Acta Univ. Sapientiae, Mathematica, **12**, 1 (2020) 54–84

DOI: 10.2478/ausm-2020-0004

Composition iterates, Cauchy, translation, and Sincov inclusions

Włodzimierz Fechner
Institute of Mathematics,
Lodz University of Technology,
90-924 Łódź, Poland
email: wlodzimierz.fechner@p.lodz.pl

Árpád Száz
Institute of Mathematics,
University of Debrecen,
H-4002 Debrecen, Hungary
email: szaz@science.unideb.hu

Abstract. Improving and extending some ideas of Gottlob Frege from 1874 (on a generalization of the notion of the composition iterates of a function), we consider the composition iterates φ^n of a relation φ on X, defined by

$$\varphi^0 = \Delta_X, \qquad \varphi^n = \varphi \circ \varphi^{n-1} \quad \text{if} \quad n \in \mathbb{N}, \qquad \text{and} \qquad \varphi^\infty = \bigcup_{n=0}^{\infty} \varphi^n.$$

In particular, by using the relational inclusion $\varphi^n \circ \varphi^m \subseteq \varphi^{n+m}$ with $n, m \in \overline{\mathbb{N}}_0 = \{0\} \cup \mathbb{N} \cup \{\infty\}$, we show that the function α, defined by

$$\alpha(n) = \varphi^n \quad \text{for} \quad n \in \overline{\mathbb{N}}_0,$$

satisfies the Cauchy problem

$$\alpha(n) \circ \alpha(m) \subseteq \alpha(n+m), \qquad \alpha(0) = \Delta_X.$$

Moreover, the function f, defined by

$$f(n, A) = \alpha(n)[A] \qquad \text{for} \qquad n \in \overline{\mathbb{N}}_0 \quad \text{and} \quad A \subseteq X,$$

2010 Mathematics Subject Classification: Primary 39B12, 39B52, 39B62; Secondary 20N02, 26E25, 54E25
Key words and phrases: relations, groupoids, composition iterates, Cauchy, translation and Sincov inclusions

satisfies the translation problem

$$f\big(n, f(m, A)\big) \subseteq f(n+m, A), \qquad f(0, A) = A.$$

Furthermore, the function F, defined by

$$F(A, B) = \big\{ n \in \overline{\mathbb{N}}_0 : \quad A \subseteq f(n, B) \big\} \qquad \text{for} \qquad A, B \subseteq X,$$

satisfies the Sincov problem

$$F(A, B) + F(B, C) \subseteq F(A, C), \qquad 0 \in F(A, A).$$

Motivated by the above observations, we investigate a function F on the product set X^2 to the power groupoid $\mathcal{P}(U)$ of an additively written groupoid U which is supertriangular in the sense that

$$F(x, y) + F(y, z) \subseteq F(x, z)$$

for all $x, y, z \in X$. For this, we introduce the convenient notations

$$R(x, y) = F(y, x) \qquad \text{and} \qquad S(x, y) = F(x, y) + R(x, y),$$

and

$$\Phi(x) = F(x, x) \qquad \text{and} \qquad \Psi(x) = \bigcup_{y \in X} S(x, y).$$

Moreover, we gradually assume that U and F have some useful additional properties. For instance, U has a zero, U is a group, U is commutative, U is cancellative, or U has a suitable distance function; while F is nonpartial, F is symmetric, skew symmetric, or single-valued.

1 A few basic facts on relations

In [40], a subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation on X to itself is called a *relation on* X. More specially, $\Delta_X = \{(x, x) : \ x \in X\}$ is called the *identity relation on* X.

If F is a relation on X to Y, then by the above definitions we can also state that F is a relation on $X \cup Y$. However, for our present purposes, the latter view of the relation F would also be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{y \in Y : \ (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images of* x *and* A *under* F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write xFy. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

The sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain and range of* F, respectively. If in particular $D_F = X$, then we say that F is a *relation of* X *to* Y, or that F is a *nonpartial relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

In particular, a function $\star$ of X to itself is called a *unary operation on* X, while a function $*$ of X^2 to X is called a *binary operation on* X. In this case, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then we can easily see that $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *inverse* F^{-1} can be defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$. Moreover, if G is a relation on Y to Z, then the *composition* $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$.

If F is a relation on X to Y, then a relation Φ of D_F to Y is called a *selection relation of* F if $\Phi \subseteq F$, i.e., $\Phi(x) \subseteq F(x)$ for all $x \in D_F$. By using the Axiom of Choice, it can be seen that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ of X to $\mathcal{P}(Y)$ and Φ of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi(x) = F(x)$ for all $x \in X$ and $\Phi(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y, while functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. They were briefly called *co-relations on* X *to* Y in [40].

In particular, a relation R on X can be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R can be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder* (*tolerance*) *relation*, and a symmetric (antisymmetric) preorder relation may be called an *equivalence* (*partial order*) *relation.*

For $A \subseteq X$, *Pervin's relation* $R_A = A^2 \cup A^c \times X$, with $A^c = X \setminus A$, is an important preorder on X. While, for a *pseudometric* d on X, *Weil's surrounding* $B_r = \{(x, y) \in X^2 : d(x, y) < r\}$, with $r > 0$, is an important tolerance on X.

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \big(A^c\big)^2$ is already an equivalence on X. And, more generally if $\mathcal{A}$ is a *partition of* X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is an equivalence on X which can, to some extent, be identified with $\mathcal{A}$.

2 A few basic facts on ordered sets and groupoids

If $\leq$ is a relation on X, then motivated by Birkhoff [5, p. 1] the ordered pair $X(\leq) = (X, \leq)$ is called a *goset* (generalized ordered set) [39]. In particular, it is called a *proset* (preordered set) if the relation $\leq$ is a preorder on X.

Quite similarly, a goset $X(\leq)$ is called a *poset* (partially ordered set) if the relation $\leq$ is a partial order on X. The importance of posets lies mainly in the fact that any family of sets forms a poset with set inclusion.

A function f of one goset $X(\leq)$ to another $Y(\leq)$ is called *increasing* if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in X$. The function f can now be briefly called *decreasing* if it is increasing as a function of $X(\leq)$ to the dual $Y(\geq)$.

An increasing function φ of the goset $X = X(\leq)$ to itself is called a *projection* (*involution*) *operation* on X if it is *idempotent* (*involutive*) in the sense that $\varphi \circ \varphi = \varphi$ $\big(\varphi \circ \varphi = \Delta_X\big)$. Note that $\varphi \circ \varphi = \Delta_X$ if and only if $\varphi^{-1} = \varphi$.

Moreover, a projection operation φ on a poset X is called a *closure operation on* X if it is *extensive* in the sense that $\Delta_X \leq \varphi$. That is, $x \leq \varphi(x)$ for all $x \in X$. The *interior operations* can again be most briefly defined by dualization.

If f is a function of one goset X to another Y and g is a function of Y to X such that, for any $x \in X$ and $y \in Y$, we have $f(x) \leq y$ if and only $x \leq g(y)$, then g is called a *Galois adjoint of* f [12, p. 155].

