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Fixed point and a Cantilever beam problem in a partial b-metric space

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Abstract. We determine the common fixed point of two maps satisfying Hardy-Roger type contraction in a complete partial b-metric space without exploiting any variant of continuity or commutativity, which is indispensable in analogous results. Towards the end, we give examples and an application to solve a Cantilever beam problem employed in the distortion of an elastic beam in equilibrium to substantiate the utility of these improvements.

1 Introduction and preliminaries

Fixed point theory is a major tool in nonlinear analysis, having applications in many real-world problems, which emerged in 1837 with the article of Liouville [10] on solutions of differential equations. In 1890, Picard [13] developed it further as a process of successive approximations which were conceptualized and extracted by Banach [2] as a fixed point result in a complete normed space in 1922. On the other hand, Shukla [16] familiarized partial b-metric blending

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partial metric (Matthews [11]) and $\mathfrak{b}$-metric (Bakhtin [1] and Czerwik [6]) to establish a fixed point via Banach contraction [2] and Kannan contraction [9].

The aim of the current work is to demonstrate the survival of one and only one common fixed point of two maps satisfying classical Hardy-Rogers type contraction [7] in a complete partial $\mathfrak{b}$-metric space without exploiting any variant of continuity [17] or commutativity [18], which is indispensable in analogous results. We support our theoretical consequences by illustrative examples and conclude the paper by giving an application to solve a Cantilever beam problem employed in the distortion of an elastic beam in equilibrium to substantiate the utility of these improvements.

It is worth mentioning here that in numerous cable-driven docile mechanisms, like a fixed pulley or a cable routing channel in a segmented disk, the need for controlled motion in the flexible frameworks often mandates the actuation cables to pass through a fixed point to compel the force angle on the cable. This situation may be modeled as the large deflection problem of a cantilever beam with two parameters. Recently Zeng et al. [19] emphasized the numerical analysis of the large deflection problem of the cantilever beam subjected to a constraint force pointing at a fixed point which permitted widespread analysis of the impact of diverse factors, including the fixed point position, the force magnitude, and the beam length, on the behaviour of the cantilever beam put to a constraint force pointing at a fixed point. This work permitted mathematical model-based design optimization of docile frameworks in areas such as soft robotics and smart materials.

Definition 1 [16] *A function* $\mathfrak{p}_\mathfrak{b} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *on a nonempty set* $\mathcal{X}$ *is a partial* $\mathfrak{b}$*-metric if* $\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathcal{X}$,

1. $\mathfrak{u} = \mathfrak{v}$ *iff* $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{u}) = \mathfrak{p}_\mathfrak{b}(\mathfrak{v}, \mathfrak{v})$;
2. $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{u}) \leq \mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v})$;
3. $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{p}_\mathfrak{b}(\mathfrak{v}, \mathfrak{u})$;
4. $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) \leq \mathfrak{s}[\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{w}) + \mathfrak{p}_\mathfrak{b}(\mathfrak{w}, \mathfrak{v})] - \mathfrak{p}_\mathfrak{b}(\mathfrak{w}, \mathfrak{w})$.

The pair $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$ *is a partial* $\mathfrak{b}$*-metric space and* $\mathfrak{s} \geq 1$ *is the coefficient of* $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$.

Example 1 *Let* $\mathcal{X} = [0, 10]$ *and* $\mathfrak{p}_\mathfrak{b} : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ *be defined as:* $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{w}|^2 + 2$*. By routine calculation, one may verify that* $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$ *is a partial* $\mathfrak{b}$*-metric space for* $\mathfrak{s} = 2$*. However,* $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$ *is not a partial metric*

space. Since for $\mathfrak{u} = 0$, $\mathfrak{v} = 10$ *and* $\mathfrak{w} = 5$, *we obtain*
$p_b(0, 10) = |0 - 10|^2 + 2 = 102$,

$$\begin{aligned} p_b(0,5) + p_b(5,10) - p_b(5,5) &= |0-5|^2 + 2 + |5-10|^2 + 2 - 2 \\ &= 25 + 2 + 25 \\ &= 52. \end{aligned}$$

Therefore, $p_b(0, 10) > p_b(0, 5) + p_b(5, 10) - p_b(5, 5)$. *Noticeably,* $(\mathcal{X}, p_b)$ *is also not a* $\mathfrak{b}$*-metric space.*

Definition 2 [12] *A sequence* $\{\mathfrak{u}_n\}$ *in a partial* $\mathfrak{b}$*-metric space* $(\mathcal{X}, p_b)$ *is*

1. *convergent to* $\mathfrak{u} \in \mathcal{X}$ *if* $p_b(\mathfrak{u}, \mathfrak{u}) = \lim_{n\to\infty} p_b(\mathfrak{u}, \mathfrak{u}_n)$.

2. *Cauchy sequence if* $\lim_{n\to\infty} p_b(\mathfrak{u}_n, \mathfrak{u}_m)$ *exists and is finite.*

A partial $\mathfrak{b}$-metric space $(\mathcal{X}, p_b)$ is complete [16] if each p_b-Cauchy sequence in $\mathcal{X}$ converges to $\mathfrak{u} \in \mathcal{X}$, i.e., $p_b(\mathfrak{u}, \mathfrak{u}) = \lim_{n\to\infty} p_b(\mathfrak{u}, \mathfrak{u}_n) = \lim_{n,m\to\infty} p_b(\mathfrak{u}_n, \mathfrak{u}_m)$.
One may notice that the limit of a convergent sequence is not essentially unique in a partial $\mathfrak{b}$-metric space.

