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Construction of (M, N)-hypermodule over (R, S)-hyperring

S. Mirvakili
Department of Mathematics,
Payame Noor University (PNU), Yazd, Iran
email: saeed_mirvakili@pnu.ac.ir

S. M. Anvariyeh
Department of Mathematics,
Yazd University, Yazd, Iran
email: anvariyeh@yazd.ac.ir

B. Davvaz
Department of Mathematics,
Yazd University, Yazd, Iran
email: davvaz@yazd.ac.ir

Abstract. The aim of this paper is to introduce a new class of hypermodules that may be called (M, N)-hypermodules over (R, S)-hyperrings. Then, we investigate some properties of this new class of hyperstructures. Since the main tools in the theory of hyperstructures are the fundamental relations, we give some results about them with respect to the fundamental relations.

1 (M, N)-hypermodule over (R, S)-hyperring

One knows the construction of a hypergroup K having as core a fixed hypergroup H. In [10], the aforesaid construction is generalized to a large class of hypergroups obtained from a group and from a family of fixed sets, and its properties are analyzed especially in the finite case. We recall the following notions from [4, 10]. Let $(M, \oplus)$ be a hypergroup and $(N, \uplus)$ be a group with a neutral element 0_N. Also, let $\{A_n\}_{n\in N}$ be a family of non-empty subsets indexed in N such that for all $x, y \in N$, $x \neq y$, $A_x \cap A_y = \emptyset$, and $A_{0_N} = M$. We set $P = \bigcup_{n\in N} A_n$ and we define the hyperoperation $\bar{\oplus}$ in P in the following way:

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(1) for every $(x, y) \in M^2$, $x \bar{\oplus} y = x \oplus y$,

(2) for every $(x, y) \in A_{n_1} \times A_{n_2} \neq H^2$, $x \bar{\oplus} y = A_{n_1 \uplus n_2}$.

The hyperstructure $(P, \bar{\oplus})$ is a hypergroup [4, 10]. In [14], Spartalis presented a way to obtain new hyperrings, starting with other hyperrings. We recall the following notions from [8, 14]. Let $(S, \dagger, \cdot)$ be a hyperring and let $\{B_i\}_{i \in R}$ be a family of non-empty sets such that:

(1) $(R, +, \star)$ is a ring,

(2) $B_{0_R} = S$,

(3) for every $i \neq j$, $B_i \cap B_j = \emptyset$.

Let $T = \bigcup_{i \in R} B_i$ and define the following hyperoperations on T: for every $(x, y) \in B_i \times B_j$:

$$x \ddagger y = \begin{cases} x \dagger y, & \text{if } (i, j) = (0_R, 0_R) \\ B_{i+j}, & \text{if } (i, j) \neq (0_R, 0_R) \end{cases} \quad \text{and} \quad x \odot y = \begin{cases} x \cdot y, & \text{if } (i, j) = (0_R, 0_R) \\ B_{i \star j}, & \text{if } (i, j) \neq (0_R, 0_R). \end{cases}$$

The structure $(T, \ddagger, \odot)$ is a hyperring [8, 14].

Now, we introduce a way to obtain new hypermodules, starting with other hypermodules.

Definition 1 *Let $(M, \oplus, \bullet)$ be a hypermodule over a hyperring $(S, \dagger, \cdot)$ and let $\{A_n\}_{n \in N}$ and $\{B_i\}_{i \in R}$ be two families of non-empty sets such that:*

(1) *$(N, \uplus, *)$ be a module over a ring $(R, +, \star)$,*

(2) *$A_{0_N} = M$ and $B_{0_R} = S$,*

(3) *for every $m, n \in \mathbb{N}$, $m \neq n$, $A_m \cap A_n = \emptyset$ and for every $i, j \in \mathbb{N}$, $i \neq j$, $B_i \cap B_j = \emptyset$.*

Let $P = \bigcup_{n \in N} A_n$ and $T = \bigcup_{i \in R} B_i$. We define the hyperoperation $\bar{\oplus}$ on P and the hyperoperations $\ddagger$ and $\odot$ on T similar to the above mentioned definitions. Also, we define a map $\bar{\bullet} : T \times P \longrightarrow \wp^*(P)$ as follows:

$$t \bar{\bullet} x = \begin{cases} t \bullet x, & \text{if } (i, n) = (0_R, 0_M) \\ A_{i * n}, & \text{if } (i, n) \neq (0_R, 0_M), \end{cases}$$

for every $(t, x) \in B_i \times A_n$.

Theorem 1 *The structure* $(P, \bar{\oplus}, \bar{\bullet})$ *over the hyperring* $(T, \ddagger, \odot)$ *is a hypermodule.*

Proof. According to [10, 14], $(P, \bar{\oplus})$ is a hypergroup and $(T, \ddagger, \odot)$ is a hyperring. We show that for every $r, s \in T$ and $x, y \in P$:

(1) $r\bar{\bullet}(x\bar{\oplus}y) = r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x$,

(2) $(r\ddagger s)\bar{\bullet}x = r\bar{\bullet}x\ddagger s\bar{\bullet}x$,

(3) $(r \odot s)\bar{\bullet}x = r\bar{\bullet}(s\bar{\bullet}x)$.