Hence, by taking $\varphi = g \circ f$, one can easily see that, for any $u, v \in X$, we have $f(u) \leq f(v)$ if and only if $u \leq \varphi(v)$. Moreover, if X and Y are prosets, then it can be shown that f is increasing, φ is a closure and $f = f \circ \varphi$ [39].

If $+$ is a binary operation on a set X, then the ordered pair $X(+) = (X, +)$ is called an *additive groupoid*. Recently, groupoids are usually called *magmas*, not to be confused with *Brandt groupoids* [6].

If X is a groupoid, then for any $A, B \subseteq X$ we may also naturally define $A + B = \{x + y : \ x \in A, \ y \in B\}$. Thus, by identifying singletons with their elements, X may be considered as a subgoupoid of its *power groupoid* $\mathcal{P}(X)$.

In a groupoid X, for any $n \in \mathbb{N}$ and $x \in X$ we may also naturally define $nx = x$ if $n = 1$, and $nx = (n-1)x + x$ if $n > 1$. Thus, for any $n \in \mathbb{N}$ and $A \subseteq X$, we may also naturally define $nA = \{nx : \ x \in A\}$.

If X is a *semigroup* (associative groupoid), then we have $(n+m)x = nx + mx$ and $(nm)x = n(mx)$ for all $n, m \in \mathbb{N}$ and $x \in X$. However, the equality $n(x+y) = nx + ny$ requires the elements $x, y \in X$ to be commuting [19].

If the groupoid X has a zero element 0, then we also naturally define $0x = 0$ for all $x \in X$. Moreover, if X is a group, then we also naturally define $(-n)x = n(-x)$ for all $n \in \mathbb{N}$ and $x \in X$. And thus also kA for all $k \in \mathbb{Z}$ and $A \subseteq X$.

Concerning the corresponding operations in $\mathcal{P}(X)$, we must be very careful.

Namely, in general, we only have $(n + m)A \subseteq nA + mA$ and $nA \subseteq \sum_{i=1}^{n} A$ for all $n, m \in \mathbb{N}$ and $A \subseteq X$. However, $\mathcal{P}(X)$ has a richer structure than X.

In particular, an element x of a groupoid X is called *left-cancellable* if $x+y = x+z$ implies $y = z$ for all $y, z \in X$. Moreover, the groupoid X is called *left-cancellative* if every element of X is left-cancellable.

"Right-cancellable" and *"right-cancellative"* are to be defined quite similarly. Moreover, for instance, the groupoid X is to be called *cancellative* if it is both left-cancellative and right-cancellative.

A semigroup X can be easily embedded in a *monoid* (semigroup with zero element), by adjoining an element 0 not in X, and defining $0 + x = x + 0 = x$ for all $x \in X$. Important monoids will be $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$.

3 The finite composition iterates of a relation

Notation 1 *In the sequel, we shall assume that X is a set, Δ is the identity function of X and φ is a relation on X.*

Note that the family $\mathcal{P}(X^2)$ of all relations on X forms a semigroup, with identity element Δ, with respect to the composition of relations. Therefore, we may naturally use the following

Definition 1 Define $\varphi^0 = \Delta$, and for any $n \in \mathbb{N}$

$$\varphi^n = \varphi \circ \varphi^{n-1}.$$

Remark 1 Thus, for each $n \in \mathbb{N}_0$, φ^n is also a relation on X which is called the nth composition iterate of φ.

Now, as a particular case of a more general theorem on monoids, we can state the following theorem whose direct proof is included here only for the reader's convenience.

Theorem 1 *For any $n, m \in \mathbb{N}_0$, we have*

$$\varphi^{n+m} = \varphi^n \circ \varphi^m.$$

Proof. For fixed $m \in \mathbb{N}_0$, we shall prove, by induction, that

$$\varphi^{m+n} = \varphi^n \circ \varphi^m$$

for all $n \in \mathbb{N}_0$. Hence, by the commutativity of the addition in $\mathbb{N}_0$, the assertion of the theorem follows.

By Definition 1, we evidently have $\varphi^{m+0} = \varphi^m = \Delta \circ \varphi^m = \varphi^0 \circ \varphi^m$. Therefore, the required equity is true for $n = 0$.

Let us suppose now that the required equality is true for some $n \in \mathbb{N}_0$. Then, by Definition 1, the above assumption, and the corresponding associativities, we have

$$\begin{aligned}\varphi^{m+(n+1)} &= \varphi^{(m+n)+1} = \varphi \circ \varphi^{m+n} \\ &= \varphi \circ (\varphi^n \circ \varphi^m) = (\varphi \circ \varphi^n) \circ \varphi^m = \varphi^{n+1} \circ \varphi^m.\end{aligned}$$

Therefore, the required equality is also true for $n + 1$. □

Remark 2 This theorem shows that the family $\{\varphi^n\}_{n=0}^{\infty}$ also forms a semigroup, with identity element Δ, with respect to composition.

By induction, we can also easily prove the less trivial part of the following

Theorem 2 *The following assertions are equivalent:*
(1) $\Delta \subseteq \varphi$; (2) $\varphi^n \subseteq \varphi^{n+1}$ *for all* $n \in \mathbb{N}_0$.

Remark 3 This theorem shows that the sequence $\left(\varphi^n\right)_{n=0}^{\infty}$ is increasing, with respect to set inclusion, if and only if the relation φ is reflexive on X.

Note that if in particular φ is reflexive on X and φ is a function, then we necessarily have $\varphi = \Delta$, and thus also $\varphi^n = \Delta$ for all $n \in \mathbb{N}_0$.

Therefore, in the important particular case when φ is a function of X to itself, Theorem 2 cannot have any significance.

4 The infinite composition iterate of a relation

In addition to Definition 1, we may also naturally use the following

Definition 2 Define

$$\varphi^{\infty} = \bigcup_{n=0}^{\infty} \varphi^n.$$

Remark 4 Moreover, the relations

$$\underline{\lim}_{n\to\infty} \varphi^n = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} \varphi^k \qquad \text{and} \qquad \overline{\lim}_{n\to\infty} \varphi^n = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} \varphi^k$$

may also be naturally investigated.

Note that if in particular the sequence $(\varphi^n)_{n=0}^{\infty}$ is increasing with respect to set inclusion, then these relations coincide with φ^∞.

The relation φ^∞ is called the *preorder hull (closure) of* φ. Namely, we have

Theorem 3 *φ^∞ is the smallest preorder relation on X containing φ.*

Proof. By Definition 2, it is clear that $\Delta \subseteq \varphi^\infty$ and $\varphi \subseteq \varphi^\infty$. Thus, φ^∞ is reflexive and contains φ.

Moreover, if $(x, y) \in \varphi^\infty$ and $(y, z) \in \varphi^\infty$, then by Definition 2 there exist $m, n \in \mathbb{N}_0$ such that $(x, y) \in \varphi^m$ and $(y, z) \in \varphi^n$. Hence, by using Theorem 1, we can infer that $(x, z) \in \varphi^n \circ \varphi^m = \varphi^{n+m}$. Thus, by Definition 2, we also have $(x, z) \in \varphi^\infty$. Therefore, φ^∞ is also transitive.