2 Main results

Theorem 1 *Let* $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be self maps of a complete partial* $\mathfrak{b}$*-metric space* $(\mathcal{X}, p_b)$ *so that* $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$ *and*

$$p_b(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \mathfrak{a}p_b(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}p_b(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{c}p_b(\mathfrak{u}, \mathcal{T}\mathfrak{v}) + \mathfrak{d}p_b(\mathfrak{v}, \mathcal{S}\mathfrak{u}) + \mathfrak{e}p_b(\mathfrak{u}, \mathfrak{v}), \quad (1)$$

$\forall\, \mathfrak{u}, \mathfrak{v} \in \mathcal{X}$ *and* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ *are positive reals satisfying* $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{e} + \mathfrak{d}(2s - 1) \leq 1$ *and* $s > 1$. *Then* $\mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point in* $\mathcal{X}$.

Proof. Assume $\mathfrak{u}_0 \in \mathcal{X}$ and since $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$, so we may inductively define a sequence $\{\mathfrak{u}_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ as

$$\mathfrak{u}_n = \mathcal{T}\mathfrak{u}_{n-1} \text{ and } \mathfrak{u}_{n+1} = \mathcal{S}\mathfrak{u}_n, \quad (2)$$

for $n = 0, 1, 2, \ldots$. If $\mathfrak{u}_n = \mathfrak{u}_{n+1}$, i.e., $\mathfrak{u}_n = \mathcal{S}\mathfrak{u}_n$, i.e., $\mathfrak{u}_n$ is a fixed point of $\mathcal{S}$.

Since, $u_n = u_{n+1} \Longrightarrow u_{n+1} = \mathcal{S}u_n = \mathcal{S}u_{n+1}$. So

$$\begin{aligned} p_b(u_{n+1}, u_{n+2}) &= p_b(\mathcal{S}u_{n+1}, \mathcal{T}u_n) \\ &\leq \mathfrak{a}p_b(u_{n+1}, \mathcal{S}u_{n+1}) + \mathfrak{b}p_b(u_n, \mathcal{T}u_n) + \mathfrak{c}p_b(u_{n+1}, \mathcal{T}u_n) \\ &\quad + \mathfrak{d}p_b(u_n, \mathcal{S}u_{n+1}) + \mathfrak{e}p_b(u_{n+1}, u_n) \\ &= \mathfrak{a}p_b(u_{n+1}, u_{n+2}) + \mathfrak{b}p_b(u_n, u_{n+1}) + \mathfrak{c}p_b(u_{n+1}, u_{n+1}) \\ &\quad + \mathfrak{d}p_b(u_n, u_{n+2}) + \mathfrak{e}p_b(u_{n+1}, u_n) \\ &\leq \mathfrak{a}p_b(u_{n+1}, u_{n+2}) + \mathfrak{b}p_b(u_n, u_{n+1}) + \mathfrak{c}p_b(u_{n+1}, u_{n+1}) \\ &\quad + \mathfrak{d}s[p_b(u_n, u_{n+1}) + p_b(u_{n+1}, u_{n+2})] - \mathfrak{d}p_b(u_{n+1}, u_{n+1}) \\ &\quad + \mathfrak{e}p_b(u_{n+1}, u_n), \end{aligned}$$

$$\begin{aligned} \text{i.e.,}\quad & (1 - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) + \mathfrak{d}p_b(u_{n+1}, u_{n+1}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e})p_b(u_n, u_{n+1}) \\ & + \mathfrak{c}p_b(u_{n+1}, u_{n+1}), \\ \text{i.e.,}\quad & (1 - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) + \mathfrak{d}p_b(u_{n+1}, u_{n+2}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e})p_b(u_n, u_{n+1}) \\ & + \mathfrak{c}p_b(u_n, u_{n+1}), \\ \text{i.e.,}\quad & (1 + \mathfrak{d} - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e} + \mathfrak{c})p_b(u_n, u_{n+1}), \\ \text{i.e.,}\quad & (1 + \mathfrak{d} - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e} + \mathfrak{c})p_b(u_{n+1}, u_{n+2}), \\ \text{i.e.,}\quad & (1 - \mathfrak{a} - \mathfrak{b} - \mathfrak{c} - \mathfrak{e} - \mathfrak{d}(2s - 1))p_b(u_{n+1}, u_{n+2}) \leq 0, \\ \text{i.e.,}\quad & p_b(u_{n+1}, u_{n+2}) \leq 0 \Longrightarrow p_b(u_{n+1}, u_{n+2}) = 0, \end{aligned}$$

i.e., $\mathcal{T}u_n = u_{n+1} = u_{n+2}$ and $u_n = u_{n+1} \Longrightarrow \mathcal{T}u_n = u_n$, i.e., u_n is a fixed point of $\mathcal{T}$.
Also, $u_n = u_{n+1} = u_{n+2} = \dots,$ i.e., u_n is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.
So, presume that for even n, $u_n \neq u_{n+1}$. Then