First, we prove (1). Let $r \in T$ and $x, y \in P$. Then, we have the following cases:

(i) $r \in B_{0_R} = S$ and $x, y \in A_{0_N} = M$. Then, we have $r\bar{\bullet}(x\bar{\oplus}y) = r \bullet (x \oplus y) = r \bullet x \oplus r \bullet x = r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x$,

(ii) $r \in B_j$, where $0_R \neq j \in R$, and $x, y \in A_{0_N}$. Then, we have $r\bar{\bullet}(x\bar{\oplus}y) = r\bar{\bullet}(x \oplus y) = A_{j*0_N} = A_{0_N}$ and $r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x = A_{j*0_N}\bar{\oplus}A_{j*0_N} = A_{0_N} \oplus A_{0_N} = A_{0_N}$. So (1) is true.

(iii) $r \in B_{0_R}$ and $(x, y) \in A_a \times A_b$, where $(0_R, 0_R) \neq (a, b)$. Then, it is not difficult to see that $r\bar{\bullet}(x\bar{\oplus}y) = A_{0_N}$ and $r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x = A_{0_N}$.

(iv) $r \in B_j$, where $0_R \neq j \in R$, and $(x, y) \in A_a \times A_b$, where $(0_N, 0_N) \neq (a, b)$. Then, it is not difficult to see that $r\bar{\bullet}(x\bar{\oplus}y) = A_{j*(a\uplus b)}$ and $r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x = A_{j*a\uplus j*b}$. Since $(N, \uplus, *)$ is a module over a ring $(R, +, \star)$, then $j*(a\uplus b) = j * a \uplus j * b$ and so (1) is true.

Therefore, we show that (1). Similarly, we can prove (2) and (3). □

Example 1 *Let* $N = (\mathbb{Z}_3, +)$ *be a module over the ring* $R = (\mathbb{Z}_3, +, \cdot)$, $M = (\mathbb{Z}_2, \oplus)$ *be a hypermodule over a hyperring* $S = (\mathbb{Z}_2, \oplus, \cdot)$, *where* $0 \oplus 0 = 0$, $0 \oplus 1 = 1 \oplus 1 = 1$ *and* $1 \oplus 1 = \{0, 1\}$ *and set* $A_0 = B_0 = \mathbb{Z}_2$, $A_1 = B_1 = \{a, b\}$ *and* $A_2 = B_2 = \{c\}$. *Now, we have* $P = T = \{0, 1, a, b, c, d, e\}$. *Then, we obtain* $\bar{\oplus} = \ddagger$ *and* $\bar{\bullet} = \odot$. *Also, we have*

$$\begin{array}{lll} 0\bar{\oplus}1 = 1, & a\bar{\oplus}a = b\bar{\oplus}a = a\bar{\oplus}b = b\bar{\oplus}b = \{c\}, & c\bar{\oplus}c = \{a, b\}, \\ 0\bar{\oplus}0 = 0, & 0\bar{\oplus}a = 1\bar{\oplus}a = 0\bar{\oplus}b = 1\bar{\oplus}b = \{a, b\}, & 0\bar{\oplus}c = 1\bar{\oplus}c = \{c\}, \\ 1\bar{\oplus}1 = \{0, 1\}, & c\bar{\oplus}a = c\bar{\oplus}a = c\bar{\oplus}b = c\bar{\oplus}b = \{0, 1\}. & \end{array}$$

and

$$\begin{array}{lll} 0\bar{\bullet}1 = 0, & a\bar{\bullet}a = b\bar{\bullet}a = a\bar{\bullet}b = b\bar{\bullet}b = \{a, b\}, & c\bar{\bullet}c = \{a, b\}, \\ 0\bar{\bullet}0 = 0, & 0\bar{\bullet}a = 1\bar{\bullet}a = 0\bar{\bullet}b = 1\bar{\bullet}b = \{0, 1\}, & 0\bar{\bullet}c = 1\bar{\bullet}c = \{0, 1\}, \\ 1\bar{\bullet}1 = 1, & c\bar{\bullet}a = c\bar{\bullet}a = c\bar{\bullet}b = c\bar{\bullet}b = \{c\}, & \end{array}$$

Let $(H, +)$ be a hypergroup. We consider the fundamental relation β on H as follows: $x\beta y$ if and only if $\{x, y\} \subseteq \sum_{i=1}^{n} x_i$, for some $x_i \in H$. Let β^* be the transitive closure of β. The fundamental relation β^* is the smallest equivalence relation such that the quotient H/β^* is a group. This relation introduced by Koskas [12] and studied by others, for example see [3, 4, 5, 12, 16]. Also, we recall the definition of the fundamental relation γ on a hypergroup H as follows: $x\gamma y$ if and only if $x \in \sum_{i=1}^{n} x_i$, $y \in \sum_{i=1}^{n} x_{\sigma(i)}$, $x_i \in H$, $\sigma \in \mathbb{S}_n$. Let γ^* be the transitive closure of γ. The fundamental relation γ^* is the smallest equivalence relation such that the quotient H/γ^* is an abelian group [11], also see [6, 7].

