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Differential subordination results for some classes of the family $\zeta(\varphi, \vartheta)$ associated with linear operator

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Abstract. For some classes of family of real valued functions defined in a unit disk, we use a linear operator to obtain some interesting differential subordination results.

1 Introduction and preliminaries

Let E^+_α denote the family of all functions $F(z)$, in the unit disk U, of the form

$$F(z) = 1 + \sum_{n=1}^{\infty} a_n z^{n-n/\alpha},\ \alpha = \{2, 3, 4 \ldots\} \tag{1}$$

satisfying $F(0) = 1$.

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Let E_α^- denote the family of all functions $F(z)$, in the unit disk U, of the form

$$F(z) = 1 - \sum_{n=1}^{\infty} a_n z^{n-n/\alpha},\ \alpha = \{2, 3, 4 \ldots\} \tag{2}$$

which satisfy the condition $F(0) = 1$.

We know that if functions f and g are analytic in U, then f is called subordinate to g if there exists a Schwarz function $w(z)$, analytic in U such that $f(z) = g(w(z))$, and $z \in U = \{z : z \in C,\ |z| < 1\}$.

Then we denote this subordination by $f(z) \prec g(z)$ or simply $f \prec g$, but in a special case if g is univalent in U then above subordination is equivalent to $f(0) = g(0)$, and $f(U) \subset g(U)$.

Let $\phi : C^3 \times U \to C$ and let h analytic in U. Assume that p, ϕ are analytic and univalent in U and p satisfies the differential superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z). \tag{3}$$

Then p is called a solution of the differential superordination.

An analytic function q is called a subordinant if $q \prec p$, for all p satisfying equation (3). A univalent function q such that $p \prec q$ for all subordinants p of equation (3) is said to be the best subordinant.

Let E^+ be the class of analytic functions of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathcal{U},\ a_n, b_n \geq 0.$$

Let $f, g \in E^+$ where

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = 1 + \sum_{n=1}^{\infty} a_n b_n z^n,\ z \in \mathcal{U}.$$

Juneja et al. [1] define the family $\varepsilon(\phi, \psi)$ so that

$$\text{Re}\left(\frac{f(z) * \phi(z)}{f(z) * \psi(z)}\right) > 0, z \in \mathcal{U}$$

where

$$\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n$$

and

$$\psi(z) = 1 + \sum_{n=1}^{\infty} \psi_n z^n$$

are analytic in U with the conditions $\phi_n,\ \mathtt{psi}_n \geq 0$, $\phi_n \geq \psi_n$ and $\phi(z) * \psi(z) \neq 0$.

Definition 1 *Let* $\zeta_\alpha^+(\varphi, \vartheta)$ *be the class of family of all* $F(z) \in E_\alpha^+$ *such that*

$$\text{Re}\left(\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)}\right) > 0, z \in \mathcal{U}$$

where

$$\varphi(z) = 1 + \sum_{n=2}^{\infty} \varphi_n z^{n-n/\alpha} \text{ and } \vartheta(z) = 1 + \sum_{n=2}^{\infty} \vartheta_n z^{n-n/\alpha}$$

are analytic in U *with specific conditions,* $\varphi_n, \vartheta_n \geq 0$, $\varphi_n \geq \vartheta_n$ *and* $F(z) * \vartheta(z) \neq 0$ *and for all* $n \geq 0$.

Definition 2 *Let* $\zeta_\alpha^-(\varphi, \vartheta)$ *be the class of family of all* $F(z) \in E_\alpha^-$ *such that*

$$\text{Re}\left(\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)}\right) > 0,\ z \in \mathcal{U}$$

where

$$\varphi(z) = 1 - \sum_{n=2}^{\infty} \varphi_n z^{n-n/\alpha} \text{ and } \vartheta(z) = 1 - \sum_{n=2}^{\infty} \vartheta_n z^{n-n/\alpha}$$

are analytic in U *with specific conditions,* $\varphi_n, \vartheta_n \geq 0$, $\varphi_n \geq \vartheta_n$ *and* $F(z) * \vartheta(z) \neq 0$ *and for all* $n \geq 0$.

