



Finite Difference Modeling of Orthotropic Materials

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Abstract: This paper presents a finite-difference computational method for the integration of differential equations with partial derivatives which describe the plane state of displacement or stress of the anisotropic, orthotropic and isotropic materials. The paper examines the anisotropic case, and the relations can be particularized for isotropic materials.

We start from the classic idea of the Airy stress function. The second order partial derivatives of this describe the stress field, but with the help of the stresses, and with the equations of the material the specific strains can be determined. The disadvantage of using the Airy function is that all the boundary conditions must be given in stresses, because the displacement cannot be expressed in a direct way.

We discovered through analogy, that a “potential function” of the displacement can be used, which makes the prescription of the mixed boundary conditions possible. The partial derivatives of this function give the displacement in the direction of the coordinate axes. The derivatives of the displacement, namely the derivatives of superior order of the function of the displacement give the specific strains, and through the application of the material equations, these derivatives of superior order will lead to the stress field. This, points to the fact that the description of the boundary conditions under the form of prescribed stresses (of the load distribution on the boundary) becomes possible, because there is a direct relation (differential equations) between the displacements and stresses. These relations are estimated with finite differences. The disadvantage of the method is that we can have body forces only in one direction.

Keywords: finite-difference method, plane stress and strain, orthotropic materials

1. Introduction

In the linear elasticity theory it is assumed that the relations between stress and strain are linear. Using matrix formulation, this can be described by the formula (Hooke's law):

$$\{\sigma\} = [E] \{\varepsilon\}, \quad (1)$$

where $\{\sigma\} = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}]^T$ and $\{\varepsilon\} = [\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{zx}]^T$ contain each 6 components of the stress, respectively of the strain, and $[E]$ is the 6-by-6 elasticity matrix, which contains material constants [9].

For orthotropic materials in plane stress state, the matrix value $[E]$ has the following form [5], [6]:

$$[E] = \begin{bmatrix} \frac{E_x}{1 - \mu_{xy} \cdot \mu_{yx}} & \frac{\mu_{xy} \cdot E_y}{1 - \mu_{xy} \cdot \mu_{yx}} & 0 \\ \frac{\mu_{yx} \cdot E_x}{1 - \mu_{xy} \cdot \mu_{yx}} & \frac{E_y}{1 - \mu_{xy} \cdot \mu_{yx}} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}, \quad (2)$$

and for plane strain state, as follows:

$$[E] = \begin{bmatrix} \frac{E_x \cdot (1 - \mu_{yz} \cdot \mu_{zy})}{\delta} & \frac{E_x \cdot (\mu_{yx} + \mu_{zx} \cdot \mu_{yz})}{\delta} & 0 \\ \frac{E_y \cdot (\mu_{xy} + \mu_{xz} \cdot \mu_{zy})}{\delta} & \frac{E_y \cdot (1 - \mu_{xz} \cdot \mu_{zx})}{\delta} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}, \quad (3)$$

where

$$\delta = 1 - \mu_{xy} \mu_{yx} - \mu_{yz} \mu_{zy} - \mu_{zx} \mu_{xz} - \mu_{xy} \mu_{yz} \mu_{zx} - \mu_{yx} \mu_{zy} \mu_{xz}. \quad (4.a)$$

In these formulae E_i are the Young's moduli and μ_{ij} are the Poisson's ratios, each defined in the directions of the used coordinate system. The shear modulus can be expressed as

$$G_{xy} = G_{yx} = \frac{E_x \cdot E_y}{E_x \cdot (1 + \mu_{yx}) + E_y \cdot (1 + \mu_{xy})}, \quad (4.b)$$

and there exists a compatibility equation

$$\mu_{yx} \cdot E_x = \mu_{xy} \cdot E_y, \quad (4.c)$$

so in the plane case we have only three independent material constants and both (2) and (3) are symmetric.

The two elasticity matrices $[E]$ of above are valid only if the directions of orthotropy coincide with the directions of the coordinate axes. Otherwise, these must be rotated with the angle θ , that is measured from the first direction of the orthotropy 1 to the used x axis. This transformation leads to a full matrix [5]:

$$[E] = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix}, \quad (5)$$

of which components are:

$$\begin{aligned} \bar{E}_{11} &= E_{11} \cos^4 \theta + E_{22} \sin^4 \theta + (E_{12} + E_{21} + 4 E_{33}) \sin^2 \theta \cos^2 \theta, \\ \bar{E}_{12} &= E_{12} \cos^4 \theta + E_{21} \sin^4 \theta + (E_{11} + E_{22} + 4 E_{33}) \sin^2 \theta \cos^2 \theta, \\ \bar{E}_{13} &= (E_{11} - E_{12} - 2 \cdot E_{33}) \cdot \sin \theta \cdot \cos^3 \theta + (E_{21} - E_{22} + 2 \cdot E_{33}) \cdot \sin^3 \theta \cdot \cos \theta, \\ \bar{E}_{21} &= E_{12} \sin^4 \theta + E_{21} \cos^4 \theta + (E_{11} + E_{22} + 4 E_{33}) \cdot \sin^2 \theta \cdot \cos^2 \theta, \\ \bar{E}_{22} &= E_{11} \sin^4 \theta + E_{22} \cos^4 \theta + (E_{21} + E_{12} + 4 E_{33}) \sin^2 \theta \cos^2 \theta, \\ \bar{E}_{23} &= (E_{11} - E_{12} - 2 \cdot E_{33}) \cdot \sin^3 \theta \cdot \cos \theta + (E_{21} - E_{22} + 2 \cdot E_{33}) \cdot \sin \theta \cdot \cos^3 \theta, \\ \bar{E}_{31} &= (E_{11} - E_{21} - 2 \cdot E_{33}) \cdot \sin \theta \cdot \cos^3 \theta + (E_{12} - E_{22} + 2 \cdot E_{33}) \cdot \sin^3 \theta \cdot \cos \theta, \\ \bar{E}_{32} &= (E_{11} - E_{21} - 2 \cdot E_{33}) \cdot \sin^3 \theta \cdot \cos \theta + (E_{12} - E_{22} + 2 \cdot E_{33}) \cdot \sin \theta \cdot \cos^3 \theta, \\ \bar{E}_{33} &= (E_{11} + E_{12} + E_{21} + E_{22} - 2 \cdot E_{33}) \sin^2 \theta \cos^2 \theta + E_{33} (\sin^4 \theta + \cos^4 \theta), \end{aligned} \quad (6)$$

where E_{ij} are the members of the $[E]$ elasticity matrix (eq. 2 or 3).

Therefore, when the directions of orthotropy do not coincide with the use coordinate axes, the elasticity matrix contains nine nonzero elements (but only three independent ones) and it is symmetric ($\bar{E}_{ij} = \bar{E}_{ji}$). In the general case of plane anisotropy, the elasticity matrix is also full and symmetric, but now it contains six independent elements.

2. Formulation for finite-difference solution

Starting from the idea of Airy stress function, let us suppose that in the case of orthotropic materials there is a function $\Psi(x, y)$ of which partial derivatives give the projections of the displacement as it follows [7]:

$$\begin{aligned} u &= \alpha_1 \cdot \frac{\partial^2 \Psi}{\partial x^2} + \alpha_2 \cdot \frac{\partial^2 \Psi}{\partial x \cdot \partial y} + \alpha_3 \cdot \frac{\partial^2 \Psi}{\partial y^2}, \\ v &= \alpha_4 \cdot \frac{\partial^2 \Psi}{\partial x^2} + \alpha_5 \cdot \frac{\partial^2 \Psi}{\partial x \cdot \partial y} + \alpha_6 \cdot \frac{\partial^2 \Psi}{\partial y^2}, \end{aligned} \quad (7)$$

where u is the projection on the x axis, respectively v on the y axis, and the α_i are some coefficients to be determined. It can be shown that the (7) relations satisfy the compatibility equations of the strains

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y} = 0. \quad (8)$$

We transcribe the

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + f_y = 0 \quad (9)$$

equilibrium equations using Hooke's law in strains, considering that the f body force has only vertical component ($f_x = 0$):

$$\begin{aligned} \frac{\partial(E_{11} \cdot \varepsilon_x + E_{12} \cdot \varepsilon_y + E_{13} \cdot \gamma_{xy})}{\partial x} + \frac{\partial(E_{31} \cdot \varepsilon_x + E_{32} \cdot \varepsilon_y + E_{33} \cdot \gamma_{xy})}{\partial y} &= 0, \\ \frac{\partial(E_{21} \cdot \varepsilon_x + E_{22} \cdot \varepsilon_y + E_{23} \cdot \gamma_{xy})}{\partial y} + \frac{\partial(E_{31} \cdot \varepsilon_x + E_{32} \cdot \varepsilon_y + E_{33} \cdot \gamma_{xy})}{\partial x} + f_y &= 0. \end{aligned} \quad (10)$$