The fundamental relation Γ on a hyperring was introduced by Vougiouklis at the fourth AHA congress (1990) [15] as follows: $x\Gamma y$ if and only if $\exists n \in \mathbb{N}$, $\exists (k_1, \ldots, k_n) \in \mathbb{N}^n$, and $[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, (i = 1, \ldots, n)]$ such that $\{x, y\} \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$. The fundamental relation Γ on a hyperring is defined as the smallest equivalence relation so that the quotient would be the (fundamental) ring. Note that the commutativity with respect to both sum and product in the fundamental ring are not assumed. In [9], Davvaz and Vougiouklis introduced a new strongly regular equivalence relation on a hyperring such that the set of quotients is an ordinary commutative ring. We recall the following definition from [9].

Definition 2 [9] *Let R be a hyperring. We define the relation α as follows: $x\alpha y$ if and only if $\exists n \in \mathbb{N}$, $\exists (k_1, \ldots, k_n) \in \mathbb{N}^n$, $\exists \sigma \in \mathbb{S}_n$ and $[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$, $\exists \sigma_i \in \mathbb{S}_{k_i}$, $(i = 1, \ldots, n)]$ such that $x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in \sum_{i=1}^{n} A_{\sigma(i)}$, where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.*

If α^* is the transitive closure of α, then α^* is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$, and the quotient R/α^* is a commutative ring [9], also see [13].

Now, consider Definition 1 and Theorem 1. Then:

Theorem 2 *We have*

(1) $P/\beta_P^* \cong N$ *(group isomorphism).*

(2) $P/\gamma_P^* \cong N/\gamma_N^*$ *(group isomorphism) and if N is commutative then $P/\gamma_P^* \cong N$.*

(3) $T/\Gamma_T^* \cong R$ *(ring isomorphism).*

(4) $T/\alpha_T^* \cong R/\alpha_R^*$ *(ring isomorphism) and if R is commutative (with respect to the both operations) then $T/\alpha_T^* \cong R$.*

Proof. (1) We define $\phi : P/\beta^*_P \longrightarrow N$, with $\phi(\beta^*_P(a_n)) = n$, where $a_n \in A_n$ and $n \in \mathbb{N}$. Since β^*_P is a regular relation, so $(\beta^*_P(a_n))(\beta^*_P(a_m)) = (\beta^*_P(a_n a_m))$ and ϕ is a homomorphism. Let $(\beta^*_P(a_n)) = 0_N$. Then, $n = 0_N$ and so $\mathsf{Ker}\phi = (\beta^*_P(a_{0_N}))$. Hence, ϕ is one to one. Clearly, ϕ is onto.

(2) We define $\psi : P/\gamma^*_P \longrightarrow N/\gamma^*_N$, with $\psi(\gamma^*_P(a_n)) = \gamma_N(a_n)$, where $a_n \in A_n$ and $n \in \mathbb{N}$. Since γ^*_P and γ^*_N are regular relations, so $(\gamma^*_P(a_n))(\gamma^*_P(a_m))$ $= (\gamma^*_N(n)\gamma^*_N(m)) = (\gamma^*_N(nm)) = (\gamma^*_P(a_n a_m))$. Then, ϕ is a homomorphism. Let $(\gamma^*_P(a_n)) = 0_{N/\gamma^*_N} = \gamma^*_N(0_N)$. Then, $n = 0_N$ and so $\mathsf{Ker}\psi = (\gamma^*_P(a_{0_N}))$. Hence, ψ is one to one. Clearly, ψ is onto.

(3) We define $\lambda : T/\gamma^*_T \longrightarrow R$, with $\lambda(\Gamma^*_T(b_i)) = i$, where $b_i \in A_i$ and $i \in \mathbb{N}$. Since Γ^*_T is a regular relation, so $(\Gamma^*_P(a_n))(\Gamma^*_P(a_m)) = (\Gamma^*_P(a_n a_m))$ and λ is a homomorphism. Let $(\Gamma^*_P(a_i)) = 0_R$. Then, $i = 0_R$ and so $\mathsf{Ker}\lambda = (\Gamma^*_P(a_{0_R}))$. Hence λ is one to one. Clearly, λ is onto.

(4) We define $\mu : T/\alpha^*_T \longrightarrow R$, with $\mu(\alpha^*_T(b_i)) = \alpha^*_R(i)$, where $b_i \in A_i$ and $i \in \mathbb{N}$. Since α^*_T and α^*_R are regular relations, so $(\alpha^*_P(a_i))(\alpha^*_P(a_j)) = (\alpha^*_R(i))(\alpha^*_R(j)) = (\alpha^*_R(ij)) = (\alpha^*_P(a_i a_j))$. Then, μ is a homomorphism. Let $(\alpha^*_P(a_i)) = 0_{R/\alpha^*_R}$. Then, $i = 0_R$ and so $\mathsf{Ker}\mu = (\alpha^*_P(a_{0_R}))$. Thus, μ is one to one. Clearly, μ is onto. □