The aim of the present paper is to propose some sufficient conditions for all functions $F(z)$ belongs to the new classes E_α^+ and E_α^- to satisfy

$$\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)} \prec q(z),\ z \in U.$$

Where $q(z)$ is a given univalent function in U such that $q(0) = 1$.

Define the function $\varphi_\alpha(a, c; z)$ by

$$\varphi_\alpha(a, c; z) = 1 + \sum_{1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-n/\alpha},\ z \in U,\ c \in \Re \setminus \{0, -1, -2 \dots\}$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)(a+2)\cdots(a+n-1) & \text{if } n \in N \end{cases}$$

Corresponding to the function $\varphi_\alpha(a, c; z)$, define a linear operator $I_\alpha(a, c)$, by

$$I_\alpha(a, c)F(z) = I_\alpha(a, c; z) * F(z),\quad F(z) \in E_\alpha^+,$$

or equivalently by

$$I_\alpha(a, c)F(z) = 1 + \sum_{1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-n/\alpha}, z \in U, c \in \Re \setminus \{0, -1, -2 \dots\}$$

Different authors have used this linear operator for various types of classes of univalent functions namely, Uralgaddi and Somanatha [4], Cho, Kwon and Srivastava [5], Saitoh [6], and Sokol and Spelina [7], respectively.

The classes E_α^+ and E_α^- defined above exhibit some interesting properties. We need the following lemmas.

Lemma 1 [3]. *Let* $q(z)$ *be univalent in the unit U disk and* $\theta(z)$ *be analytic in a domain D containing q(U). If* $zq'(z)\theta(q)$ *is starlike in U , and*

$$zp'(z)\theta(p(z)) \prec zq'(z)\theta(q(z))$$

then $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Theorem 1 *Let the function* $q(z)$ *be univalent in the unit disk* U *such that* $q'(z) \neq (0)$ *and* $\frac{zq'(z)}{q(z)} \neq 0$ *is starlike in* U*, if* $F(z) \in E_\alpha^+$ *satisfies the subordination*

$$b\left[\frac{z(I_\alpha(a, c)\phi(z))'}{I_\alpha(a, c)\phi(z)} - \frac{z(I_\alpha(a, c)\psi(z))'}{I_\alpha(a, c)\psi(z)}\right] \prec \frac{bzq'(z)}{q(z)}$$

then,

$$\left[\frac{I_\alpha(a, c)\phi(z)}{I_\alpha(a, c)\psi(z)}\right] \prec q(z)$$

Then is $q(z)$ *the best dominant.*

Proof. First we defined the function $p(z)$,

$$p(z) = \left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$$

then,

$$\frac{bzp'(z)}{p(z)} = b\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \tag{4}$$

By setting, $\theta(\omega) = \dfrac{b}{\omega}$, it can easily observed that $\theta(\omega)$ is analytic in $C \setminus \{0\}$. Then we obtain that,

$$\theta(p(z)) = \frac{b}{p(z)} \;\; and \;\; \theta(q(z)) = \frac{b}{q(z)}.$$

So from equation (4), we have

$$zp'(z)\theta(p(z)) \preceq b\frac{q'(z)}{q(z)} = zq'(z)\theta(q(z)),$$

this implies,

$$zp'(z)\theta(p(z)) \prec zq'(z)\theta(q(z))$$

from lemma (1), we have

$$p(z) \prec q(z)$$

this implies,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec q(z)$$

□

Corollary 1 *If* $F(z)$ *satisfies the subordination*

$$b\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{b(A-B)z}{(1+Az)(1+BZ)}\right]$$

then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{1+Az}{1+Bz}\right], \quad -1 \le A \le B \le 1,$$

and $\dfrac{(1+Az)}{(1+Bz)}$ *is the best dominant.*

Corollary 2 *If* $F(z)$ *satisfies the subordination*

$$b\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{2bz}{(1+z)(1+z)}\right]$$

then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{1+z}{1-z}\right], \quad -1 \le A \le B \le 1,$$

and $\frac{(1+z)}{(1+z)}$ *is the best dominant.*

Lemma 2 [2]. *Let* $q(z)$ *be convex in the unit disk* U *with* $q(0) = 1$ *and* $\Re(q) > 1/2$*,* $z \in U$*. If* $0 \le U < 1$*,* p *is analytic function in with* $p(0) = 1$ *and if*