Then with the

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad (11)$$

geometrical equations, using the (7) expressions of the displacements, we obtain the followings:

$$\begin{aligned}
& (\alpha_1 \cdot E_{11} + \alpha_4 \cdot E_{13}) \cdot \frac{\partial^4 \Psi}{\partial x^4} + \\
& + (\alpha_1 \cdot E_{13} + \alpha_1 \cdot E_{31} + \alpha_2 \cdot E_{11} + \alpha_4 \cdot E_{12} + \alpha_4 \cdot E_{33} + \alpha_5 \cdot E_{13}) \cdot \frac{\partial^4 \Psi}{\partial x^3 \cdot \partial y} + \\
& + (\alpha_1 \cdot E_{33} + \alpha_2 \cdot E_{13} + \alpha_2 \cdot E_{31} + \alpha_3 \cdot E_{11} + \\
& \quad + \alpha_4 \cdot E_{32} + \alpha_5 \cdot E_{12} + \alpha_5 \cdot E_{33} + \alpha_6 \cdot E_{13}) \cdot \frac{\partial^4 \Psi}{\partial x^2 \cdot \partial y^2} + \\
& + (\alpha_2 \cdot E_{33} + \alpha_3 \cdot E_{13} + \alpha_3 \cdot E_{31} + \alpha_5 \cdot E_{32} + \alpha_6 \cdot E_{12} + \alpha_6 \cdot E_{33}) \cdot \frac{\partial^4 \Psi}{\partial x \cdot \partial y^3} + \\
& + (\alpha_3 \cdot E_{33} + \alpha_6 \cdot E_{32}) \cdot \frac{\partial^4 \Psi}{\partial y^4} = 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
& (\alpha_1 \cdot E_{31} + \alpha_4 \cdot E_{33}) \cdot \frac{\partial^4 \Psi}{\partial x^4} + \\
& + (\alpha_1 \cdot E_{21} + \alpha_1 \cdot E_{33} + \alpha_2 \cdot E_{31} + \alpha_4 \cdot E_{23} + \alpha_4 \cdot E_{32} + \alpha_5 \cdot E_{33}) \cdot \frac{\partial^4 \Psi}{\partial x^3 \cdot \partial y} + \\
& + (\alpha_1 \cdot E_{23} + \alpha_2 \cdot E_{21} + \alpha_2 \cdot E_{33} + \alpha_3 \cdot E_{31} + \\
& \quad + \alpha_4 \cdot E_{22} + \alpha_5 \cdot E_{23} + \alpha_5 \cdot E_{32} + \alpha_6 \cdot E_{33}) \cdot \frac{\partial^4 \Psi}{\partial x^2 \cdot \partial y^2} + \\
& + (\alpha_2 \cdot E_{23} + \alpha_3 \cdot E_{21} + \alpha_3 \cdot E_{33} + \alpha_5 \cdot E_{22} + \alpha_6 \cdot E_{23} + \alpha_6 \cdot E_{32}) \cdot \frac{\partial^4 \Psi}{\partial x \cdot \partial y^3} + \\
& + (\alpha_3 \cdot E_{23} + \alpha_6 \cdot E_{22}) \cdot \frac{\partial^4 \Psi}{\partial y^4} + f_y = 0.
\end{aligned} \tag{13}$$

We wish to determine the α coefficients in such manner to get all multipliers of the partial derivatives of the first equation equal to zero. In this case any function Ψ is a solution of the first equation (12):

$$\begin{aligned}
& \alpha_1 \cdot E_{11} + \alpha_4 \cdot E_{13} = 0 \\
& \alpha_1 \cdot E_{13} + \alpha_1 \cdot E_{31} + \alpha_2 \cdot E_{11} + \alpha_4 \cdot E_{12} + \alpha_4 \cdot E_{33} + \alpha_5 \cdot E_{13} = 0 \\
& \alpha_1 \cdot E_{33} + \alpha_2 \cdot E_{13} + \alpha_2 \cdot E_{31} + \alpha_3 \cdot E_{11} + \alpha_4 \cdot E_{32} + \alpha_5 \cdot E_{12} + \alpha_5 \cdot E_{33} + \alpha_6 \cdot E_{13} = 0 \quad (14) \\
& \alpha_2 \cdot E_{33} + \alpha_3 \cdot E_{13} + \alpha_3 \cdot E_{31} + \alpha_5 \cdot E_{32} + \alpha_6 \cdot E_{12} + \alpha_6 \cdot E_{33} = 0 \\
& \alpha_3 \cdot E_{33} + \alpha_6 \cdot E_{32} = 0.
\end{aligned}$$