Now, we recall the definition of the fundamental relation ϵ on M from [16]. Let M be an R-hypermodule. Then $x\epsilon y$ if and only if $\{x, y\} \subseteq \sum_{i=1}^{n} m'_i$, where $m'_i = m_i$ or $m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} x_{ijk})m_i$, $r_{ijk} \in R$. The fundamental relation ϵ^* is defined to be the smallest equivalence relation such that the quotient M/ϵ^* is a module over the ring R/Γ^*. Also, according to [1, 2] we can consider the fundamental relation θ on hypermodules as follows: $x\theta y$ if and only if $\exists\ n \in \mathbb{N}$, $\exists(m_1, \ldots, m_n) \in M^n$, $\exists(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$, $\exists\sigma \in \mathbb{S}_n$, $\exists(x_{i1}, x_{i2}, \ldots, x_{ik}) \in R^{k_i}$, $\exists\sigma_i \in \mathbb{S}_{n_i}$, $\exists\sigma_{ij} \in \mathbb{S}_{k_{ij}}$, such that $x \in \sum_{i=1}^{n} m'_i$, $m'_i = m_i$ or $m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} x_{ijk})m_i$ and $y \in \sum_{i=1}^{n} m'_{\sigma(i)}$, where $m'_{\sigma(i)} = m_{\sigma(i)}$ if $m'_i = m_i$; $m'_{\sigma(i)} = B_{\sigma(i)}m_{\sigma(i)}$ if $m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} x_{ijk})m_i$, such that $B_i = \sum_{j=1}^{n_i} A_{i\sigma_i(j)}$ and $A_{ij} = \prod_{k=1}^{k_{ij}} x_{ij\sigma_{ij}(k)}$. Then, the (abelian group) M/θ^* is an R/α^*- module, where R/α^* is a commutative ring.

Theorem 3 (1) *The module* P/ϵ^*_P *over the ring* T/Γ^*_T *is isomorphic to the module* N *over the ring* R.

(2) *The module* P/θ^*_P *over the ring* T/α^*_T *is isomorphic to the module* N/θ^*_N *over the ring* R/α^*_R.

Proof. (1) Let $x \in P$. Then, there exists $n \in N$ such that $x \in A_n$. If $x\ \epsilon\ y$, then there exist $r_{ijk} \in T$ and $m_k \in P$ such that $\{x, y\} \subseteq \sum_{k=1}^{l} m'_k$, where $m'_k = m_k$

or $m'_k = (\sum \prod r_{ijk})m_k$. From the definition of the hyperoperations $\bar{\oplus}$, $\bar{\bullet}$, $\ddagger$ and $\odot$ it follows that $\sum_{k=1}^{l} m'_k = A_m$ for some $m \in N$. Hence, $x \in A_n \cap A_m$ and so $m = n$. Then, $y \in A_n$. Now, if $y \in \epsilon^*(x)$, then there exist $z_1, z_2, \ldots, z_s \in P$ such that $x \ \epsilon \ z_1 \ \epsilon \ z_2 \ldots z_s \ \epsilon \ y$. From $x \ \epsilon \ z_1$ and $x \in A_n$, we have $z_1 \in A_n$, so $z_2 \in A_n$ and finally we obtain $y \in A_n$. Therefore, $\epsilon^*(x) \subseteq A_n$.

Conversely, suppose that $y \in A_n$. If $n = 0$ then set $v \in A_m$ and $w \in A_{-m}$, where $m \in N - \{0\}$. Then, $\{x, y\} \subseteq A_0 = v \bar{\oplus} w$. Thus, $y \in \epsilon^*(x)$. If $n \neq 0$, then we consider $v \in A_n$ and $w \in A_0$, so $\{x, y\} \subseteq A_n = v \bar{\oplus} w$. Therefore, $y \in \epsilon^*(x)$ and consequently $A_n \subseteq \epsilon^*(x)$.

Finally, we consider the maps $\Psi : P/\epsilon^* \to N$ by $\epsilon^*(x) \to n$, where $x \in A_n$, and $\psi : T/\Gamma^* \to R$ by $\Gamma^*(r) \to i$, where $r \in B_i$. Then, Ψ is a module isomorphism and ψ is a ring isomorphism. □

The following theorem from [16] gives us a connection between the fundamental relations of β^* and ϵ^*.

Theorem 4 [16]. *If for any $a \in T$ and $p \in P$, there exists $u \in P$ such that $\Gamma^*(a).\beta^*(p) \subseteq \beta^*(u)$, then $\epsilon = \beta$.*

Also, in a similar way we have:

Theorem 5 *If for any $a \in T$ and $p \in P$, there exists $u \in P$ such that $\alpha^*(a).\gamma^*(p) \subseteq \gamma^*(u)$, then $\theta = \gamma$.*

Corollary 1 *Let for any $a \in T$ and $p \in P$, there exists $u \in P$ such that $\Gamma^*(a).\beta^*(p) \subseteq \beta^*(u)$.*

(1) *The module P/β^*_P over the ring T/Γ^*_T is isomorphic to the module N over the ring R.*

(2) *The module P/γ^*_P over the ring T/α^*_T is isomorphic to the module N/θ^*_N over the ring R/α^*_R.*

By the proof of Theorem 3, we have:

Theorem 6 *For every $m_1, \ldots, m_k \in P$ and $r_{ijk} \in T$ where $k \geq 1$, one of the following cases is verified.*