$$\begin{aligned} p_b(u_{n+1}, u_n) &= p_b(Su_n, Tu_{n-1}) \\ &\leq \mathfrak{a}p_b(u_n, Su_n) + \mathfrak{b}p_b(u_{n-1}, Tu_{n-1}) + \mathfrak{c}p_b(u_n, Tu_{n-1}) \\ &\qquad + \mathfrak{d}p_b(u_{n-1}, Su_n) + \mathfrak{e}p_b(u_n, u_{n-1}) \\ &\leq \mathfrak{a}p_b(u_n, u_{n+1}) + \mathfrak{b}p_b(u_{n-1}, u_n) + \mathfrak{c}p_b(u_n, u_n) \\ &\qquad + \mathfrak{d}p_b(u_{n-1}, u_{n+1}) + \mathfrak{e}p_b(u_n, u_{n-1}) \\ &\leq \mathfrak{a}p_b(u_n, u_{n+1}) + \mathfrak{b}p_b(u_{n-1}, u_n) + \mathfrak{c}p_b(u_n, u_n) + \mathfrak{d}s[p_b(u_{n-1}, u_n) \\ &\qquad + p_b(u_n, u_{n+1})] - \mathfrak{d}p_b(u_n, u_n) + \mathfrak{e}p_b(u_n, u_{n-1}) \\ &\leq \mathfrak{a}p_b(u_n, u_{n+1}) + \mathfrak{b}p_b(u_{n-1}, u_n) + \mathfrak{c}p_b(u_n, u_n) + \mathfrak{d}sp_b(u_{n-1}, u_n) \\ &\qquad + \mathfrak{d}sp_b(u_n, u_{n+1}) - dp_b(u_n, u_n) + \mathfrak{e}p_b(u_n, u_{n-1}), \end{aligned}$$

i.e., $(1-\mathfrak{a}-\mathfrak{d}s)p(u_n,u_{n+1}) \le (\mathfrak{b}+\mathfrak{d}s+\mathfrak{e})p_b(u_{n-1},u_n)+(\mathfrak{c}-\mathfrak{d})p_b(u_n,u_n)$,

i.e., $(1-\mathfrak{a}-\mathfrak{d}s)p(u_n,u_{n+1})+\mathfrak{d}p_b(u_n,u_n) \le (\mathfrak{b}+\mathfrak{d}s+\mathfrak{e})p_b(u_n,u_{n-1})$
$+\mathfrak{c}p_b(u_n,u_n)$,

i.e., $(1-\mathfrak{a}-\mathfrak{d}s)p(u_n,u_{n+1})+\mathfrak{d}p_b(u_n,u_{n+1}) \le (\mathfrak{b}+\mathfrak{d}s+\mathfrak{e})p_b(u_nu_{n-1})$
$+\mathfrak{c}p_b(u_n,u_{n-1})$

$(1+\mathfrak{d}-\mathfrak{a}-\mathfrak{d}s)p_bu_n,u_{n+1}) \le (\mathfrak{b}+\mathfrak{c}+\mathfrak{e}+\mathfrak{d}s)p_b(u_n,u_{n+1})$,

i.e., $p_b(u_n,u_{n+1}) \le \dfrac{\mathfrak{b}+\mathfrak{c}+\mathfrak{e}+\mathfrak{d}s}{1+\mathfrak{d}-\mathfrak{a}-\mathfrak{d}s}p_b(u_n,u_{n+1})$,

$$\text{i.e.,}\quad p_b(u_n,u_{n+1}) \le kp_b(u_n,u_{n-1}), \quad \text{where,}\ k=\frac{\mathfrak{b}+\mathfrak{c}+\mathfrak{e}+\mathfrak{d}s}{1+\mathfrak{d}-\mathfrak{a}-\mathfrak{d}s} \le 1. \tag{3}$$

If n is odd, the same inequality (3) can be obtained analogously.
Continuing this process, we attain
$p_b(u_n,u_{n+1}) \le k^n p_b(u_0,u_1)$.
We assert that $\{u_n\}$is a Cauchy sequence in $\mathcal{X}$. For $m > n$ and $m,n \in N$, consider

$$\begin{aligned}
p_b(u_n,u_m) &\le s[p_b(u_n,u_{n+1})+p_b(u_{n+1},u_m)]-p_b(u_{n+1},u_{n+1})\\
&\le s[p_b(u_n,u_{n+1})+p_b(u_{n+1},u_m)]\\
&\le sp_b(u_n,u_{n+1})+s[s\{p_b(u_{n+1},u_{n+2})\\
&\qquad +p_b(u_{n+2},u_m)\}-p_b(u_{n+2},u_{n+2})]\\
&\le sp_b(u_n,u_{n+1})+s[s\{p_b(u_{n+1},u_{n+2})+p_b(u_{n+2},u_m)\}]\\
&\le sk^np_b(u_0,u_1)+s^2k^{n+1}p_b(u_0,u_1)+....\\
&\le sk^np_b(u_0,u_1)[1+sk+(sk)^2+....]\\
&\le \frac{sk^n}{1-sk}p_b(u_0,u_1) \longrightarrow 0 \ \text{as}\ \ n \longrightarrow \infty,
\end{aligned}$$

i.e., $\{u_n\}$ is a Cauchy sequence. Using completeness of $\mathcal{X}$, $\{u_n\}$ converges to $u^* \in \mathcal{X}$ and we have $\lim_{n,m\longrightarrow\infty} p_b(u_n,u_m) = \lim_{n\longrightarrow\infty} p_b(u_n,u^*) = p_b(u^*,u^*) = 0$. Further, we assert that u^* is a fixed point of $\mathcal{S}$. Let $\{u_{n_i}\}_{i=1}^{\infty}$ be a subsequence of $\{u_n\}$.