$$(1-\mu)p^2(z) + (2\mu - 1)p(z) - \mu + (1-\mu)zp'(z)$$

$$\prec (1-\mu)q^2(z) + (2\mu - 1)q(z) - \mu + (1-\mu)zq'(z)$$

then $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Theorem 2 *Let* $q(z)$ *be convex in the unit disk* U *with* $q(0) = 1$ *and* $\Re(q) > 1/2$*. If* $F(z) \in E_\alpha^+$ *and* $\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$ *is an analytic function in* U *satisfies the subordination*

$$\begin{aligned}
&(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + (2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - \mu + \\
+ \quad &(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \\
\prec \quad &(1-\mu)q^2(z) + (2\mu-1)q(z) - \mu + (1-\mu)zq'(z)
\end{aligned}$$

Then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec q(z)$$

and $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) = \left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right], \; z \in U$$

since $p(0) = 1$, therefore

$$\begin{aligned}
&(1-\mu)p^2(z) + (2\mu-1)p(z) - \mu + (1-\mu)zp'(z) = \\
&= (1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + (2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - \mu + \\
&\quad + (1-\mu)z\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]' = \\
&= [1-\mu]\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + [2\mu-1]\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - [\mu] + \\
&\quad + (1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \\
&\prec (1-\mu)q^2(z) + (2\mu-1)q(z) - \mu + (1-\mu)zq'(z)
\end{aligned}$$

now by using the Lemma 2, we have

$$p(z) \prec q(z)$$

implies that,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec q(z)$$

and $q(z)$ is the best dominant. □

Corollary 3 *If* $F(z) \in E_\alpha^+$ *and* $\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$ *is an analytic function in* U *satisfying the subordination*

$$\begin{aligned}
&(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + (2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - \mu + \\
&\quad + (1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \\
&\prec (1-\mu)\left[\frac{1+Az}{1+Bz}\right]^2 + (2\mu-1)\left[\frac{1+Az}{1+Bz}\right] - \mu + \\
&\quad + (1-\mu)\left[\frac{1+Az}{1+Bz}\right]\left[\frac{(A-B)z}{(1+Az)(1+Bz)}\right]
\end{aligned}$$

Then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{(1+Az)}{(1+Bz)}\right]$$

and $\left[\frac{1+Az}{1+Bz}\right]$ *is the best dominant.*

Proof. Let us define $q(z)$ by

$$q(z)=\left[\frac{1+Az}{1+Bz}\right],\ z\in U$$

this implies that $q(0)=1$ and $\Re(q)>1/2$ for arbitrary $A,B,z\in U$ where

$$\frac{zq'(z)}{q(z)}=\frac{(A-B)z}{(1+Az)(1+Bz)}$$

Then applying the Theorem 2, we obtain the result. □

Corollary 4 *If* $F(z)\in E_\alpha^+$ *and* $\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$ *is an analytic function in* U *satisfying the subordination*

$$(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2+(2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]-\mu+$$
$$+(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)}-\frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right]\prec$$
$$\prec(1-\mu)\left[\frac{1+z}{1-z}\right]^2+(2\mu-1)\left[\frac{1+z}{1-z}\right]-\mu+(1-\mu)\left[\frac{1+z}{1-z}\right]\left[\frac{2z}{(1+z)(1-z)}\right]$$

Then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\prec\frac{(1+z)}{(1-z)}$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

Proof. Let the function $q(z)$ be defined by

$$q(z)=\left[\frac{1+z}{1-z}\right],\ z\in U,$$

then in view of Theorem 2 we obtain the result. □

Definition 3 *The fractional integral of order* α *is defined, for a function* $f(z)$ *by*

$$I_z^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z f(z)(z-\zeta)^{\alpha-1} d\zeta, \quad 0 \le \alpha < 1$$

where, the function $f(z)$ *is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of* $(z-\zeta)^{\alpha-1}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $(z-\zeta) > 0$. *Note that* $I_z^{\alpha} f(z) = f(z) \times z^{\alpha-1}/\Gamma(\alpha)$ *for* $z > 0$ *and 0 (see [8, 9, 10, 11]). Let*