(1) *There exists $t \in N$ such that $\sum_{l=1}^{k} m'_l = A_t$, where $m'_l = m_l$ or $m'_l = (\sum \prod r_{ijl})m_l$.*

(2) *There exists $B \in \wp^*(M)$ such that $\sum_{l=1}^{l} m'_l = B$, where $m'_l = m_l$ or $m'_l = (\sum \prod r_{ijl})m_l$.*

Proof. Let $m_1,\dots,m_k \in P$ and $r_{ijk} \in T$. Set $m'_l = m_l$ or $m'_l = (\sum\prod r_{ijl})m_l$. Since P is a hypermodule so $\sum_{l=1}^{k} m'_l \subseteq P$. Let $m_l \in A_{n_l}$ and $r_{ijl} \in B_{t_{ijl}}$. If $n_l \neq 0_N$ or $t_{ijl} \neq 0_R$ then by definition of the (M,N)-hypermodule over the (R,S)-hyperring, there exists $t \in N$ such that $\sum_{l=1}^{k} m'_l = A_t$. Else, for every l, i and j, we have$m_l \in A_{0_N} = M$ and $r_{ijl} \in B_{0_R} = S$. Therefore, $\sum_{l=1}^{k} m'_l \subseteq A_{0_N} = M$ and so there exists $B \in \wp^*(M)$ such that $\sum_{l=1}^{k} m'_l = B$. □

Theorem 7 (1) *For every* $x \in N$ *and* $a \in A_x$, $C_\epsilon(a) = A_i$.

(2) $w_P = M$.

Proof.

(1) By Theorem 6, it follows that for any $i \in N$, A_i is a complete part. On the other hand for any $i \in N$, there exists $(y,z) \in P^2$ such that $y\bar{\oplus}z = A_{y\uplus z} = A_i$.

(2) It obtains immediately from (1).

□

Theorem 8 *Let* $(P,\bar{\oplus},\bar{\bullet})$ *be an* (M,N)*-hypermodule over an* (R,S)*-hyperring* $(T,\ddagger,\odot)$. *Then* $\bar{\oplus}$ *is commutative if and only if* $\oplus$ *is commutative.*

Proof. It is straightforward. □

Lemma 1 *Let* $(P,\bar{\oplus},\bar{\bullet})$ *be an* (M,N)*-hypermodule over an* (R,S)*-hyperring* $(T,\ddagger,\odot)$. *Let* N *has an element* 1_N *such that for every* $r \in R$, $r * 1_N = r$. *Then,* $B_r \subseteq A_r$ *for every* $r \in R$ *if and only if for every* $t \in T$, $t \in t\bar{\bullet}u$, *for all* $u \in A_{1_N}$.

Proof. If N has an element 1_N such that $r * 1_N = r$, for every $r \in R$, then $R \subseteq N$ and so $B_0 \subseteq A_0$. Let $r \in R^*$, $t \in B_r$ and $u \in A_{1_N}$. Then, $t\bar{\bullet}u = A_{r*1_N} = A_r \supseteq B_r \ni t$.

Conversely, let $r \in R$ and $t \in B_r$. Then for every $u \in A_{1_N}$ we have $t \in t\bar{\bullet}u = A_{r*1_N} = A_r$ and so $B_r \subseteq A_r$. □

Let $(M,+,\circ)$ be a hypermodule over a hyperring $(R,+,\cdot)$ such that M has zero element 0. If $A \subseteq M$ and $B \subseteq R$ then we define the following notations:

$$(0 :_R A) = \{r \in R \mid \forall x \in A,\ r\circ x = 0\} = Ann_R(M),$$
$$(B :_M 0) = \{x \in M \mid \forall r \in B,\ r\circ x = 0\}.$$

A faithful module M is one where the action of each $r \neq 0_R$ in R on M is non-trivial (i.e., $rx \neq 0_N$ for some x in M). Equivalently, the annihilator of $M(Ann_R(M))$ is the zero hyperideal.

Lemma 2 *Let* $(M, +, \circ)$ *be a hypermodule over a hyperring* $(R, +, \cdot)$ *such that* M *has zero element* 0.

(1) *If* A *be a non-empty subset of* M*, then* $(0 :_R A)$ *is a hyperideal of* R.

(2) *If* B *be a non-empty subset of* R*, then* $(B :_M 0)$ *is a subhypermodule of* R.

Theorem 9 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$.

(1) *Let* N *has an element* 1_N *such that* $r * 1_N = r$ *for every* $r \in R$, $t \in t\bar{\bullet}u$ *for every* $t \in T$ *and* $u \in A_{1_N}$*. Set* $E((P, \bar{\bullet})) = \{e \in P \mid \forall t \in T,\ t \in t\bar{\bullet}e\}$*. Then* $E((P, \bar{\bullet})) = \bigcup_{x \in (R :_N 0)} A_{x+1_N}$.

(2) *Let* R *has an element* 1_R *such that* $1_R * x = x$ *for every* $x \in N$*, and* $E((T, \bar{\bullet})) = \{\varepsilon \in T \mid \forall x \in P,\ \varepsilon \in \varepsilon\bar{\bullet}x\}$*. Then* $E((T, \bar{\bullet})) = \bigcup_{a \in Ann_R(N)} B_{a+1_R}$.