So,

$$\begin{aligned}
p_b(u^*,\mathcal{S}u^*) &\le s[p_b(u^*,u_{n_i})+p_b(u_{n_i},\mathcal{S}u^*)]-p_b(u_{n_i},u_{n_i})\\
&\le sp_b(u^*,u_{n_i})+sp_b(\mathcal{T}u_{n-1_i},\mathcal{S}u^*)\\
&\le sp_b(u^*,u_{n_i})+s[p_b(\mathcal{S}u^*,\mathcal{T}u_{n-1_i})]\\
&\le sp_b(u^*,u_{n_i})+s[\mathfrak{a}p_b(u^*,Su^*)+\mathfrak{b}p_b(u_{n-1_i},\mathcal{T}u_{n-1_i})
\end{aligned}$$

$$+ \mathfrak{c}p_\mathfrak{b}(u^*, \mathcal{T}u_{n-1_i}) + \mathfrak{d}p_\mathfrak{b}(u_{n-1_i}, \mathcal{S}u^*) + \mathfrak{e}p_\mathfrak{b}(u^*, u_{n-1_i})]$$
$$\leq sp_\mathfrak{b}(u^*, u_{n_i}) + s[\mathfrak{a}p_\mathfrak{b}(u^*, \mathcal{S}u^*) + \mathfrak{b}p_\mathfrak{b}(u_{n-1_i}, u_{n_i}) \tag{4}$$
$$+ \mathfrak{c}p_\mathfrak{b}(u^*, u_{n_i}) + \mathfrak{d}p_\mathfrak{b}(u_{n-1_i}, \mathcal{S}u^*) + \mathfrak{e}p_\mathfrak{b}(u^*, u_{n-1_i})].$$

As $n \to \infty$, $p_\mathfrak{b}(u^*, \mathcal{S}u^*) \leq s(\mathfrak{a}+\mathfrak{d})p_\mathfrak{b}(u^*, Su^*)$, which gives a contradiction. So, $u^* = \mathcal{S}u^* \Rightarrow u^*$ is fixed point of $\mathcal{S}$.
Furthermore, we assert that u^* is a fixed point of $\mathcal{T}$. Let $\{u_{n+1_i}\}_{i=1}^{\infty}$ be a subsequence of $\{u_n\}$.
So,

$$\begin{aligned} p_\mathfrak{b}(u^*, \mathcal{T}u^*) &\leq s[p_\mathfrak{b}(u^*, u_{n+1_i}) + p_\mathfrak{b}(u_{n+1_i}, \mathcal{T}u^*)] - p_\mathfrak{b}(u_{n+1_i}, u_{n+1_i}) \\ &\leq sp_\mathfrak{b}(u^*, u_{n+1_i}) + sp_\mathfrak{b}(\mathcal{S}u_{n_i}, \mathcal{T}u^*) \\ &\leq sp_\mathfrak{b}(u^*, u_{n+1_i}) + s[\mathfrak{a}p_\mathfrak{b}(u_{n_i}, \mathcal{S}u_{n_i}) + \mathfrak{b}p_\mathfrak{b}(u^*, \mathcal{T}u^*) \\ &\qquad + \mathfrak{c}p_\mathfrak{b}(u_{n_i}, \mathcal{T}u^*) + \mathfrak{d}p_\mathfrak{b}(u^*, \mathcal{S}u_{n_i}) + \mathfrak{e}p_\mathfrak{b}(u_{n_i}, u^*)] \\ &\leq sp_\mathfrak{b}(u^*, u_{n+1_i}) + s[\mathfrak{a}p_\mathfrak{b}(u_{n_i}, u_{n+1_i}) + \mathfrak{b}p_\mathfrak{b}(u^*, \mathcal{T}u^*) \\ &\qquad + \mathfrak{c}p_\mathfrak{b}(u_{n_i}, \mathcal{T}u^*) + \mathfrak{d}p_\mathfrak{b}(u^*, u_{n+1_i}) + \mathfrak{e}p_\mathfrak{b}(u_{n_i}, u^*)]. \end{aligned}$$

As $n \to \infty$, $p_\mathfrak{b}(u^*, \mathcal{T}u^*) \leq s(\mathfrak{b}+\mathfrak{c})p_\mathfrak{b}(u^*, \mathcal{T}u^*)$, which gives a contradiction.
Therefore, $u^* = \mathcal{T}u^* \Rightarrow u^*$ is a fixed point of $\mathcal{T}$.
If u and u^* are two different common fixed points of $\mathcal{S}$ and $\mathcal{T}$, then we have $\mathcal{S}u = \mathcal{T}u = u$ and $\mathcal{S}u^* = \mathcal{T}u^* = u^*$. Consider

$$\begin{aligned} p_\mathfrak{b}(u, u^*) &= p_\mathfrak{b}(Su, \mathcal{T}u^*) \\ &\leq \mathfrak{a}p_\mathfrak{b}(u, \mathcal{S}u) + \mathfrak{b}p_\mathfrak{b}(u^*, \mathcal{T}u^*) + \mathfrak{c}p_\mathfrak{b}(u, \mathcal{T}u^*) + \mathfrak{d}p_\mathfrak{b}(u^*, \mathcal{S}u) + \mathfrak{e}p_\mathfrak{b}(u, u^*) \\ &\leq \mathfrak{a}p_\mathfrak{b}(u, u) + \mathfrak{b}p_\mathfrak{b}(u^*, u^*) + \mathfrak{c}p_\mathfrak{b}(u, u^*) + \mathfrak{d}p_\mathfrak{b}(u^*, u) + \mathfrak{e}p_\mathfrak{b}(u, u^*) \\ &\leq (\mathfrak{c}+\mathfrak{d}+\mathfrak{e})p_\mathfrak{b}(u, u^*), \end{aligned}$$

a contradiction, i.e., $u = u^* \Rightarrow \mathcal{S}$ and $\mathcal{T}$ has a unique common fixed point in $\mathcal{X}$. □

Next, we provide a non-trivial illustration to exhibit the significance of Theorem 1.