On the other hand, if ψ is a relation on X such that $\varphi \subseteq \psi$, then we can note that $\varphi^n \subseteq \psi^n$ for all $n \in \mathbb{N}_0$, and thus by Definition 2 we have $\varphi^\infty \subseteq \psi^\infty$. Moreover, if ψ is reflexive, then $\psi^0 \subseteq \psi$. And, if ψ is transitive, then $\psi^n \subseteq \psi$ for all $n \in \mathbb{N}$. Therefore, if ψ is both reflexive and transitive, then by Definition 2 we have $\psi^\infty \subseteq \psi$, and thus also $\varphi^\infty \subseteq \psi$. □

Now, as an immediate consequence of this theorem, we can also state

Corollary 1 *The following assertions are equivalent:*
(1) $\varphi^\infty = \varphi$; (2) *φ is a preorder on X.*

Remark 5 From the above results, it is clear that ∞ is a closure operation on the poset $\mathcal{P}(X^2)$.

In general, it is not even finitely union preserving. However, it is compatible with the inversion of relations [18].

Moreover, in addition to Theorem 1, we can also easily prove the following

Theorem 4 *For any $n, m \in \overline{\mathbb{N}}_0$, we have*

$$\varphi^n \circ \varphi^m \subseteq \varphi^{n+m}.$$

Moreover, if φ is reflexive on X, then the corresponding equality is also true.

Proof. If in particular $n, m \in \mathbb{N}_0$, then by Theorem 1 the corresponding equality is true even if φ is not assumed to be reflexive.

Moreover, by using Definition 2 and Theorem 3, we can see that

$$\varphi^n \circ \varphi^\infty \subseteq \varphi^\infty \circ \varphi^\infty \subseteq \varphi^\infty = \varphi^{n+\infty}.$$

Furthermore, if φ is reflexive, then it is clear that we also have

$$\varphi^{\infty} = \Delta \circ \varphi^{\infty} \subseteq \varphi^{n} \circ \varphi^{\infty}.$$

Therefore, in this case, $\varphi^{n} \circ \varphi^{\infty} = \varphi^{\infty} = \varphi^{n+\infty}$ also holds. The case "$\infty + m$" can be treated quite similarly. □

Remark 6 Now, in addition to Theorem 2, we can only state that $\varphi^{n} \subseteq \varphi^{\infty}$ for all $n \in \overline{\mathbb{N}}_0$.

However, by [30], we may naturally say that φ is n-*well-chained* if $\varphi^{n} = X^2$. And, φ is n-*connected* if $\varphi \cup \varphi^{-1}$ is n–well-chained.

Moreover, under the notation $\mathcal{T}_{\varphi} = \{A \subseteq X : \ \varphi[A] \subseteq A\}$ of [24], we have $\varphi^{\infty} = \bigcap_{A \in \mathcal{T}_{\varphi}} R_A$. And, φ^{∞} is the largest relation on X such that $\mathcal{T}_{\varphi^{\infty}} = \mathcal{T}_{\varphi}$.

5 From the composition iterates to a Cauchy inclusion

Now, extending an idea of Frege [15, 16], we may also naturally introduce

Definition 3 For any $n \in \overline{\mathbb{N}}_0$, define

$$\alpha(n) = \varphi^{n}.$$

Thus, α may be considered as a relation on $\overline{\mathbb{N}}_0$ to X^2, or as a function of $\overline{\mathbb{N}}_0$ to $\mathcal{P}(X^2)$, which can be proved to satisfy a Cauchy type inclusion.

First of all, by Theorem 1, we evidently have the following

Theorem 5 *For any* $n, m \in \mathbb{N}_0$*, we have*

$$\alpha(n+m) = \alpha(n) \circ \alpha(m).$$

Proof. By Definition 3 and Theorem 1, it is clear that

$$\alpha(n+m) = \varphi^{n+m} = \varphi^{n} \circ \varphi^{m} = \alpha(n) \circ \alpha(m).$$

□

Remark 7 In addition to this theorem, it is also worth noticing that $\alpha(0) = \Delta$.

Moreover, by Theorem 2, we can also at once state the following

Theorem 6 *The following assertions are equivalent:*
(1) $\Delta \subseteq \varphi$; (2) $\alpha(n) \subseteq \alpha(n+1)$ *for all* $n \in \mathbb{N}_0$.

Remark 8 Thus, the restriction of the set-valued function α to $\mathbb{N}_0$ is increasing, with respect to set inclusion, if and only if the relation φ is reflexive on X.

By using Theorem 4 instead of Theorem 1, we can also easily establish

Theorem 7 *For any* $n, m \in \overline{\mathbb{N}}_0$ *we have*

$$\alpha(n) \circ \alpha(m) \subseteq \alpha(n+m).$$

Moreover, if φ *is reflexive on* X*, then the corresponding equality is also true.*

Remark 9 Now, in addition to Theorem 6, we can also state that $\alpha(n) \subseteq \alpha(\infty)$ for all $n \in \overline{\mathbb{N}}_0$.

Thus, in particular, the set-valued function α is increasing, with respect to set inclusion, if and only if the relation φ is reflexive on X.

6 From a Cauchy inclusion to a translation inclusion

Now, as an extension of our former observations, we may naturally start with

Notation 2 *Suppose that* U *is a additive groupoid and* α *is a relation on* U *to* X^2 *such that*

$$\alpha(u) \circ \alpha(v) \subseteq \alpha(u+v)$$

for all $u, v \in U$.

Thus, extending an idea of Frege [15, 16], we may also naturally introduce

Definition 4 For any $u \in U$ and $A \subseteq X$, define

$$f(u, A) = \alpha(u)[A].$$

Thus, f may be considered a relation on $U \times \mathcal{P}(X)$ to X, or as a function of $U \times \mathcal{P}(X)$ to $\mathcal{P}(X)$, which can be proved to satisfy a translation inclusion.

Theorem 8 *For any* $u, v \in U$ *and* $A \subseteq X$*, we have*

$$f\big(u, f(v, A)\big) \subseteq f(u+v, A).$$

Proof. By Definition 4 and the assumed superadditivity property of α, we have

$$f\big(u, f(v, A)\big) = \alpha(u)\big[f(v, A)\big] = \alpha(u)\big[\alpha(v)[A]\big] \\ = \big(\alpha(u) \circ \alpha(v)\big)[A] \subseteq \alpha(u+v)[A] = f(u+v, A).$$

□

Remark 10 Thus, by identifying singleton with their elements, we may also write

$$f\big(u, f(v, x)\big) \subseteq f(u+v, x)$$

for all $u, v \in U$ and $x \in X$.

Now, to illustrate the appropriateness of Definition 4, we can also state

Example 1 *If in particular α is as in Definition 3, then by Definition 4 we have*

$$f(n, A) = \alpha(n)[A] = \varphi^n[A]$$

for all $n \in \overline{\mathbb{N}}_0$ and $A \subseteq X$. Thus, in particular $f(0, A) = A$ for all $A \subseteq X$.

7 From a translation inclusion to a Sincov inclusion

Now, as an extension of our former observations, we may also naturally start with the following

Notation 3 *Suppose that U is an additive groupoid, X is a goset and f is a function of $U \times X$ to X such that f is increasing in its second variable and*

$$f\big(u, f(v, x)\big) \leq f(u+v, x)$$

for all $u, v \in U$ and $x \in X$.