Proof. (1) By Lemma 1, we have $B_r \subseteq A_r$ for every $r \in R$. For every $t \in T$ there exists $r \in R$ such that $t \in B_r$. Now, let $u \in \bigcup_{x \in (R :_N 0)} A_{x+1_N}$. Then, there exists $z \in (R :_N 0)$ such that $u = A_{z+1_N}$. Thus, $t\bar{\bullet}u = B_r\bar{\bullet}A_{z+1_N} = A_{r*(z+1_N)} = A_r \supseteq B_r \ni t$. Therefore, $u \in E((P, \bar{\bullet}))$.

Conversely, suppose that $e \in E((P, \bar{\bullet}))$. Then, for every $t \in T$, $t \in t\bar{\bullet}e$. Let $t \in B_j$ and $e \in A_n$. Then, $t \in A_{j*n}$. But $t \in t\bar{\bullet}A_{1_N} = A_{j*1_N} = A_j$ so $A_j = A_{j*n}$. Therefore, $j = j*n$ for every $j \in R$. Thus, $j(n - 1_N) = 0_N$ and $n - 1_N \in (R :_N 0)$. Therefore, there exists $z \in (R :_N 0)$ such that $n = z + 1_N$.

(2) Let $t \in B_{1_R+a}$, where $a \in (0 :_R A)$. For all $x \in P$, if $x \in A_n$, then $t\bar{\bullet}x = A_{(1_R+a)*n} = A_{(1_R*n+a*n)} = A_{n+0} = A_n \ni x$. Hence, $t \in E((T, \bar{\bullet}))$. Conversely, suppose that $b \in E((T, \bar{\bullet}))$. Then, there exists $r \in R^*$, such that $b \in B_r$. Let $z \in B_{1_R}$. So, for every $n \in N$ and $x \in A_n$ we have $x \in z\bar{\bullet}x \in A_{1_R} * n = A_n$ and $x \in b\bar{\bullet}x \in A_{r*n}$. Therefore, for every $A_n \cap A_{r*n} \neq \emptyset$ and $r * n = n$ for every $n \in N$. Therefore, $(r - 1_R) * n = 0$ and $r - 1_R \in (0 :_R A)$ and there exists $a \in (0 :_R A)$ such that $r = 1_R + a$. □

Corollary 2 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$*. If* N *has an element* 1_N *such that* $t \in t\bar{\bullet}1_N$ *for every* $t \in T$ *and* R *is a unitary ring, then* $E((P, \bar{\bullet})) = A_{1_N}$.

Corollary 3 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$*. If* R *has an element* 1_R *such that* $1_R * x = x$ *for every* $x \in N$*, and* N *is a faithful module over the ring* R*, then* $E((T, \bar{\bullet})) = B_{1_R}$.

Lemma 3 *Let* $(M, +, \circ)$ *be a hypermodule over a commutative hyperring* $(R, +, \cdot)$ *and for every* $a \in R$ *set* $Q = a \circ M$. *Then* Q *is a subhypermodule.*

Proof. We show that $R \circ Q \subseteq Q$ and for all $q \in Q$, $Q + q = q + Q = Q$. Let $r \in R$ and $q \in Q$. Then, there exists $m \in M$ such that $q = a \circ m$. Now, we have $r \circ q = r \circ (a \circ m) = (r \cdot a) \circ m = (a \cdot r) \circ m = a \circ (r \circ m) \subseteq a \circ M = Q$. Also, $Q + q = a \circ M + a \circ m = a \circ (M + m) = a \circ M = Q$ and $q + Q = a \circ m + a \circ M = a \circ (m + M) = a \circ M = Q$. Therefore, Q is a subhypermodule of M. □

Theorem 10 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$. *Set* $P_t = t \bar{\bullet} P$. *If* S *is a commutative hyperring, then* P_t *is a subhypermodule of* P. *Also, for every* $r \in R$, $P_r = 0$, *for every* $t \in (0 :_P t)$, $P_t = 0$.

Lemma 4 [8]. *Let* $(R, +, \cdot)$ *be a hyperring and let* $x \in R$. *Let* $I = K \cdot x$. *Then* I *is a left hyperideal of* R *if and only if for every* $y \in I$, $I \cdot y = y \cdot I = I$.

Corollary 4 *Let* $(R, +, \cdot)$ *be a commutative hyperring and let* $x \in R$. *If we set* $I = K \cdot x$ *then* I *is a hyperideal of* R *if and only if for every* $y \in I$, $I \cdot y = I$. *Moreover,* $(I, +, \circ)$ *is a hyperring.*

Theorem 11 [8] *Let* $(T, \ddagger, \odot)$ *be an* (R, S)*-hyperring and* S *be commutative. Then* $T_t = T \odot t$ *is a hyperideal of* T *and* $(T_t, \ddagger, \odot)$ *is a commutative hyperring.*

Lemma 5 *Let* $(M, +, \circ)$ *be a hypermodule over a commutative hyperring* $(R, +, \cdot)$ *and for every* $a, b \in R$ *set* $M_a = a \circ M$ *and* $R_b = R \cdot b$. *Then* M_a *is a hypermodule over a hyperring* R_b *if and only if for every* $x \in R_b$, $R_b \cdot x = R_b$.