Since six coefficients cannot be determined from these five equations, we must prescribe one of the values [1]. Therefore, we assign $\alpha_2 = 1$, and the five

remaining coefficients are found by solving the system of equations (14). This can be resolved using numerical methods. With the obtained α coefficients the second equilibrium equation (13) became:

$$\beta_1 \cdot \frac{\partial^4 \Psi}{\partial x^4} + \beta_2 \cdot \frac{\partial^4 \Psi}{\partial x^3 \cdot \partial y} + \beta_3 \cdot \frac{\partial^4 \Psi}{\partial x^2 \cdot \partial y^2} + \beta_4 \cdot \frac{\partial^4 \Psi}{\partial x \cdot \partial y^3} + \beta_5 \cdot \frac{\partial^4 \Psi}{\partial y^4} = \beta_0 \cdot f_y, \quad (15)$$

of which solution is the potential function sought by us. The coefficients of this equation are:

$$\begin{aligned} \beta_1 &= \alpha_1 \cdot E_{31} + \alpha_4 \cdot E_{33}, \\ \beta_2 &= \alpha_1 \cdot (E_{21} + E_{33}) + \alpha_2 \cdot E_{31} + \alpha_4 \cdot (E_{23} + E_{32}) + \alpha_5 \cdot E_{33}, \\ \beta_3 &= \alpha_1 \cdot E_{23} + \alpha_2 \cdot (E_{21} + E_{33}) + \alpha_3 \cdot E_{31} + \alpha_4 \cdot E_{22} + \alpha_5 \cdot (E_{23} + E_{32}) + \alpha_6 \cdot E_{33}, \\ \beta_4 &= \alpha_2 \cdot E_{23} + \alpha_3 \cdot (E_{21} + E_{33}) + \alpha_5 \cdot E_{22} + \alpha_6 \cdot (E_{23} + E_{32}), \\ \beta_5 &= \alpha_3 \cdot E_{23} + \alpha_6 \cdot E_{22}, \\ \beta_0 &= -1. \end{aligned} \quad (16)$$

When the orthotropy directions coincide with the x and y axes, the expressions of coefficients β_i are simpler: if the angle θ is an integer multiple of the right angle, the coefficients β_2 and β_4 are equal to zero. In case of isotropic materials the relations will be more simplified. For isotropic materials, when $f_y = 0$, the equation (13) becomes biharmonic.

The problem is ultimately reduced to solving the equation (15): we propose a method using finite differences. If the partial derivatives of the equation are replaced by centered finite differences, we achieve the molecule presented in Fig. 1. Therefore, in the (i, j) node of the grid we can write the following equation:

	$-2 \cdot h$	$-h$	0	h	$2 \cdot h$
$2 \cdot k$		$-\frac{\beta_4}{4 \cdot h \cdot k^3}$	$\frac{\beta_2}{k^4}$	$\frac{\beta_4}{4 \cdot h \cdot k^3}$	
k	$-\frac{\beta_2}{4 \cdot h^3 \cdot k}$	$\frac{\beta_2}{2 \cdot h^2 \cdot k} + \frac{\beta_3}{h^2 \cdot k^3} + \frac{\beta_4}{2 \cdot h \cdot k^3}$	$-2 \cdot \frac{\beta_2}{h^2 \cdot k^3} - 4 \cdot \frac{\beta_1}{k^4}$	$-\frac{\beta_2}{2 \cdot h^2 \cdot k} + \frac{\beta_3}{h^2 \cdot k^3} - \frac{\beta_4}{2 \cdot h \cdot k^3}$	$\frac{\beta_2}{4 \cdot h^3 \cdot k}$
0	$\frac{\beta_1}{h^4}$	$-4 \cdot \frac{\beta_1}{h^4} - 2 \cdot \frac{\beta_3}{h^2 \cdot k^3}$	$6 \cdot \frac{\beta_1}{h^4} + 4 \cdot \frac{\beta_3}{h^2 \cdot k^3} + 6 \cdot \frac{\beta_1}{k^4}$	$-4 \cdot \frac{\beta_1}{h^4} - 2 \cdot \frac{\beta_3}{h^2 \cdot k^3}$	$\frac{\beta_1}{h^4}$
$-k$	$\frac{\beta_2}{4 \cdot h^3 \cdot k}$	$-\frac{\beta_2}{2 \cdot h^2 \cdot k} + \frac{\beta_3}{h^2 \cdot k^3} - \frac{\beta_4}{2 \cdot h \cdot k^3}$	$-2 \cdot \frac{\beta_2}{h^2 \cdot k^3} - 4 \cdot \frac{\beta_1}{k^4}$	$\frac{\beta_2}{2 \cdot h^2 \cdot k} + \frac{\beta_3}{h^2 \cdot k^3} + \frac{\beta_4}{2 \cdot h \cdot k^3}$	$-\frac{\beta_2}{4 \cdot h^3 \cdot k}$
$-2 \cdot k$		$\frac{\beta_4}{4 \cdot h \cdot k^3}$	$\frac{\beta_2}{k^4}$	$-\frac{\beta_4}{4 \cdot h \cdot k^3}$	

Figure 1: The approximation with finite-difference method of the equation (15).