Example 2 *Let* $\mathcal{X} = [-10, 10]$ *and* $p_\mathfrak{b} : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ *be defined as:* $p_\mathfrak{b}(u, \mathfrak{v}) = (|u| + |\mathfrak{v}| + 2)^2$*. Then* $(\mathcal{X}, p_\mathfrak{b})$ *is a complete partial* $\mathfrak{b}$*-metric space and* $s = 2$*. Define* $\mathcal{S}, \mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ *as:* $\mathcal{S}u = \frac{u}{6}$ *and* $\mathcal{T}u = \frac{u}{10}$*. Let* $u \geq \mathfrak{v}$*. Then*

$$\begin{aligned} p_\mathfrak{b}(\mathcal{S}u, \mathcal{T}\mathfrak{v}) = p_\mathfrak{b}\left(\frac{u}{6}, \frac{\mathfrak{v}}{10}\right) &= \left(\frac{|u|}{6} + \frac{|\mathfrak{v}|}{10} + 2\right)^2 \\ &= \left(\frac{10|u| + 6|\mathfrak{v}| + 120}{60}\right)^2 \quad \textit{and} \end{aligned} \tag{5}$$

$$
\begin{aligned}
&\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{S}\mathfrak{u})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{T}\mathfrak{v})+\mathfrak{d}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{S}\mathfrak{u})+\mathfrak{e}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})\\
&=\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\frac{\mathfrak{u}}{6})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\frac{\mathfrak{v}}{10})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\frac{\mathfrak{v}}{10})+\mathfrak{d}p_{\mathfrak{b}}(\mathfrak{v},\frac{\mathfrak{u}}{6})+\mathfrak{e}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})\\
&=\mathfrak{a}\big(|\mathfrak{u}|+|\frac{\mathfrak{u}}{6}|+2\big)^2+\mathfrak{b}\big(|\mathfrak{v}|+|\frac{\mathfrak{v}}{10}|+2\big)^2+\mathfrak{c}\big(\mathfrak{u}+\frac{\mathfrak{v}}{10}+2\big)^2+\mathfrak{d}\big(\mathfrak{v}+\frac{\mathfrak{u}}{6}+2\big)^2\\
&\quad+\mathfrak{e}\big(\mathfrak{u}+\mathfrak{v}+2\big)^2 \qquad (6)\\
&=\mathfrak{a}\left(\frac{7|\mathfrak{u}|+12}{6}\right)^2+\mathfrak{b}\left(\frac{11|\mathfrak{v}|+20}{10}\right)^2+\mathfrak{c}\left(\frac{10|\mathfrak{u}|+|\mathfrak{v}|+20}{10}\right)^2\\
&\quad+\mathfrak{d}\left(\frac{6|\mathfrak{v}|+|\mathfrak{u}|+12}{6}\right)^2+\mathfrak{e}(|\mathfrak{u}|+|\mathfrak{v}|+2)^2.
\end{aligned}
$$

From equations (5) and (6) it is clear that for $\mathfrak{a}=\mathfrak{b}=\mathfrak{e}=\frac{1}{6}$, $\mathfrak{c}=\frac{1}{3}$, *and* $\mathfrak{d}=\frac{1}{9}$,

$$p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{v})\le\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{S}\mathfrak{u})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{T}\mathfrak{v})+\mathfrak{d}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{S}\mathfrak{u})+\mathfrak{e}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v}).$$

Consequently, all postulates of Theorem 1 are verified, and 0 *is the unique common fixed point of* $\mathcal{S}$ *and* $\mathcal{T}$.

Corollary 1 *Inference of Theorem 1 is valid if* $\mathfrak{c}=\mathfrak{d}=0$.

Proof. The proof follows the pattern of Theorem 1. □

Next, we present two examples to understand and support the result proved herein. In one example involved maps are continuous and commutative and in another maps are discontinuous and noncommutative. It is worth mentioning that continuity is difficult to be fulfilled in some daily life applications and is an ideal property.

Example 3 *Let* $\mathcal{X}=\mathbb{R}^+$ *and* $p_{\mathfrak{b}}:\mathcal{X}\times\mathcal{X}\longrightarrow[0,\infty)$ *be defined as:* $p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})=\max\{\mathfrak{u},\mathfrak{v}\}^2+|\mathfrak{u}-\mathfrak{v}|^2$. *Then* $(\mathcal{X},p_{\mathfrak{b}})$ *is a complete partial* $\mathfrak{b}$*-metric space and* $s=4$. *Define* $\mathcal{S},\ \mathcal{T}:\mathcal{X}\longrightarrow\mathcal{X}$ *as:* $\mathcal{S}\mathfrak{u}=\frac{\mathfrak{u}}{4}$ *and* $\mathcal{T}\mathfrak{u}=\frac{\mathfrak{u}}{5}$. *Let* $\mathfrak{u}\ge\mathfrak{v}$. *Then*

$$
\begin{aligned}
p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{v})&=p_{\mathfrak{b}}\left(\frac{\mathfrak{u}}{4},\frac{\mathfrak{v}}{5}\right)\\
&=\max\left\{\frac{\mathfrak{u}}{4},\frac{\mathfrak{v}}{5}\right\}^2+\left|\frac{\mathfrak{u}}{4}-\frac{\mathfrak{v}}{5}\right|^2 \qquad (7)\\
&=\frac{\mathfrak{u}^2}{16}+\left|\frac{5\mathfrak{u}-4\mathfrak{v}}{25}\right|^2 \quad \textit{and}
\end{aligned}
$$