Theorem 12 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$ *and let* $a, b \in T$. *If* S *is a commutative hyperring then* $(a \bar{\bullet} P, \bar{\oplus}, \bar{\bullet})$ *is a hypermodule over a hyperring* $(T \odot b, \ddagger, \odot)$.

Proof. It obtains from Theorems 10 and 11 and Lemma 5. □

Example 2 *Let* $(M, +, \circ)$ *be a hypermodule over a commutative hyperring* $(R, +, \cdot)$ *and for every* $a \in R$ *set* $Q = a \circ M$, *and* $Q + q \neq Q$.

Lemma 6 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$. *Then* S *has a weak neutral element if and only if* P *has a weak neutral element.*

Proof. Let $e \in P$ be a weak neutral element of P. So for every $p \in P$ we have $p \in e\bar{\oplus}p \cap p\bar{\oplus}e$. Let $e \in A_n$. We show that $n = 0_N$. If $n \neq 0_N$, then $e \in e\bar{\oplus}e = A_{n+n}$ which implies that $e \in A_n \cap A_{n+n}$ and $A_n = A_{n+n}$. Thus, $n + n = n$ and $n = 0_N$. Therefore, $e \in A_{0_N} = M$.

Conversely, let $e \in M$ be a weak neutral element of M. Then, for every $p \in A_n$ when $n \neq 0_N$, we have $p\bar{\oplus}e \in A_{n+0_N} = A_n$ and so $p \in p\bar{\oplus}e$. In a similar way, we obtain $p \in e\bar{\oplus}p$. Therefore, e is a weak neutral element of P. □

Theorem 13 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$*. If* R *is a field and* N *is a unitary* R*-module, then* P/ϵ_P^* *is a hypervector space over the field* T/Γ_T^**.*

Proof. Since R is a field, T is a hyperfield. Since N is a unitary R-module, P/ϵ_P^* is a unitary T/Γ_T^*-module. Therefore, P/ϵ_P^* is a hypervector space over the field T/Γ_T^*. □

Let us denote $P_{\bar{\oplus}}$ and $P_{\bar{\bullet}}$, the sets of scalars of the (M, N)-hypermodule over the (R, S)-hyperring with respect to the hyperoperations $\bar{\oplus}$ and $\bar{\bullet}$, respectively, i.e., $P_{\bar{\oplus}} = \{u \in P \mid \text{card}(u\bar{\oplus}x) = 1, \text{ for all } x \in P\}$ and $P_{\bar{\bullet}} = \{u \in P \mid \text{card}(t\bar{\bullet}u) = 1,$ for all $t \in T\}$.

Theorem 14 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$*. Then:*

(1) *If* $P_{\bar{\oplus}} \cap (P - M) \neq \emptyset$ *and* $P_{\bar{\oplus}} \cap (P - M) \neq \emptyset$*, then* $\bar{\oplus}$ *and* $\bar{\bullet}$ *are operations.*

(2) *If* $P_{\bar{\oplus}} \neq \emptyset$ *and* $P_{\bar{\oplus}} \cap (P - M) = \emptyset$*, then* $\text{card}A_n = 1$ *for all* $n \in N - \{0_N\}$*.*

Proof. (1) Let $u \in P_{\bar{\oplus}} \cap (P - M)$, i.e., $u \in A_n \neq N$. Then, for all $m \in N$, A_m is singleton, because by taking $y \in A_{m-1_N}$, we get the singleton $u\bar{\oplus}y = A_m$. Consequently, $\bar{\bullet}$ and $\bar{\oplus}$ are operations.

(2) By hypothesis, we have $P_{\bar{\oplus}} \subseteq M$. Moreover, if $u \in P_{\bar{\oplus}}$, then $u \in A_{0_N}$. For all $n \in N - \{0_N\}$, we consider $y \in A_n$. Then, we get the singleton $u\bar{\oplus}y = A_n$. □

An (M, N)-hypermodule over an (R, S)-hyperring $(T, \ddagger, \odot)$ is called a $(0, N)$-hypermodule, when M is a singleton set.

Theorem 15 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule. We have*

(1) $P_{\bar{\bullet}} \neq \emptyset$*, if and only if* P *is a* $(0, N)$*-hypermodule.*

(2) *If* $P_{\bar{\bullet}} \cap A_n \neq \emptyset$*, for some* $n \in N$*, then* $A_n \subseteq P_{\bar{\bullet}}$ *and we have* $\text{card}A_k = 1$ *and* $A_k \subseteq P_{\bar{\bullet}}$*, for all* $k \in R * n$*.*

Proof. (1) Let $y \in P_{\bar{\bullet}}$. If $y \in M$, then for $t \in B_i \neq S$ we have $M = t\bar{\bullet}y$ is a singleton set. If $y \in P - M$, then for $s \in S = B_{0_R}$, we have $M = A_{0_N} = t\bar{\bullet}y$ is a singleton set. Hence, P is a $(0, N)$-hypermodule. Conversely, if M is a singleton set, then $P_{\bar{\bullet}} \neq \emptyset$.