Thus, improving an idea of Frege [15, 16], we may also naturally introduce

Definition 5 For any $x, y \in X$, define

$$F(x, y) = \big\{u \in U : \quad x \leq f(u, y)\big\}.$$

Thus, F may be considered as a relation on X^2 to U, or as a function of X^2 to $\mathcal{P}(U)$, which can be proved to satisfy a Sincov type inclusion.

Theorem 9 *For any* $x, y, z \in X$*, we have*

$$F(x, y) + F(y, z) \subseteq F(x, z).$$

Proof. If

$$u \in F(x, y) \qquad \text{and} \qquad v \in F(y, z),$$

then by Definition 5 we get

$$x \leq f(u, y) \qquad \text{and} \qquad y \leq f(v, z).$$

Hence, by using the assumed increasingness and translation property of f, we can infer that

$$x \leq f(u, y) \leq f\big(u, f(v, z)\big) \leq f(u + v, z).$$

Therefore, by Definition 5, we have

$$u + v \in F(x, z).$$

Thus, the required inclusion is true. □

Now, to illustrate the appropriateness of Definition 5, we can also state

Example 2 *If* f *is as in Example 1, then by Definition 5 we have*

$$F(A, B) = \big\{ n \in \overline{\mathbb{N}}_0 : \quad A \subseteq f(n, B) \big\} = \big\{ n \in \overline{\mathbb{N}}_0 : \quad A \subseteq \varphi^n[B] \big\}$$

for all $A, B \subseteq X$*. Thus, in particular* $0 \in F(A, A)$ *for all* $A \subseteq X$*.*

Remark 11 By Aczél [1, pp. 223, 303 and 353], Sincov's functional equation and its generalizations have been investigated by a surprisingly great number of authors.

For some more recent investigations, see [4, 33, 38, 27, 34, 35, 7, 8, 3, 14, 13]. The most relevant ones are the set-valued considerations of Smajdor [38] and Augustová and Klapka [3].

Moreover, it is noteworthy that, by using the famous partial operation

$$(x, y) \bullet (y, z) = (x, z),$$

the above Sincov inclusion can be turned into a restricted Cauchy inclusion.

Therefore, some of the methods of the theory of superadditive functions and relations [28, 17, 29, 19] can certainly be applied to investigate the corresponding Sincov inequalities and inclusions.

8 Some immediate consequences of a Sincov inclusion

Now, motivated by our former observations, we may also naturally introduce the following notations and definitions.

Notation 4 *In what follows, we shall also assume that* X *is a set and* U *is an additive groupoid. Moreover, we shall suppose that* F *is a relation on* X^2 *to* U.

Definition 6 The relation F will be called *supertriangular* if

$$F(x, y) + F(y, z) \subseteq F(x, z)$$

for all $x, y, z \in X$.

Remark 12 Now, the relation F may also be naturally called *subtriangular* if the reverse inclusion holds. Moreover, F may be naturally called *triangular* if it is both subtriangular and supertriangular.

Subtriangular relations are certainly more important than the supertriangular ones. Namely, if a function d of X^2 to $[0, +\infty]$ satisfies the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$

for all $x, y, z \in X$, then the relation F, defined such that

$$F(x, y) = \big[0,\ d(x, y)\big] \qquad \Big(\ F(x, y) = \ [-d(x, y),\ d(x, y)]\ \Big)$$

for all $x, y \in X$, can, in general, be proved to be only subtriangular [2].

The $y = x$, $y = z$ and $z = x$ particular cases of the inclusion considered in Definition 6 strongly suggest the introduction of the following

Definition 7 For any $x, y \in X$, define

$$R(x, y) = F(y, x) \qquad \text{and} \qquad S(x, y) = F(x, y) + R(x, y).$$

Moreover, for any $x \in X$, define

$$\Phi(x) = F(x, x) \qquad \text{and} \qquad \Psi(x) = \bigcup_{y \in X} S(x, y).$$

Thus, R and S may be considered as relations on X^2 to U, and Φ and Ψ may be considered as relations on X to U.

Concerning these relations, we can easily prove the following

Theorem 10 *For any* $x, y \in X$ *we have*

(1) $\Phi(x) + \Phi(x) \subseteq \Psi(x)$;

(2) $R(x, x) = \Phi(x)$; (3) $S(x, x) = \Phi(x) + \Phi(x)$.

Proof. By Definition 7, we evidently have

$$R(x, x) = F(x, x) = \Phi(x),$$

and thus also

$$S(x, x) = F(x, x) + R(x, x) = \Phi(x) + \Phi(x).$$

Hence, by using the definition of Ψ, we can also easily note that

$$\Phi(x) + \Phi(x) = S(x, x) \subseteq \bigcup_{y \in X} S(x, y) = \Psi(x).$$

Therefore, assertions (2), (3) and (1) are true. □

Now, as a counterpart of [38, Lemma 1] of Wilhelmina Smajdor, we can also prove the following

Theorem 11 *If* F *is supertriangular, then for any* $x, y \in X$ *we have*

(1) $\Psi(x) \subseteq \Phi(x)$;

(2) $\Phi(x) + F(x, y) \subseteq F(x, y)$; (3) $F(x, y) + \Phi(y) \subseteq F(x, y)$.

Proof. By using Definition 7 and the corresponding particular cases of the inclusion considered in Definition 6, we can easily see that

$$\Phi(x) + F(x, y) = F(x, x) + F(x, y) \subseteq F(x, y)$$

and

$$F(x, y) + \Phi(y) = F(x, y) + F(y, y) \subseteq F(x, y).$$

Moreover,

$$S(x, y) = F(x, y) + R(x, y) = F(x, y) + F(y, x) \subseteq F(x, x) = \Phi(x),$$

and thus also

$$\Psi(x) = \bigcup_{y \in X} S(x, y) \subseteq \bigcup_{y \in X} \Phi(x) \subseteq \Phi(x).$$

Therefore, assertions (2), (3) and (1) are true even if only some consequences of the assumed inclusion property of F are supposed to hold. □

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 2 *If* F *is supertriangular, then for any* $x, y \in X$ *we have*

(1) $\Phi(x) + \Phi(x) \subseteq \Phi(x)$; (2) $\Psi(x) + \Psi(x) \subseteq \Psi(x)$;
(3) $\Psi(x) + \Phi(x) \subseteq \Psi(x)$; (4) $\Phi(x) + \Psi(x) \subseteq \Psi(x)$;
(5) $\Psi(x) + F(x, y) \subseteq F(x, y)$; (6) $F(x, y) + \Psi(y) \subseteq F(x, y)$.

Remark 13 By [8], in addition to Definition 6, the *separability equation*

$$F(x, y) + F(y, z) = F(x, z) + \Phi(y)$$

may also be naturally investigated.

Moreover, if in particular U is a group, then in addition to Definition 7, the *disymmetry relation* D *of* F, defined such that $D(x, y) = F(x, y) - R(x, y)$ for all $x, y \in X$, may also be naturally investigated.

9 The particular case when U has a zero element

Theorem 12 *If* F *is supertriangular,* U *has a one-sided zero element* 0 *and* $x \in X$ *is such that* $0 \in \Phi(x)$, *then*

(1) $\Phi(x) = \Psi(x)$; (2) $\Phi(x) = \Phi(x) + \Phi(x)$.