$$\begin{aligned}
& \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot \Psi(i-1, j-2) + \frac{\beta_5}{k^4} \cdot \Psi(i, j-2) - \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot \Psi(i+1, j-2) + \\
& + \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i-2, j-1) - \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} - \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i-1, j-1) - \\
& - \left(2 \cdot \frac{\beta_3}{h^2 \cdot k^2} + 4 \cdot \frac{\beta_5}{k^4} \right) \cdot \Psi(i, j-1) + \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i+1, j-1) - \\
& - \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i+2, j-1) + \frac{\beta_1}{h^4} \cdot \Psi(i-2, j) - \left(4 \cdot \frac{\beta_1}{h^4} + 2 \cdot \frac{\beta_3}{h^2 \cdot k^2} \right) \cdot \Psi(i-1, j) + \\
& + \left(6 \cdot \frac{\beta_1}{h^4} + 4 \cdot \frac{\beta_3}{h^2 \cdot k^2} + 6 \cdot \frac{\beta_5}{k^4} \right) \cdot \Psi(i, j) - \left(4 \cdot \frac{\beta_1}{h^4} + 2 \cdot \frac{\beta_3}{h^2 \cdot k^2} \right) \cdot \Psi(i+1, j) + \quad (17) \\
& + \frac{\beta_1}{h^4} \cdot \Psi(i+2, j) - \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i-2, j+1) + \\
& + \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} + \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i-1, j+1) - \left(2 \cdot \frac{\beta_3}{h^2 \cdot k^2} + 4 \cdot \frac{\beta_5}{k^4} \right) \cdot \Psi(i, j+1) - \\
& - \left(\frac{\beta_2}{2 \cdot h^3 \cdot k} - \frac{\beta_3}{h^2 \cdot k^2} + \frac{\beta_4}{2 \cdot h \cdot k^3} \right) \cdot \Psi(i+1, j+1) + \frac{\beta_2}{4 \cdot h^3 \cdot k} \cdot \Psi(i+2, j+1) - \\
& - \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot f(i-1, j+2) + \frac{\beta_5}{k^4} \cdot \Psi(i, j+2) + \frac{\beta_4}{4 \cdot h \cdot k^3} \cdot \Psi(i, j+2) = f_y(i, j).
\end{aligned}$$

At each node of the finite-difference grid, excepting those on the boundary, we write this equation. In these appear the values of the Ψ function taken in the neighbouring nodes, resulting in a system of equations to be solved in the $\Psi(i, j)$ nodal values.

For the boundary points we can't apply the molecule from *Fig. 1*, because in (17) appear values of Ψ in some non-existing external nodes. These values also appear when we apply (17) for the nodes next to the boundary ones: these external nodes define a new virtual boundary beyond the physical one, increasing the number of the unknowns to be determined. It is to note, that the external grid points, situated in the diagonally opposite convex corners, do not belong to this virtual boundary (*Fig. 2*).

The system of equations can be solved only by writing boundary conditions: we will give these conditions in all boundary nodes, as prescribed displacements and loading forces.

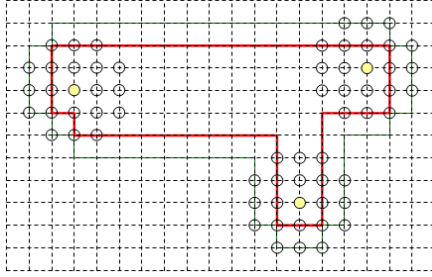


Figure 2: The virtual boundary that appears due to finite-difference approximation.

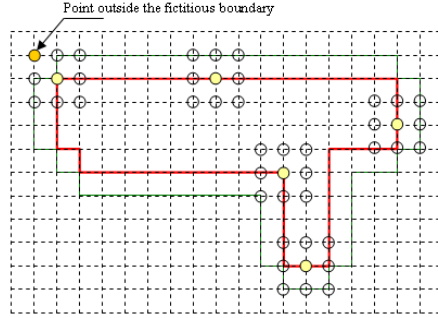


Figure 3: Cells used in approximation of u and v with centered differences.