$$\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{S}\mathfrak{u})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})=\mathfrak{a}p_{\mathfrak{b}}\left(\mathfrak{u},\frac{\mathfrak{u}}{4}\right)+\mathfrak{b}p_{\mathfrak{b}}\left(\mathfrak{v},\frac{\mathfrak{v}}{5}\right)+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})$$
$$=\mathfrak{a}\left[\max\left\{\mathfrak{u},\frac{\mathfrak{u}}{4}\right\}^2+\left|\mathfrak{u}-\frac{\mathfrak{u}}{4}\right|^2\right]+\mathfrak{b}\left[\max\left\{\mathfrak{v},\frac{\mathfrak{v}}{5}\right\}^2+\left|\mathfrak{v}-\frac{\mathfrak{v}}{5}\right|^2\right]$$
$$+\mathfrak{c}\left[\max\{\mathfrak{u},\mathfrak{v}\}^2+|\mathfrak{u}-\mathfrak{v}|^2\right]$$
$$=\mathfrak{a}\left[\mathfrak{u}^2+\frac{9}{16}\mathfrak{u}^2\right]+\mathfrak{b}\left[\mathfrak{v}^2+\frac{16}{25}\mathfrak{v}^2\right]+\mathfrak{c}\left[\mathfrak{u}^2+|\mathfrak{u}-\mathfrak{v}|^2\right]$$
$$=\frac{25}{16}\mathfrak{a}\mathfrak{u}^2+\frac{41}{25}\mathfrak{b}\mathfrak{v}^2+\mathfrak{c}[\mathfrak{u}^2+|\mathfrak{u}-\mathfrak{v}|^2]. \tag{8}$$

From Equations (7) and (8) it is clear that for $\mathfrak{a}=\frac{1}{3}$, $\mathfrak{b}=\mathfrak{c}=\frac{1}{9}$,

$$p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{v})\leq\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{S}\mathfrak{u})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v}).$$

Hence, all postulates of Corollary 1 are verified, and 0 *is the unique common fixed point of* $\mathcal{S}$ *and* $\mathcal{T}$.

Example 4 *Let* $\mathcal{X}=\mathbb{R}^+$ *and* $p_{\mathfrak{b}}:\mathcal{X}\times\mathcal{X}\longrightarrow[0,\infty)$ *be defined as:* $p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})=\max\{\mathfrak{u},\mathfrak{v}\}+|\mathfrak{u}-\mathfrak{v}|^2$. *Then* $(\mathcal{X},p_{\mathfrak{b}})$ *is a complete partial* $\mathfrak{b}$*-metric space and* $s=4$. *Define* $\mathcal{S}$, $\mathcal{T}:\mathcal{X}\longrightarrow\mathcal{X}$ *as:* $\mathcal{S}\mathfrak{u}=\begin{cases}\frac{\mathfrak{u}}{2}, & \mathfrak{u}\in[0,1]\\ 0, & \textit{otherwise}\end{cases}$ *and* $\mathcal{T}\mathfrak{u}=\begin{cases}\frac{\mathfrak{u}^2-\mathfrak{u}}{2}, & \mathfrak{u}\in[0,1]\\ 0, & \textit{otherwise}\end{cases}$. *Let* $\mathfrak{u},\mathfrak{v}\in[0,1]$ *and* $\mathfrak{u}\geq\mathfrak{v}$. *Therefore,*

$$\begin{aligned}p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{v})&=p_{\mathfrak{b}}\left(\frac{\mathfrak{u}}{2},\frac{\mathfrak{v}^2-\mathfrak{v}}{2}\right)\\&=\max\left\{\frac{\mathfrak{u}}{2},\frac{\mathfrak{v}^2-\mathfrak{v}}{2}\right\}+\left|\frac{\mathfrak{u}}{2}-\frac{\mathfrak{v}^2-\mathfrak{v}}{2}\right|^2\\&=\frac{\mathfrak{u}}{2}+\left|\frac{\mathfrak{u}+\mathfrak{v}-\mathfrak{v}^2}{2}\right|^2\quad\textit{and}\end{aligned}\tag{9}$$

$$\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{S}\mathfrak{u})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})=\mathfrak{a}p_{\mathfrak{b}}\left(\mathfrak{u},\frac{\mathfrak{u}}{2}\right)+\mathfrak{b}p_{\mathfrak{b}}\left(\mathfrak{v},\frac{\mathfrak{v}^2-\mathfrak{v}}{2}\right)+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})$$
$$=\mathfrak{a}\left[\max\left\{\mathfrak{u},\frac{\mathfrak{u}}{2}\right\}+\left|\mathfrak{u}-\frac{\mathfrak{u}}{2}\right|^2\right]+\mathfrak{b}\left[\max\left\{\mathfrak{v},\frac{\mathfrak{v}^2-\mathfrak{v}}{2}\right\}+\left|\mathfrak{v}-\frac{\mathfrak{v}^2-\mathfrak{v}}{2}\right|^2\right]$$
$$\mathfrak{c}\left[\max\{\mathfrak{u},\mathfrak{v}\}+\left|\mathfrak{u}-\mathfrak{v}\right|^2\right]$$
$$=\mathfrak{a}\left[\mathfrak{u}+\frac{1}{4}\mathfrak{u}^2\right]+\mathfrak{b}\left[\mathfrak{v}+\frac{1}{4}(3\mathfrak{v}-\mathfrak{v}^2)]\right]+\mathfrak{c}[\mathfrak{u}+|\mathfrak{u}-\mathfrak{v}|^2\Big]. \tag{10}$$