Proof. If 0 is a right zero element of U, then by using Theorems 10 and 11 we can see that

$$\Phi(x) = \Phi(x) + \{0\} \subseteq \Phi(x) + \Phi(x) \subseteq \Psi(x) \subseteq \Phi(x).$$

While, if 0 is a left zero element of U, then we can quite similarly see that

$$\Phi(x) = \{0\} + \Phi(x) \subseteq \Phi(x) + \Phi(x) \subseteq \Psi(x) \subseteq \Phi(x).$$

Therefore, in both cases, the required equalities are true. □

Remark 14 Note that if in particular F is as in Example 2, then $0 \in \Phi(A)$ holds for all $A \subseteq X$. Therefore, the above theorem can be applied.

Now, by using a somewhat more complicated argument, we can also prove

Theorem 13 *If* F *is supertriangular,* U *has a one-sided zero element* 0 *and* $x, y \in X$ *are such that*

$$0 \in F(x, y) \cap F(y, x),$$

then

(1) $\Phi(x) = \Psi(x) = F(x, y) = S(x, y)$; (2) $\Phi(x) = \Phi(x) + \Phi(y)$.

Proof. If 0 is a right zero element of U, then by using Theorem 11 we can see that

$$\begin{aligned}\Phi(x) &= \Phi(x) + \{0\} \subseteq \Phi(x) + F(x,y) \subseteq F(x,y) = F(x,y) + \{0\} \\ &\subseteq F(x,y) + F(y,x) = F(x,y) + R(x,y) = S(x,y) \subseteq \Psi(x) \subseteq \Phi(x).\end{aligned}$$

While, if 0 is a left zero element of U, then we can quite similarly obtain

$$\begin{aligned}\Phi(x) &= \{0\} + \Phi(x) \subseteq F(y,x) + \Phi(x) \subseteq F(y,x) = \{0\} + F(y,x) \\ &\subseteq F(x,y) + F(y,x) = F(x,y) + R(x,y) = S(x,y) \subseteq \Psi(x) \subseteq \Phi(x).\end{aligned}$$

Therefore, in both cases, assertion (1) is true.

Now, assertion (2) can be easily derived from assertion (1), by noticing that

$$\Phi(x) = S(x,y) = F(x,y) + R(x,y) = F(x,y) + F(y,x) = \Phi(x) + \Phi(y).$$

□

From this theorem, it is clear that in particular we also have the following

Corollary 3 *If F is supertriangular and U has a one-sided zero element 0 such that $0 \in F(x,y)$ for all $x, y \in X$, then for any $x, y \in X$ we have*
(1) $\Phi(x) = \Psi(x) = F(x,y) = S(x,y)$; (2) $\Phi(x) = \Phi(x) + \Phi(y)$.

10 The particular case when U is a group

By using an argument of Frege [15, 16] and Sincov [36, 23], we can prove

Theorem 14 *If F is a nonpartial, triangular function and U is a group, then there exists a function ξ of X to U such that*

$$F(x,y) = \xi(x) - \xi(y)$$

for all $x, y \in X$.

Proof. By choosing $z \in X$, and defining

$$\xi(x) = F(x,z)$$

for all $x \in X$, we can see that

$$F(x,y) + \xi(y) = F(x,y) + F(y,z) = F(x,z) = \xi(x),$$

and thus $F(x,y) = \xi(x) - \xi(y)$ for all $x, y \in X$. □

Remark 15 If F is nonpartial and supertriangular and U is a group, then by using a similar argument we can only prove that

$$F(x, y) \subseteq \bigcap_{z \in X} \big(F(x, z) - F(y, z) \big)$$

for all $x, y \in X$.

Now, analogously to [38, Theorem 1] of Wilhelmina Smajdor, we can also prove

Theorem 15 *If* F *is nonpartial and supertriangular,* U *is a commutative group and* ϕ *is a triangular selection function of* F*, then*

$$F(x, y) = \phi(x, y) + \Phi(x)$$

for all $x, y \in X$.

Proof. Define

$$G(x, y) = -\phi(x, y) + F(x, y)$$

for all $x, y \in X$.

Then, because of $\phi(x, y) \in F(x, y)$, we evidently have

$$0 = -\phi(x, y) + \phi(x, y) \in -\phi(x, y) + F(x, y) = G(x, y)$$

for all $x, y \in X$. Moreover, by using the assumed triangularity properties of ϕ and F, we can easily see that

$$\begin{aligned} G(x, y) + G(y, z) = -\phi(x, y) + F(x, y) - \phi(y, z) + F(y, z) = \\ -\big(\phi(x, y) + \phi(y, z)\big) + F(x, y) + F(y, z) \subseteq -\phi(x, z) + F(x, z) = G(x, z) \end{aligned}$$

for all $x, y, z \in X$.

Hence, by using Corollary 3 and the simple observation that

$$\phi(x, x) + \phi(x, x) = \phi(x, x),$$

and thus $\phi(x, x) = 0$ for all $x \in X$, we can already infer that

$$G(x, y) = G(x, x) = -\phi(x, x) + F(x, x) = \Phi(x),$$

and thus

$$-\phi(x, y) + F(x, y) = \Phi(x)$$

for all $x, y \in X$. Therefore, the required equality is also true. □

Remark 16 It can be easily seen that a converse of Theorem 14 is also true. Therefore, if F is nonpartial and U is a group, then to find a triangular selection function ϕ of F, it is enough to find only a function ξ of X to U such that

$$\xi(x) - \xi(y) \in F(x, y)$$

for all $x, y \in X$.

11 The particular case when U is a commutative groupoid

Theorem 16 *If F is supertriangular and U is commutative, then R is also supertriangular.*

Proof. By Definitions 6 and 7 and the commutativity of U, we have

$$\begin{aligned} R(x, y) + R(y, z) &= F(y, x) + F(z, y) \\ &= F(z, y) + F(y, x) \subseteq F(z, x) = R(x, z) \end{aligned}$$

for all $x, y, z \in X$. □

Theorem 17 *If U is commutative, then for any $x, y, z \in X$ we have*
(1) $S(x, y) = S(y, x)$; (2) $S(x, y) \subseteq \Psi(x) \cap \Psi(y)$.

Proof. By Definition 7 and the commutativity of U, we have

$$\begin{aligned} S(x, y) &= F(x, y) + R(x, y) = R(y, x) + F(y, x) \\ &= F(y, x) + R(y, x) = S(y, x). \end{aligned}$$

Moreover, by the definition of Ψ, it is clear that $S(x, y) \subseteq \Psi(x)$. Hence, by using the above symmetry property of S, we can already infer that

$$S(x, y) = S(y, x) \subseteq \Psi(y),$$

and thus $S(x, y) \subseteq \Psi(x) \cap \Psi(y)$ also holds. □

Remark 17 Thus, if U is commutative, then S is already *pointwise symmetric* in the sense that $S(x, y) = S(y, x)$ for all $x, y \in X$.