3. The boundary conditions as prescribed displacements

For easier application of this method, let's approximate the physical boundary with one which is made from horizontal and vertical lines adapted to the grid. In this case we define the boundary conditions as the projections of the displacement, as prescribed values of u and v . According to the relations (7) these projections are obtained by deriving the function Ψ . If we express u from the relation (7) with centered differences, we obtain the molecule from Fig. 4. and the following equation:

$$\begin{aligned} & \frac{\alpha_2}{4 \cdot h \cdot k} \cdot \Psi(i-1, j-1) + \frac{\alpha_3}{k^2} \cdot \Psi(i, j-1) - \frac{\alpha_2}{4 \cdot h \cdot k} \cdot \Psi(i+1, j-1) + \\ & + \frac{\alpha_1}{h^2} \cdot \Psi(i-1, j) - \left(2 \cdot \frac{\alpha_1}{h^2} + 2 \cdot \frac{\alpha_3}{k^2} \right) \cdot \Psi(i, j) + \frac{\alpha_1}{h^2} \cdot \Psi(i+1, j) - \\ & - \frac{\alpha_2}{4 \cdot h \cdot k} \cdot \Psi(i-1, j+1) + \frac{\alpha_3}{k^2} \cdot \Psi(i, j+1) + \frac{\alpha_2}{4 \cdot h \cdot k} \cdot \Psi(i+1, j+1) = u(i, j). \end{aligned} \quad (18)$$

	$-h$	0	h
k	$-\frac{\alpha_2}{4 \cdot h \cdot k}$	$\frac{\alpha_3}{k^2}$	$\frac{\alpha_2}{4 \cdot h \cdot k}$
0	$\frac{\alpha_1}{h^2}$	$-2 \cdot \frac{\alpha_1}{h^2} - 2 \cdot \frac{\alpha_3}{k^2}$	$\frac{\alpha_1}{h^2}$
$-k$	$\frac{\alpha_2}{4 \cdot h \cdot k}$	$\frac{\alpha_3}{k^2}$	$-\frac{\alpha_2}{4 \cdot h \cdot k}$

Figure 4: The approximation of the displacements with centered differences.

For v we obtain the same scheme and formula, the index of the α -s have to be increased by 3.

We can observe that applying the molecule for a grid node positioned on the boundary, it will be based on three points that are on the imaginary boundary. Straight edges and concave corners will not raise issues or difficulties, however in the convex corners this scheme would include a point that does not belong to the virtual boundary (Fig. 3). In this case instead of centered difference approximation, we apply the derivatives' approximations with the help of forward or backward differences, depending on the corner position (Fig. 5. - in this figure **X** denotes the position of the node that does not belong to the virtual boundary).

	$-h$	0	h	$-h$	0	h
k	X	$\frac{\alpha_2}{k^2}$			$\frac{\alpha_2}{k^2}$	X
0	$\frac{\alpha_1}{h^2}$	$-2 \cdot \frac{\alpha_1}{h^2} - \frac{\alpha_2}{h \cdot k} - 2 \cdot \frac{\alpha_3}{k^2}$	$\frac{\alpha_1}{h^2} + \frac{\alpha_2}{h \cdot k}$	$\frac{\alpha_1}{h^2} - \frac{\alpha_2}{h \cdot k}$	$-2 \cdot \frac{\alpha_1}{h^2} + \frac{\alpha_2}{h \cdot k} - 2 \cdot \frac{\alpha_3}{k^2}$	$\frac{\alpha_1}{h^2}$
$-k$		$\frac{\alpha_2}{h \cdot k} + \frac{\alpha_3}{k^2}$	$-\frac{\alpha_2}{h \cdot k}$	$\frac{\alpha_2}{h \cdot k}$	$-\frac{\alpha_2}{h \cdot k} + \frac{\alpha_3}{k^2}$	
k		$-\frac{\alpha_2}{h \cdot k} + \frac{\alpha_3}{k^2}$	$\frac{\alpha_2}{h \cdot k}$	$-\frac{\alpha_2}{h \cdot k}$	$\frac{\alpha_2}{h \cdot k} + \frac{\alpha_3}{k^2}$	
0	$\frac{\alpha_1}{h^2}$	$-2 \cdot \frac{\alpha_1}{h^2} + \frac{\alpha_2}{h \cdot k} - 2 \cdot \frac{\alpha_3}{k^2}$	$\frac{\alpha_1}{h^2} - \frac{\alpha_2}{h \cdot k}$	$\frac{\alpha_1}{h^2} + \frac{\alpha_2}{h \cdot k}$	$-2 \cdot \frac{\alpha_1}{h^2} - \frac{\alpha_2}{h \cdot k} - 2 \cdot \frac{\alpha_3}{k^2}$	$\frac{\alpha_1}{h^2}$
$-k$	X	$\frac{\alpha_3}{k^2}$			$\frac{\alpha_3}{k^2}$	X

Figure 5: The approximation of displacements in the convex corners (upper left, upper right, lower left and lower right corners; the double line shows the boundary).

4. The boundary conditions as loading

Distributed stress, that loads the boundary is defined by its projections according to x and y directions, noted as p_x and p_y . These projections generally are described by arbitrary functions. During the grid generation, these functions are replaced by step functions. Concentrated loads also will be replaced by constant distributed loads (Fig. 6).

