpha(x_i)}) = H_0$, where $H_0 \in \mathcal{H}$. Therefore, we have $Y \subseteq f(\cup_{i=1}^{\infty} \gamma(U_{\alpha(x_i)})) \cup f(H_0) \subseteq \cup_{i=1}^{\infty} \delta(V_{\alpha(x_i)}) \cup f(H_0)$. Hence $(Y, n, f(\mathcal{H}))$ is $\delta f(\mathcal{H})$-Lindelöf. □

Definition 10 [11] *A function* $f : (X, m) \to (Y, n)$ *is said to be* M*-closed if for each* m*-closed set* F *of* X*,* $f(F)$ *is* n*-closed in* Y*.*

Theorem 9 *Let* $f : (X, m) \to (Y, n, \mathcal{H})$ *be an* M*-closed surjective function. If for every* $y \in Y$*,* $f^{-1}(y)$ *is Strongly* $f^{-1}(\mathcal{H})$*-Lindelöf in* X*, then* $f^{-1}(A)$ *is Strongly* $f^{-1}(\mathcal{H})$*-Lindelöf relative to* X *whenever* A *is Strongly* $\mathcal{H}$*-Lindelöf relative to* Y *and* $A \setminus U \in \mathcal{H}$ *for every* $U \in n$*.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of m-open subsets of X such that $f^{-1}(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in f^{-1}(\mathcal{H})$. For each $y \in A$ there exists a countable subset $\Delta_0(y)$ of Δ such that $f^{-1}(y) \setminus \cup\{V_\alpha : \alpha \in \Delta_0(y)\} \in f^{-1}(\mathcal{H})$. Let $V_y = \cup\{V_\alpha : \alpha \in \Delta_0(y)\}$. Each V_y is an m-open set in (X, m) and $f^{-1}(y) \setminus V_y \in f^{-1}(\mathcal{H})$.

Now each set $f(X - V_y)$ is n-closed in Y and hence, $U(y) = Y - f(X - V_y)$ is n-open in Y. Note that $f^{-1}(U(y)) \subseteq V_y$. Thus, $\{U(y) : y \in A\}$ is a family of n-open subsets of Y such that $A \setminus \cup\{U(y) : y \in A\} \in \mathcal{H}$. Since A is Strongly $\mathcal{H}$-Lindelöf relative to Y, there exists a countable subset $\{U(y_i) : i \in \mathbb{N}\}$ such that $A \setminus \cup\{U(y_i) : i \in \mathbb{N}\} \in \mathcal{H}$ and hence $f^{-1}[A \setminus \cup\{U(y_i) : i \in \mathbb{N}\}] = f^{-1}(A) \setminus \cup\{f^{-1}(U(y_i)) : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Since $f^{-1}(A) \setminus \cup\{V_{y_i} : i \in \mathbb{N}\} \subseteq f^{-1}(A) \setminus \cup\{f^{-1}(U(y_i)) : i \in \mathbb{N}\}$, then $f^{-1}(A) \setminus \cup\{V_{y_i} : i \in \mathbb{N}\} = f^{-1}(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0(y_i), i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Hence, $f^{-1}(A)$ is Strongly $f^{-1}(\mathcal{H})$-Lindelöf relative to X. □

A subset K of an m-space is said to be m-compact [14] if every cover of K by m-open sets of X has a finite subcover.

Theorem 10 *Let* $f : (X, m) \to (Y, n, \mathcal{H})$ *be an* M*-closed surjective function. If for every* $y \in Y$*,* $f^{-1}(y)$ *is* m*-compact in* X*, then* $f^{-1}(A)$ *is* $f^{-1}(\mathcal{H})$*-Lindelöf relative to* X *whenever* A *is* $\mathcal{H}$*-Lindelöf relative to* Y*.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of $f^{-1}(A)$ by m-open sets of X. For each $y \in A$ there exists a finite subset $\Delta_0(y)$ of Δ such that $f^{-1}(y) \subseteq \cup\{V_\alpha : \alpha \in \Delta_0(y)\}$. Let $V_y = \cup\{V_\alpha : \alpha \in \Delta_0(y)\}$. Each V_y is an m-open set in (X, m) and $f^{-1}(y) \subseteq V_y$. Since f is M-closed, by Theorem 3.1 of [11] there exists an n-open set U_y containing y such that $f^{-1}(U_y) \subseteq V_y$. The collection $\{U_y : y \in A\}$ is a cover of A by n-open sets of Y. Hence, there exists a countable subcollection $\{U_{y(i)} : i \in \mathbb{N}\}$ such that $A \setminus \cup\{U_{y(i)} : i \in \mathbb{N}\} \in \mathcal{H}$. Then $f^{-1}(A \setminus \cup\{U_{y(i)} : i \in \mathbb{N}\}) = f^{-1}(A) \setminus \cup\{f^{-1}(U_{y(i)}) : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Since $f^{-1}(A) \setminus \cup\{V_{y(i)} : i \in \mathbb{N}\} \subseteq f^{-1}(A) \setminus \cup\{f^{-1}(U_{y(i)}) : i \in \mathbb{N}\})$, then $f^{-1}(A) \setminus \cup\{V_{y(i)} : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Thus, $f^{-1}(A)$ is $f^{-1}(\mathcal{H})$-Lindelöf relative to X. □

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