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Does another Euclidean plane exist other than the parasphere?

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Abstract. It is shown that for the question of the title the answer is yes. We construct a plane in the hyperbolic space which is Euclidean.

Let us use the symbols on the Figure 9. of Bolyai's Appendix [1] and take a fourth plane (more precisely a half plane) which intersects the $ABNM$ plane at an angle of $\frac{\pi}{2}$ and intersects the APM and the BND half planes.

According to the proof of Bolyai, the half planes APM resp. BND are intersecting each other if the angle between the APM and $ABNM$ planes is $\frac{\pi}{2}$ and the angle between BND and $ABNM$ (half) planes is arbitrarily less than $\frac{\pi}{2}$. Then it follows that the intersection lines of the fourth plane with the APM and BND half planes, respectively, are also intersecting each other. This means that in the fourth plane the Euclidean geometry is valid.

To describe the above construction in more detail, let us take a point R on the AM line and a point Z on the BN line such that the RZ line is perpendicular to BN.

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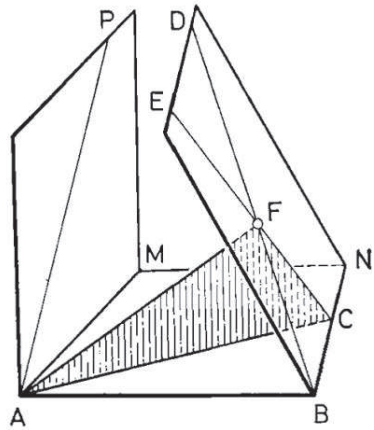

Figure 9

Take a plane containing the RZ line, which intersects the ABNM plane at $\frac{\pi}{2}$ angle (δ plane). Take a half plane containing AM intersecting the ABNM plane at $\frac{\pi}{2}$ angle (this will be denoted as AMP plane or β plane), and a half plane containing BN, which intersects the ABNM plane at an angle less than $\frac{\pi}{2}$ (BND plane or α plane). Then, according to Bolyais proof, the two half planes (α and β half plane) intersect each other and the plane through RZ (δ plane) intersects the other two, and the intersection lines of δ plane with the β plane and α plane, respectively, also intersect each other while the α half planes dihedral angle with the AMBN plane less then $\frac{\pi}{2}$. (As Bolyai has proven, if the α half plane intersects the ABNM plane at an angle arbitrarily smaller than $\frac{\pi}{2}$, then the half planes α and β intersect each other.) Then it follows: in the δ plane the Euclidean geometry seems to be valid. The intersection line of the α and β half planes does not intersect the ABNM plane, because the AM and BN lines are parallel.

Let us denote the intersection line of α and β half planes by K. The intersection lines of δ plane with the α and β half planes, respectively, both intersect K, because as Bolyai implicitly uses the statement: if there are two parallel lines in a plane and from one of the two we draw a perpendicular line in the plane, this latter line will intersect the other line. Also, K is parallel with AM and BN. If K would intersect AM or BN, then these two latter lines would also intersect each other.

Similar results were published recently by Miroslava Antic [2].

A very interesting related paper is published by Zoltán Győrfy [3].

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