Next, if $\mathfrak{u} \leq \mathfrak{v}$ *and* $\mathfrak{u}, \mathfrak{v} \in [0, 1]$,

$$p_b(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) = \frac{\mathfrak{u}}{2} + \left|\frac{\mathfrak{u} + \mathfrak{v} - \mathfrak{v}^2}{2}\right|^2 \quad \textit{and} \tag{11}$$

$$\begin{aligned} \mathfrak{a}p_b(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}p_b(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{c}p_b(\mathfrak{u}, \mathfrak{v}) &= \mathfrak{a}\left[\mathfrak{u} + \frac{1}{4}\mathfrak{u}^2\right] + \mathfrak{b}\left[\mathfrak{v} + \frac{1}{4}(3\mathfrak{v} - \mathfrak{v}^2)\right] \\ &\quad + \mathfrak{c}[\mathfrak{v} + |\mathfrak{u} - \mathfrak{v}|^2]. \end{aligned} \tag{12}$$

From Equations (9), (10), (11), *and* (12) *it is clear that for* $\mathfrak{a} = \frac{1}{3}$, $\mathfrak{b} = \frac{1}{4}$ *and* $\mathfrak{c} = \frac{1}{7}$

$$p_b(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \mathfrak{a}p_b(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}p_b(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{e}p_b(\mathfrak{u}, \mathfrak{v}), \ \mathfrak{u}, \mathfrak{v} \in [0, 1]. \tag{13}$$

Hence, all postulates of Corollary 1 are verified, and 0 *is the unique common fixed point of* $\mathcal{S}$ *and* $\mathcal{T}$.

Remark 1

(i) *Above results are also true if* $\mathcal{T}(\mathcal{X})$ *is a complete subspace instead of completeness of* $\mathcal{X}$.

(ii) *Above results become more fascinating if we appraise a better natural postulate of closures of range space, i.e.,* $\overline{\mathcal{T}(\mathcal{X})} \subseteq \mathcal{S}(\mathcal{X})$.

(iii) *Suitably choosing the values of constants* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, *and* $\mathfrak{e}$, *we get the extensions, improvements, generalizations of Bakhtin* [1], *Banach* [2], *Chatterjea* [3], Kannan [9], *Reich* [14], *and so on to a partial* $\mathfrak{b}$*-metric space for a noncommutative discontinuous pair of maps.*

(iv) *In Theorem 1* and *Corollary 1 (see, Example 4), a unique common fixed point exists for a pair of discontinuous self maps which does not satisfy even commutativity* ([8], [15], [17]) *and thereby extend, generalize and improve the comparable theorems present in the literature (for instance, Banach* [2], *Chatterjea* [3], *Ćirić* [4], *Czerwik* [6], *Hardy-Rogers* [7], *Kannan* [9], *Reich* [14], *and references therein).*

(v) *Following arguments of Theorem 1, we may relax continuity, commutativity, and completeness of numerous celebrated and contemporary results existing in different spaces.*

3 Solution of Cantilever beam problem

Motivated by the fact that the Cantilever structure permits overhanging constructions deprived of peripheral bracing, we solve a system of fourth-order differential equations arising in the two-point boundary value problem of bending of an elastic beam as an application of Corollary 1. Suppose $\mathcal{X} = C[I, \mathbb{R}]$ denotes the set of all continuous functions on $I = [0, 1]$. Define a partial $\mathfrak{b}$-metric $p_{\mathfrak{b}} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ as:
$p_{\mathfrak{b}}(\mathfrak{u}(t), \mathfrak{v}(t)) = \max_{t\in[0,1]} \left(\frac{|\mathfrak{u}(t)|+|\mathfrak{v}(t)|}{2}\right)^2$ with $s = 3$.

Theorem 2 *The equations of deformations of an elastic beam, one of whose end-point is free while the other is fixed, in its equilibrium state is:*

$$\begin{aligned} &\frac{d^4\mathfrak{u}}{dt^4} = \psi(t, \mathfrak{u}(t), \mathfrak{u}'(t), \mathfrak{u}''(t), \mathfrak{u}'''(t)), \\ &\mathfrak{u}(0) = \mathfrak{u}'(0) = \mathfrak{u}''(1) = \mathfrak{u}'''(1) = 0, \quad t \in [0, 1], \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\frac{d^4\mathfrak{v}}{dt^4} = \phi(t, \mathfrak{v}(t), \mathfrak{v}'(t), \mathfrak{v}''(t), \mathfrak{v}'''(t)), \\ &\mathfrak{v}(0) = \mathfrak{v}'(0) = \mathfrak{v}''(1) = \mathfrak{v}'''(1) = 0, \quad t \in [0, 1], \end{aligned} \tag{15}$$

where, $\psi,\ \phi : [0, 1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ *are continuous functions satisfying:*
$\max_{t\in[0,1]}(|\psi(t, \mathfrak{u}(t), \mathfrak{u}'(t), \mathfrak{u}''(t))|+|\phi(t, \mathfrak{v}(t), \mathfrak{v}'(t), \mathfrak{v}''(t))|)^2 \leq \exp^{-\alpha} \max_{t\in[0,1]} |\mathfrak{u}(t) + \mathfrak{v}(t)|^2 + \exp^{-\beta} \max_{t\in[0,1]} |\mathfrak{u}(t)|^2 + \exp^{-\gamma} \max_{t\in[0,1]} |\mathfrak{v}(t)|^2,\ \mathfrak{u}, \mathfrak{v} \in \mathcal{X},\ \lambda \in [1, \infty),\ t \in [0, 1)$.
Then, the Cantilever beam problem (14-15) *has a solution in* $\mathcal{X}$.