$$f(z) = \sum_0^{\infty} \phi_n z^{n-n/\beta+1-\alpha},$$

this implies that,

$$\begin{aligned} I_z^{\alpha} f(z) &= f(z) \times z^{\alpha-1}/\Gamma(\alpha) = z^{\alpha-1}/\Gamma(\alpha) \sum_0^{\infty} \phi_n z^{n-n/\beta+1-\alpha} \quad \textit{for } z > 0 \\ &= \sum_0^{\infty} a_n z^{n-n/\beta}, \quad \text{where} \quad a_n = \phi_n/\Gamma(\alpha), \end{aligned}$$

thus,

$$1 \pm I_z^{\alpha} f(z) \in M_{\alpha}^{+}(M_{\alpha}^{-})$$

then we have the following results.

Theorem 3 *Let* $q(z)$ *be convex in the unit disk* U *with* $q(0) = 1$ *and* $R(q(z)) > 1/2$. *If* $F(z) \in \mathcal{E}_{\alpha}^{+}$ *and* $\dfrac{(1 + I_z^{\alpha} f(z)) * \varphi(z)}{(1 + I_z^{\alpha} f(z)) * \vartheta(z)}$ *is an analytic function in* U *satisfies the subordination*

$$\begin{aligned} &(1-u)\left[\frac{(1 + I_z^{\alpha} f(z)) * \varphi(z)}{(1 + I_z^{\alpha} f(z)) * \vartheta(z)}\right]^2 (z) + (2u-1)\left[\frac{(1 + I_z^{\alpha} f(z)) * \varphi(z)}{(1 + I_z^{\alpha} f(z)) * \vartheta(z)}\right] - u + \\ &+ (1-u)\left[\frac{(1 + I_z^{\alpha} f(z)) * \varphi(z)}{(1 + I_z^{\alpha} f(z)) * \vartheta(z)}\right]\left[\frac{z(1 + I_z^{\alpha} f(z)) * \varphi(z))'}{(1 + I_z^{\alpha} f(z)) * \varphi(z))} - \frac{z(1 + I_z^{\alpha} f(z)) * \vartheta(z))'}{(1 + I_z^{\alpha} f(z)) * \vartheta(z))}\right] \\ &\prec (1-u)q^2(z) + (2u-1)q(z) - u + (1-u)zq'(z) \end{aligned}$$

then,

$$\left[\frac{(1 + I_z^{\alpha} f(z)) * \varphi(z)}{(1 + I_z^{\alpha} f(z)) * \vartheta(z)}\right] \prec q(z).$$

Proof. Let the function $p(z)$ be defined by

$$F(z) = \frac{(1 + I_z^{\alpha} f(z)) * \varphi(z)}{(1 + I_z^{\alpha} f(z)) * \vartheta(z)}, \quad z \in U$$

then in view of Theorem 2 we obtain the result. □

Theorem 4 *Let the function* $q(z)$ *be univalent in the unit disk* U *such that* $q'(z) \neq 0$ *and* $\frac{zq'(z)}{q(z)} \neq 0$ *is starlike in* U, *if* $(1 - I_z^{\alpha} f(z)) \in \mathcal{E}_{\alpha}^{-}$ *satisfies the subordination*

$$b\left[\frac{(1 - I_z^{\alpha} f(z)) * \varphi(z))'}{(1 - I_z^{\alpha} f(z)) * \varphi(z))} - \frac{(1 - I_z^{\alpha} f(z)) * \vartheta(z))'}{(1 - I_z^{\alpha} f(z)) * \vartheta(z))}\right] \prec \frac{bzq'(z)}{q(z)}$$

then,

$$b\left[\frac{(1 - I_z^{\alpha} f(z)) * \varphi(z)}{(1 - I_z^{\alpha} f(z)) * \vartheta(z)}\right] \prec q(z)$$

then $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$\frac{(1 - I_z^{\alpha} f(z)) * \varphi(z)}{(1 - I_z^{\alpha} f(z)) * \vartheta(z)}, \quad z \in U$$

then in view of Theorem 2 we obtain the result. □

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