Now, concerning the relation S, we can also prove the following

Theorem 18 *If* F *is supertriangular and* U *is a commutative semigroup, then* S *is also supertriangular.*

Proof. By using Definition 7, Theorem 16 and the commutativity and associativity of U, we can see that

$$\begin{aligned} S(x,y)+S(y,z) &= F(x,y)+R(x,y)+F(y,z)+R(y,z) \\ &= F(x,y)+F(y,z)+R(x,y)+R(y,z) \\ &\subseteq F(x,z)+R(x,z)=S(x,z) \end{aligned}$$

for all $x, y, z \in X$. □

12 The particular case when F is pointwise symmetric

In addition to Theorem 17, we can also prove the following

Theorem 19 *If* $x, y \in X$ *such that* $F(x,y)=F(y,x)$, *then*

(1) $R(x,y)=F(x,y)$;
(2) $S(x,y)=S(y,x)$;
(3) $S(x,y)=F(x,y)+F(x,y)$;
(4) $2F(x,y) \subseteq S(x,y) \subseteq \Psi(x) \cap \Psi(y)$.

Proof. By Definition 7 and the assumed symmetry property of F, we have

$$R(x,y)=F(y,x)=F(x,y),$$

and thus also

$$S(x,y)=F(x,y)+R(x,y)=F(x,y)+F(x,y).$$

Thus, assertions (1) and (3) are true.

Now, we can also easily see that

$$S(y,x)=F(y,x)+F(y,x)=F(x,y)+F(x,y)=S(x,y).$$

Therefore, assertion (2) is also true.

Hence, as in the proof of Theorem 17, we can already infer that

$$S(x,y) \subseteq \Psi(x) \cap \Psi(y).$$

Therefore, to complete the proof of assertion (4), it remains to note only that now

$$2F(x,y) \subseteq F(x,y) + F(x,y) = S(x,y)$$

is also true. □

Remark 18 Thus, not only the commutativity of U, but the pointwise symmetry of F also implies the pointwise symmetry of S.

By [8], in addition to the pointwise symmetry of F, one may also naturally investigate the case when F is only *weightable* in the sense that

$$w(x) + F(x,y) = R(x,y) + w(y)$$

for all $x, y \in X$ and some function (or relation) w on X to U.

However, it is now more important to note that, as an immediate consequence of our former results, we can also state

Corollary 4 *If F is supertriangular and U is commutative, then for any $x, y \in X$ we have*

$$2S(x,y) \subseteq S(x,y) + S(y,x) \subseteq S(x,x) \cap S(y,y).$$

Remark 19 Note that the latter corollary only needs the important consequence of the assumed inclusion property of F that $F(x,y) + F(y,x) \subseteq F(x,x)$ for all $x, y \in X$.

In Theorem 11, by using Definition 7, the latter property has been reformulated in the shorter form that $\Psi(x) \subseteq \Phi(x)$ for all $x \in X$. Now, this already implies that Ψ is a selection relation of Φ. Namely, if $x \in X$ such that $\Phi(x) \neq \emptyset$, then because of $\Phi(x) + \Phi(x) \subseteq \Psi(x)$, we also have $\Psi(x) \neq \emptyset$.

13 The particular case when U is a group and F is pointwise skew symmetric

Analogously to Theorem 19, we can also prove the following

Theorem 20 *If U is a group and $x, y \in X$ such that $F(x,y) = -F(y,x)$, then*

(1) $R(x,y) = -F(x,y)$;

(2) $S(x,y) = -S(y,x)$;

(3) $S(x,y) = F(x,y) - F(x,y)$;

(4) $S(x,y) \subseteq \Psi(x) \cap \big(-\Psi(y)\big)$.

Proof. To prove (4), note that now, in addition to $S(x, y) \subseteq \Psi(x)$, we also have

$$S(x, y) = -S(y, x) \subseteq -\Psi(y),$$

and thus $S(x, y) \subseteq \Psi(x) \cap \big(-\Psi(y)\big)$ also holds. □

Remark 20 If in addition to the assumptions of this theorem $F(x, y) \neq \emptyset$ also holds, then from assertion (3) we can infer that $0 \in S(x, y)$.

Now, by using the corresponding definitions and Theorem 20, we can also prove

Theorem 21 *If* U *is a group and* F *is pointwise skew symmetric, then for any* $x \in X$ *we have*
(1) $\Phi(x) = -\Phi(x)$; (2) $\Psi(x) = -\Psi(x)$.

Proof. To prove (2), note that by Definition 7 and Theorem 20 we have

$$\Psi(x) = \bigcup_{y \in X} S(x, y) = \bigcup_{y \in X} \big(-S(x, y)\big) = -\bigcup_{y \in X} S(x, y) = -\Psi(x)$$

for all $x \in X$. □

Remark 21 If in addition to the assumptions of this theorem, $\Phi(x) \neq \emptyset$ also holds, then from the inclusion

$$\Phi(x) - \Phi(x) = \Phi(x) + \Phi(x) \subseteq \Psi(x),$$

we can infer that $0 \in \Psi(x)$. Therefore, if in addition F is supertriangular, then because Theorem 11, we also have $0 \in \Phi(x)$.

Thus, by Theorem 12, we can also state the following

Theorem 22 *If* U *is a group and* F *is nonpartial, supertriangular and pointwise skew symmetric, then for any* $x \in X$ *we have*
(1) $\Phi(x) = \Psi(x)$; (2) $\Phi(x) = \Phi(x) + \Phi(x)$.

Now, by Theorems 20 and 21, we can also state the following

Theorem 23 *If* U *is a group and* F *is a nonpartial, pointwise skew symmetric function, then for any* $x, y \in X$ *we have*
(1) $S(x, y) = 0$; (2) $\Phi(x) = \Psi(x) = 0$.

The following example shows the three important consequences of the inclusion considered in Definition 6 do not imply, even in a very simple case, the validity of this inclusion itself.

Example 3 *If*

$$F(x, y) = \operatorname{sgn}(x - y)$$

for all $x, y \in \mathbb{R}$*, then* F *is a skew symmetric function of* $\mathbb{R}^2$ *to* $\mathbb{R}$ *such that, under the notation* $\Phi(x) = F(x, x)$*, for any* $x, y \in X$ *we have*

(1) $F(x, y) + F(y, x) = \Phi(x)$*;*

(2) $\Phi(x) + F(x, y) = F(x, y)$*;* (3) $F(x, y) + \Phi(y) = F(x, y)$.

However, F *is not either supertriangular nor subtriangular in both functional and relational sense.*

Namely, for instance, we have

$$F(2, 1) + F(1, 0) = 2 \qquad \textit{and} \qquad F(2, 0) = 1,$$

and

$$F(0, 1) + F(1, 2) = -2 \qquad \textit{and} \qquad F(0, 2) = -1.$$

14 The particular case when U is cancellative

Definition 8 In what follows, we shall denote by $\operatorname{lcan}(U)$ and $\operatorname{rcan}(U)$ the family of all left-cancellable and right-cancellable elements of the groupoid U, respectively.

Moreover, we shall also write $\operatorname{can}(U) = \operatorname{lcan}(U) \cap \operatorname{rcan}(U)$.

Remark 22 Thus, for any $u \in U$, we have $u \in \operatorname{lcan}(U)$ if and only if $u + v = u + w$ implies $u = w$ for all $v, w \in U$.

Moreover, for instance, we can state that U is left-cancellative if and only if $\operatorname{lcan}(U) = U$.