Proof. The Cantilever beam problem (14-15) is identical to solving the system of integral equations

$$\mathfrak{u}(t) = \int_0^1 \mathcal{G}(s, t)\psi(s, \mathfrak{u}(s), \mathfrak{u}'(s), \mathfrak{u}''(s))ds \tag{16}$$

and

$$\mathfrak{v}(t) = \int_0^1 \mathcal{G}(s, t)\phi(s, \mathfrak{v}(s), \mathfrak{v}'(s), \mathfrak{v}''(s))ds,\ t \in [0, 1],\ \mathfrak{u} \in \mathcal{X}. \tag{17}$$

Here,

$$\mathcal{G}(s, t) = \begin{cases} \frac{1}{6}s^2(3t - s) & , 0 \leq t \leq s \leq 1 \\ \frac{1}{6}t^2(3s - t) & , 0 \leq s \leq t \leq 1 \end{cases}, \tag{18}$$

is a continuous Green function on $[0,1]$. Define maps $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ and $\mathcal{T}:\mathcal{X}\longrightarrow\mathcal{X}$ as:

$$\mathcal{S}u(t)=\int_{-0}^{1}G(s,t)\psi(s,u(s),u'(s),u''(s))ds$$

and

$$\mathcal{T}u(t)=\int_{0}^{1}G(s,t)\phi(s,\mathfrak{v}(s),\mathfrak{v}'(s),\mathfrak{v}''(s))ds.$$

Then u is a solution of (14-15) iff u is a single common fixed point of $\mathcal{S}$ and $\mathcal{T}$ respectively.
Clearly, $\mathcal{S},\ \mathcal{T}:\mathcal{X}\longrightarrow\mathcal{X}$are well defined, so

$$\begin{aligned}
&p_b(\mathcal{S}u(t),\mathcal{T}\mathfrak{v}(t))=\Big(\frac{|\mathcal{S}u(t)|+|\mathcal{T}\mathfrak{v}(t)|}{2}\Big)^2\\
&=\left(\frac{\left|\int_0^1\mathcal{G}(s,t)\psi(s,u(s),u'(s),u''(s))ds\right|+\left|\int_0^1\mathcal{G}(s,t)\phi(s,\mathfrak{v}(s),\mathfrak{v}'(s),\mathfrak{v}''(s))ds\right|}{2}\right)^2\\
&\leq\left(\frac{\int_0^1\mathcal{G}(s,t)\,|\psi(s,u(s),u'(s),u''(s))|\,ds+\int_0^1\mathcal{G}(s,t)\,|\phi(s,\mathfrak{v}(s),\mathfrak{v}'(s),\mathfrak{v}''(s))|\,ds}{2}\right)^2\\
&=\frac{1}{4}\left(\int_0^1\mathcal{G}(t,s)\big(\left|\psi(s,u(s),u'(s),u''(s))\right|+\left|\phi(s,\mathfrak{v}(s),\mathfrak{v}'(s),\mathfrak{v}''(s))\right|\big)ds\right)^2\\
&\leq\frac{1}{4}\max(\left|\psi(s,u(s),u'(s),u''(s))\right|+\left|\phi(s,\mathfrak{v}(s),\mathfrak{v}'(s),\mathfrak{v}''(s))\right|)^2\left(\int_{-1}^1\mathcal{G}(t,s)ds\right)^2\\
&\leq\frac{1}{4}[\exp^{-\alpha}\max_{t\in[0,1]}|u(t)+\mathfrak{v}(t)|^2+\exp^{-\beta}\max_{t\in[0,1]}|u(t)|^2+\exp^{-\gamma}\max_{t\in[0,1]}|\mathfrak{v}(t)|^2]\\
&\qquad\left(\int_{-1}^1\mathcal{G}(t,s)ds\right)^2\\
&\leq\frac{1}{4}\left[\exp^{-\alpha}\max_{t\in[0,1]}|u(t)+\mathfrak{v}(t)|^2+\exp^{-\beta}\max_{t\in[0,1]}|u(t)|^2+\exp^{-\gamma}\max_{t\in[0,1]}|\mathfrak{v}(t)|^2\right]\frac{5}{12}\\
&\leq\exp^{-\alpha}p_b(u(t),\mathfrak{v}(t))+\exp^{-\beta}p_b(u(t),\mathcal{S}u(t))+\exp^{-\gamma}p_b(\mathfrak{v}(t),\mathcal{T}\mathfrak{v}(t)).
\end{aligned}\tag{19}$$

Hence all the postulates of Corolarry 1 are verified for $a=\exp^{-\alpha}$, $b=\exp^{-\beta}$, $f=\exp^{-\gamma}$ and the Cantilever beam problem has one and only one solution. □

4 Conclusion

We have established a common fixed point of non-continuous maps exploiting partial b-metric and without exploiting commutativity or its weaker form ([17]), which is indispensable for the survival of one and only one common fixed point in analogous theorems present in the literature. Consequently, our theorems are sharpened versions of the well-known results, wherein any variant of continuity [18] or commutativity is not essentially required for the survival of a single common fixed point. Examples and applications to solve a Cantilever beam problem employed in the distortion of an elastic beam in equilibrium substantiate the utility of these improvements and extensions.

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