(2) Let $P_{\bar{\bullet}} \cap A_n \neq \emptyset$, $n \in N$. If $n = 0_N$, then because of (1), M is a singleton set and so (2) is valid. We prove (2) for $n \in N - \{0_n\}$. Since, for all $x, y \in A_n$, $t \in T$, $t\bar{\bullet}x = t\bar{\bullet}y$, this implies that $A_n \subseteq P_{\bar{\bullet}}$. Moreover, if $x \in P_{\bar{\bullet}} \cap A_n$, then for all $r \in R$, we consider an arbitrary $t \in B_r$ and we have that $A_{r*n} = t\bar{\bullet}x$ is a singleton set. Hence, $\text{card}A_k = 1$, for all $k \in R * n$. Finally, let $A_k = \{x\}$, when $k \in R * n$. Then, for all $t \in B_r \neq S$, $t\bar{\bullet}x = A_{r*k}$ is a singleton set, because $r * k \in R * n$. Also, by (1), M is a singleton set and so $A_k \subseteq P_{\bar{\bullet}}$, when $k \in R * n$. □

Now, let $T_{\bar{\bullet}} = \{t \in T \mid \text{card}(t\bar{\bullet}u) = 1$, for all $u \in P.\}$ Then, similar to Theorem 15, we have:

Theorem 16 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule. Then:*

(1) $T_{\bar{\bullet}} \neq \emptyset$*, if and only if* P *is a* $(0, N)$*-hypermodule.*

(2) *If* $T_{\bar{\bullet}} \cap B_r \neq \emptyset$*, for some* $r \in R$*, then* $B_r \subseteq T_{\bar{\bullet}}$ *and for all* $k \in r * N$*, we have* $\text{card}A_k = 1$.

2 Quotient of an (M, N)-hypermodule over an (R, S)-hyperring

Proposition 1 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be a canonical* (M, N)*-hypermodule over the Krasner* (R, S)*-hyperring* $(T, \ddagger, \odot)$ *and* $\emptyset \neq q \subseteq P$, $\emptyset \neq I \subseteq T$*. Then:*

(1) q *is a subhypermodule of* P *if and only if* $q = \bigcup_{n\in Q} A_n$*, where* Q *is a submodule of* $(N, \uplus, *)$.

(2) h *is a hyperideal of* P *if and only if* $h = \bigcup_{r\in H} B_r$*, where* H *is an ideal of* $(S, \dagger, \cdot)$.

Proof. (1) Let q be a subhypermodule of P. Then, $0 \in q$ and $r \in R^*$ which implies that $A_0 = r\bar{\bullet}0 \subseteq q$, so $M \subseteq q$. Let there exists $n \in N^*$ such that $q \cap A_n \neq \emptyset$ and $x \in q \cap A_n$. Then $-x \in g$ and $-x \in A_{-n}$ so we have $A_{-n} \in q$. Consequently, from the closure of $\bar{\oplus}$ in q, it follows $q = \bigcup_{n\in Q} A_n$, where Q is a subgroup of $(N, \uplus, *)$. Now, let $r \in R$. Then, $B_r\bar{\bullet}A_n = A_{r*n} \subseteq q$. Hence, $r * n \in Q$ and Q is a submodule of N. The converse is verified in a simple way.

(2) It obtains similar to the part (i) of Proposition 4.1 [14]. □

Proposition 2 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be a canonical* (M, N)*-hypermodule over the Krasner* (R, S)*-hyperring* $(T, \ddagger, \odot)$*. Suppose that* G *be a submodule of* $(N, \uplus, *)$ *and* H *be an ideal of* $(R, +, \star)$*. If* $g = \bigcup_{n \in G} A_n$ *and* $h = \bigcup_{j \in H} B_j$*, then* $[P : g^*] \cong [N : G^*]$ *and* $[T : h^*] \cong [R : H^*]$*. In addition, the module* $[P : g^*]$ *over the ring* $[T : h^*]$ *is isomorphic to the module* $[M : G^*]$ *over the ring* $[R : H^*]$*.*

Proof. According to [17], $[P : g^*]$ is a hypermodule over the hyperring $[T : h^*]$ and Spartalis in [14], proved that $[T : h^*] \cong [R : H^*]$ and $\varphi : [T : h^*] \to [R : H^*]$ by $\varphi(h + t) = H + r$, is an isomorphism, where $t \in A_r$. Define the map $\phi : [P : g^*] \longrightarrow [N : G^*]$ by $g \bar{\oplus} a_i \mapsto G + i$. Then, ϕ is one to one and onto. Moreover, for every $m, n \in N$, $r, s \in R$, $x \in A_m$, $y \in A_n$, $t \in B_r$, we have $\phi((g \bar{\oplus} x) + (g \bar{\oplus} y)) = G + m + n = \phi(g \bar{\oplus} x) + \phi(g \bar{\oplus} y)$ and for any $t_r \in T$ we have $\phi((h + t) \circ (g + x)) = \phi(g + t \bar{\bullet} x) = G + rm = (H + r) \circ (G + m) = \varphi(h + t) \circ \phi(g + x)$. □

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