Lemma 1 *For any* $V, W \subseteq U$,

(1) $\operatorname{card}(V + W) \leq 1$ *and* $V \cap \operatorname{lcan}(U) \neq \emptyset$ *imply that* $\operatorname{card}(W) \leq 1$*;*

(2) $\operatorname{card}(V + W) \leq 1$ *and* $W \cap \operatorname{rcan}(U) \neq \emptyset$ *imply that* $\operatorname{card}(V) \leq 1$.

Proof. Assume that the conditions of (1) hold, $v \in V \cap \operatorname{lcan}(U)$ and $w_1, w_2 \in W$. Then, we have $v + w_1, v + w_2 \in V + W$. Hence, by using that $\operatorname{card}(V + W) \leq 1$, we can infer that $v + w_1 = v + w_2$. Moreover, since $v \in \operatorname{lcan}(U)$, we can also state that $w_1 = w_2$. Therefore, $\operatorname{card}(W) \leq 1$, and thus (1) also holds.

The proof of assertion (2) is quite similar. □

Now, by using this lemma, we can give some reasonable sufficient conditions in order that a suppertriangular relation should be a function.

Theorem 24 *If* F *is supertriangular and there exist* $x_0, y_0 \in X$ *such that*

(1) $\operatorname{card}\big(F(x_0, y_0)\big) \leq 1$*;*

(2) $F(x, y_0) \cap \operatorname{rcan}(U) \neq \emptyset$ *for all* $x \in X$*;*

(3) $F(x_0, y) \cap \operatorname{lcan}(U) \neq \emptyset$ *for all* $y \in X$*;*

then $\operatorname{card}\big(F(x, y)\big) \leq 1$ *for all* $x, y \in X$*, and thus* F *is a function.*

Proof. By the assumed inclusion property of F, we have

$$F(x_0, x) + F(x, y_0) \subseteq F(x_0, y_0)$$

for all $x \in X$. Hence, by using conditions (1) and (3) and Lemma 1, we can infer that

(a) $\operatorname{card}\big(F(x, y_0)\big) \leq 1$ for all $x \in X$.

Now, by the assumed inclusion property of F, we also have

$$F(x, y) + F(y, y_0) \subseteq F(x, y_0)$$

for all $x, y \in X$. Hence, by using assertion (a) condition (2) and Lemma 1, we can infer that

(b) $\operatorname{card}\big(F(x, y)\big) \leq 1$ for all $x, y \in X$.

Thus, the required assertion is true. □

From this theorem, by using Theorem 14, we can immediately derive

Corollary 5 *If* F *is nonpartial and supertriangular,* U *is a group and* $\operatorname{card}\big(F(x_0, y_0)\big) = 1$ *for some* $x_0, y_0 \in X$*, then there exists a function* ξ *of* X *to* U *such that*

$$F(x, y) = \xi(x) - \xi(y)$$

for all $x, y \in X$*.*

15 The particular case when U has a suitable distance function

Remark 23 A function d of X^2 to $[0, +\infty]$ is usually called a *distance function* on X.

Moreover, the extended real number

$$d(X) = \operatorname{diam}(X) = \sup\big\{ d(x, y) : \quad x, y \in X \big\}$$

is called the *diameter* of X.

Remark 24 Thus, we have $d(X) = -\infty$ if $X = \emptyset$, and $d(X) \geq 0$ if $X \neq \emptyset$. Moreover, if $X \neq \emptyset$, then $\operatorname{card}(X) = +\infty$ may also hold even if X is finite.

Definition 9 A distance function d on X will be called *admissible* if

(a) $d(X) < +\infty$;

(b) $d(x, y) = 0$ implies $x = y$ *for all* $x, y \in X$.

Moreover, the distance function d will be called *extremal* if

(c) for any $x, y \in X$ there exist $c \in \,]1, +\infty[$ and $z, w \in X$ such that

$$c d(x, y) \leq d(z, w).$$

Remark 25 If X is an additive groupoid, then to satisfy condition (c) we may naturally assume that for any $x, y \in X$, there exists $n \in \mathbb{N} \setminus \{1\}$ such that

$$n d(x, y) \leq d(nx, ny).$$

Namely, if X is a commutative abelian group and p is a function of U to $[0, +\infty]$ such that

$$n p(x) \leq p(nx)$$

for all $n \in \mathbb{N}$ and $x \in X$, then by defining

$$d(x, y) = p(-x + y)$$

for all $x, y \in X$, we have

$$\begin{aligned} n d(x, y) &= n p(-x + y) \leq p\big(n(-x + y)\big) \\ &= p\big(n(-x) + ny\big) = p(-nx + ny) = d(nx, ny) \end{aligned}$$

for all $n \in \mathbb{N}$ *and* $x, y \in U$.

The introduction of Definition 9 can only be motivated by the following

Lemma 2 *If there exists an extremal, admissible distance function* d *on* X*, then* $\operatorname{card}(X) \leq 1$.

Proof. If $X = \emptyset$, then the required assertion trivially holds. Therefore, we may assume that $X \neq \emptyset$, and thus $d(X) \neq -\infty$. Now, by condition (a), we can state that $d(X) \in \mathbb{R}$. Moreover, since d is nonnegative, we can now also note that $d(X) \geq 0$.

Thus, for every $\varepsilon > 0$, we have

$$d(X) - \varepsilon < d(X).$$

Therefore, by the definition of $d(X)$, there exist $x, y \in X$ such that $d(X) - \varepsilon < d(x, y)$, and thus

$$d(X) < d(x, y) + \varepsilon.$$

Moreover, by condition (c), there exist $c \in \,]1, +\infty[$ and and $z, w \in X$ such that

$$c d(xy) \leq d(z, w).$$

Combining the above two inequalities, we can see that

$$c d(x, y) < d(z, w) \leq d(X) < d(x, y) + \varepsilon,$$

and thus $(c-1)d(x, y) < \varepsilon$. Hence, by letting ε tend to zero, we can infer that $(c-1)d(x, y) \leq 0$. Therefore, since $c - 1 > 0$, we necessarily have $d(x, y) \leq 0$, and hence $d(x, y) \leq 0$ by the nonnegativity of d. Thus, we actually have

$$d(X) < d(x, x) + \varepsilon = \varepsilon.$$

Hence, by letting ε tend to zero, we can infer that $d(X) \leq 0$, and thus also $d(X) = 0$ by the nonnegativity of $d(X)$.

This, by condition (b), already implies that $\operatorname{card}(X) = 1$. Namely, if this is not the case, then by the assumption $X \neq \emptyset$, there exist $x, y \in X$ such that $x \neq y$. Hence, by condition (b) and the nonnegativity of d, we can infer that $d(x, y) > 0$, and thus also $d(X) > 0$ by the definition of $d(X)$. This contradiction proves that $\operatorname{card}(X) = 1$. □

Remark 26 From condition (c), by induction, we can infer that there exist sequences $(c_n)_{n=1}^{\infty}$ in $]1, +\infty[$ and $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in X such that

$$d(x, y) \prod_{i=0}^{n} c_i \leq d(x_n, y_n)$$

for all $n \in \mathbb{N}$. However, this fact cannot certainly be used to give a simpler proof for Lemma 2.