On the loaded sides, we can write some equilibrium equations, those give some relations between external loads and internal stresses (Fig. 7).

Using Hooke's law, stresses can be expressed using strains (1), then using the (11) definitions of the strains and the (7) expressions of the displacements we obtain some equations giving stresses as functions of the derivatives of Ψ , e.g. we can write:

$$\begin{aligned}
 \sigma_x &= E_{11} \epsilon_x + E_{12} \epsilon_y + E_{13} \gamma_{xy} = \\
 &= E_{11} \cdot \frac{\partial u}{\partial x} + E_{12} \cdot \frac{\partial v}{\partial y} + E_{13} \cdot \frac{\partial u}{\partial y} + E_{13} \cdot \frac{\partial v}{\partial x} = \\
 &= E_{11} \left(\alpha_1 \frac{\partial^3 \Psi}{\partial x^3} + \alpha_2 \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_3 \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} \right) + \\
 &+ E_{12} \cdot \left(\alpha_4 \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_5 \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \alpha_6 \frac{\partial^3 \Psi}{\partial y^3} \right) + \\
 &+ E_{13} \cdot \left(\alpha_1 \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_2 \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \alpha_3 \frac{\partial^3 \Psi}{\partial y^3} \right) + \\
 &+ E_{13} \cdot \left(\alpha_4 \frac{\partial^3 \Psi}{\partial x^3} + \alpha_5 \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_6 \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} \right) = \\
 &= (E_{11} \alpha_1 + E_{13} \alpha_4) \frac{\partial^3 \Psi}{\partial x^3} + \\
 &+ (E_{11} \alpha_2 + E_{12} \alpha_4 + E_{13} \alpha_1 + E_{13} \alpha_5) \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \\
 &+ (E_{11} \alpha_3 + E_{12} \alpha_5 + E_{13} \alpha_2 + E_{13} \alpha_6) \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \\
 &+ (E_{12} \alpha_6 + E_{13} \alpha_3) \cdot \frac{\partial^3 \Psi}{\partial y^3}.
 \end{aligned} \tag{19}$$

These derivatives can be replaced by finite differences. We will exemplify such boundary conditions for the vertical edge on the left side, as follows:

$$\begin{aligned}
 & - \frac{c_4}{2 \cdot k^3} \cdot \Psi(j-2, i) - \left(\frac{c_2}{2 \cdot h^2 \cdot k} + \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j-1, i-1) + \\
 & + \left(\frac{c_2}{h^2 \cdot k} + \frac{c_4}{k^3} \right) \cdot \Psi(j-1, i) - \left(\frac{c_2}{2 \cdot h^2 \cdot k} - \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j-1, i+1) +
 \end{aligned}$$

$$\begin{aligned}
& + \frac{c_3}{h \cdot k^2} \cdot \Psi(j, i-1) - \frac{c_1}{h^3} \cdot \Psi(j, i) + \left(\frac{3 \cdot c_1}{h^3} - \frac{c_3}{h \cdot k^2} \right) \cdot \Psi(j, i+1) - \\
& - \frac{3 \cdot c_1}{h^3} \cdot \Psi(j, i+2) + \frac{c_1}{h^3} \cdot \Psi(j, i+3) + \left(\frac{c_2}{2 \cdot h^2 \cdot k} - \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j+1, i-1) - \\
& - \left(\frac{c_2}{h^2 \cdot k} + \frac{c_4}{k^3} \right) \cdot \Psi(j+1, i) + \left(\frac{c_2}{2 \cdot h^2 \cdot k} + \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j+1, i+1) + \\
& + \frac{c_4}{2 \cdot k^3} \cdot \Psi(j+2, i) = p_x,
\end{aligned} \tag{20}$$

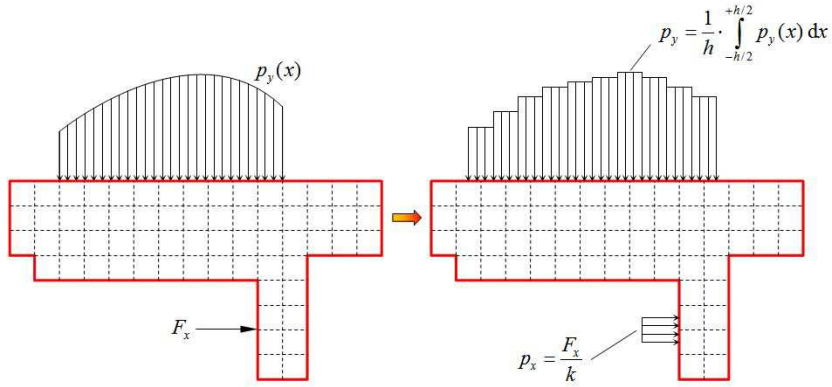


Figure 6: Load replaced by step functions.

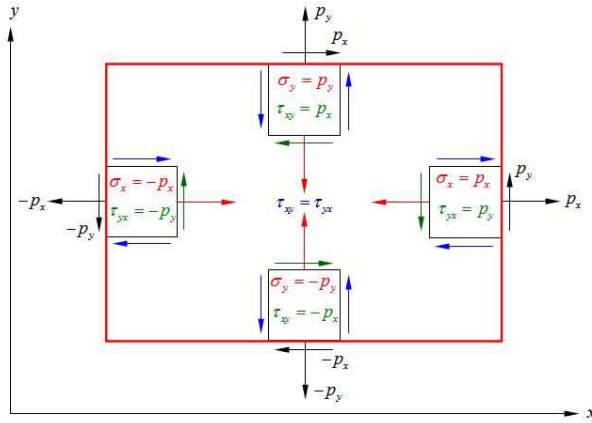


Figure 7: Equilibrium conditions on the boundary.

where

$$\begin{aligned}
 c_1 &= E_{11} \cdot \alpha_1 + E_{13} \cdot \alpha_4, \\
 c_2 &= E_{11} \cdot \alpha_2 + E_{12} \cdot \alpha_4 + E_{13} \cdot \alpha_1 + E_{13} \cdot \alpha_5, \\
 c_3 &= E_{11} \cdot \alpha_3 + E_{12} \cdot \alpha_5 + E_{13} \cdot \alpha_2 + E_{13} \cdot \alpha_6, \\
 c_4 &= E_{12} \cdot \alpha_6 + E_{13} \cdot \alpha_3.
 \end{aligned} \tag{21}$$

The corresponding molecule is shown in *Fig. 8*.

	$-h$	0	h	$2 \cdot h$	$3 \cdot h$
$2 \cdot k$		$\frac{c_4}{2 \cdot k^3}$			
k	$\frac{c_2}{2 \cdot h^2 \cdot k} - \frac{c_3}{2 \cdot h \cdot k^2}$	$-\frac{c_2}{h^2 \cdot k} - \frac{c_4}{k^3}$	$\frac{c_2}{2 \cdot h^2 \cdot k} + \frac{c_3}{2 \cdot h \cdot k^2}$		
0	$\frac{c_3}{h \cdot k^2}$	$-\frac{c_1}{h^3}$	$\frac{3 \cdot c_1}{h^3} - \frac{c_3}{h \cdot k^2}$	$-\frac{3 \cdot c_1}{h^3}$	$\frac{c_1}{h^3}$
$-k$	$-\frac{c_2}{2 \cdot h^2 \cdot k} - \frac{c_3}{2 \cdot h \cdot k^2}$	$\frac{c_2}{h^2 \cdot k} + \frac{c_4}{k^3}$	$-\frac{c_2}{2 \cdot h^2 \cdot k} + \frac{c_3}{2 \cdot h \cdot k^2}$		
$-2 \cdot k$		$-\frac{c_4}{2 \cdot k^3}$			

Figure 8: The approximation with finite differences method of the equation (20).

If we replace the elements of the matrix of elasticity in relations (21), we obtain the molecules of the stresses σ_y and τ_{xy} , leading to the equations in finite differences of the boundary conditions.

Corner nodes must be treated in different manner: because of the Cauchy-duality of the tangential stresses ($\tau_{xy} = \tau_{yx}$), external loads must be rearranged in such manner as this duality to be satisfied.

5. Conclusions

This paper presents a finite-difference computational method for the integration of differential equations with partial derivatives describing the plane state of displacement or stress of the anisotropic materials. As shown in the paper, the problem can be expressed in stress leading to Airy function, which describes the second order partial differential stress field. With stress and material equations we can determine the specific strains. This method has the disadvantage of the impossibility to express directly the displacements.

By analogy with the Airy function, we used a “potential function” of the displacement, which made it possible to write mixed boundary conditions. The partial derivatives of this function are equal with the displacements in the directions of coordinate axes. Displacement derivatives, as in the derivatives of superior order displacement function give specific strains and by using material equations these superior order derivatives will lead to the stress field. Therefore becomes possible to write outline conditions as distributed load shape, there is a direct relationship (differential equations) between displacements and stresses. These relationships are approximated by finite differences.

In approximation with finite differences the real boundary was replaced by a boundary consisting of horizontal and vertical straight lines and the boundary conditions as prescribed loading led to some equivalence relations between loads and stresses. The denser the grid is, the more accurate the modeling of the load will be and the negative effects of the approximations made in the corner points will be more reduced. The disadvantage of this method is the fact that we can have body forces only in one direction.

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