From Theorem 24, by using Lemma 2, we can immediately derive

Theorem 25 *If* F *is supertriangular and there exist* $x_0, y_0 \in X$*, such that*

(1) $F(x, y_0) \cap \text{rcan}(U) \neq \emptyset$*; for all* $x \in X$*;*

(2) $F(x_0, y) \cap \text{lcan}(U) \neq \emptyset$*; for all* $y \in X$*;*

(3) *there exists an extremal, admissible distance function on* $F(x_0, y_0)$*;*

then $\text{card}\big(F(x, y)\big) \leq 1$ *for all* $x, y \in X$*, and thus* F *is a function.*

Proof. By assumption (3) and Lemma 2, we have $\text{card}\big(F(x_0, y_0)\big) \leq 1$. Hence, by Theorem 24, we can see that the required assertion is also true. □

16 Contructions of supertriangular relations

Theorem 26 *If* V *is a subgroupoid of* U *and*

$$F(x, y) = V$$

for all $x, y \in X$*, then* F *is a supertriangular relation on* X *to* U*.*

Proof. We evidently have

$$F(x, y) + F(y, z) = V + V \subseteq V = F(x, z)$$

for all $x, y, z \in X$. □

Remark 27 Conversely, note that if F is a supertriangular relation on X^2 to U, then by Corollary 2 $\Phi(x) = F(x, x)$ is a subgroupoid of U for all $x \in X$.

Now, as a converse to Theorem 14, we can also easily prove the following

Theorem 27 *If* ξ *is a function of* X *to* U*,* U *is a group and*

$$F(x, y) = \xi(x) - \xi(y)$$

for all $x, y \in X$*, then* F *is a triangular function of* X^2 *to* U*.*

Proof. We evidently have

$$F(x, y) + F(y, z) = \xi(x) - \xi(y) + \xi(y) - \xi(z) = \xi(x) - \xi(z) = F(x, z)$$

for all $x, y, z \in X$. □

Remark 28 If ξ is only a relation of X to U, U is a group and $F(x, y) = \xi(x) - \xi(y)$ for all $x, y \in X$, then by using a similar argument we can only prove that F is a subtriangular relation of X^2 to U.

In addition to the above two theorems, it is also worth proving that the family of all supertriangular relations is closed under the usual pointwise operations.

Theorem 28 *If F is a supertriangular relation on X^2 to U and U is a commutative semigroup, then nF is also a supertriangular relation on X^2 to U for all $n \in \mathbb{N}$.*

Proof. If $n \in \mathbb{N}$, then by the corresponding definitions we have

$$\begin{aligned}\big(nF\big)(x, y) + \big(nF\big)(y, z) &= nF(x, y) + nF(y, z) \\ &= n\big(F(x, y) + F(y, z)\big) \subseteq nF(x, z) = \big(nF\big)(x, z)\end{aligned}$$

for all $x, y, z \in X$. □

Remark 29 If F is a supertriangular relation on X^2 to U and U has a zero element, then

$$\big(0F\big)(x, y) = \emptyset \quad \text{if} \quad F(x, y) = \emptyset \qquad \text{and} \qquad \big(0F\big)(x, y) = \{0\} \quad \text{if} \quad F(x, y) \neq \emptyset.$$

Therefore, $0F$ is a supertriangular function on X^2 to U.

Now, analogously to Theorem 28, we can also prove the following

Theorem 29 *If F is a supertriangular relation on X^2 to U and U is a commutative group, then kF is also a supertriangular relation on X^2 to U for all $k \in \mathbb{Z}$.*

Moreover, in addition to Theorems 28, we can also easily prove the following

Theorem 30 *If F and G are supertriangular relations on X^2 to U and U is a commutative semigroup, then $F + G$ is also a supertriangular relation on X^2 to U.*

Proof. By the corresponding definitions, it is clear that

$$\begin{aligned}&(F + G)(x, y) + (F + G)(y, z) = F(x, y) + G(x, y) + F(y, z) + G(y, z) \\ &\quad = F(x, y) + F(y, z) + G(x, y) + G(y, z) \subseteq F(x, z) + G(x, z) = (F + G)(x, z)\end{aligned}$$

for all $x, y, z \in X$. □

17 An application of the above results

Now, by using Theorems 26, 27 and 30, we can also easily establish

Theorem 31 *If* ξ *a function of* X *to* U, U *is a commutative group,* V *is a subgroupoid of* U *and*

$$F(x,y) = \xi(x) - \xi(y) + V$$

for all $x, y \in X$, *then* F *is a supertriangular relation on* X^2 *to* U *such that, under the notations of Definition 7, for any* $x, y \in X$ *we have:*

(1) $\Phi(x) = V$; (2) $S(x,y) = V + V$;

(3) $\Psi(x) = \emptyset$ *if* $X = \emptyset$ *and* $\Psi(x) = V + V$ *if* $X \neq \emptyset$.

Proof. From Theorems 26, 27 and 30, it is clear that F is supertriangular. Moreover, by the corresponding definitions, it is clear that

$$\Phi(x) = F(x,x) = \xi(x) - \xi(x) + V = V,$$

$$S(x,y) = F(x,y) + F(y,x) = \xi(x) - \xi(y) + V + \xi(y) - \xi(x) + V = V + V$$

and

$$\Psi(x) = \bigcup_{y \in X} S(x,y) = \bigcup_{y \in X} (V+V) = \begin{cases} \emptyset & \text{if} \quad X = \emptyset, \\ V+V & \text{if} \quad X \neq \emptyset. \end{cases}$$

□

Moreover, for an easy illustration of this theorem, we can also state

Example 4 *If* $r \geq 0$ *and*

$$F(x,y) = [x - y + r, \ +\infty[$$

for all $x, y \in \mathbb{R}$, *then* F *is a supertriangular relation of* $\mathbb{R}^2$ *to* $\mathbb{R}$ *such that, for any* $x, y \in X$, *we have:*

(1) $\Phi(x) = [r, +\infty[$; (2) $\Psi(x) = S(x,y) = [2r, +\infty[$.

To check this, note that, by taking $\xi = \Delta_{\mathbb{R}}$ *and* $V = [r, +\infty[$, *we have*

$$F(x,y) = [x - y + r, \ +\infty[\ = x - y + [r, +\infty[\ = \xi(x) - \xi(y) + V$$

for all $x, y \in X$. *Therefore, Theorem 31 can be applied.*

For instance, by assertion (2) of Theorem 31, we have

$$S(x,y) = V + V = [r, +\infty[\ + [r, +\infty[\ = [2r, +\infty[$$

for all $x, y \in X$.

Remark 30 Note that in the present particular case, for any $x, y \in \mathbb{R}$, we have:

(1) $0 \in \Phi(x) \iff r = 0$;
(2) $\Phi(x) = \Psi(x) \iff r = 0$;
(3) $x - y \in F(x, y) \iff r = 0$;
(4) $0 \in F(x, y) \iff r \leq y - x$;
(5) $0 \in F(x, y) \cap F(y, x) \iff r = 0, \quad x = y$.

To prove (5), note that by (4) we have

$$0 \in F(x, y) \cap F(y, x) \iff r \leq y - x,\ r \leq x - y \iff r \leq \min\{x - y,\ y - x\}.$$

Moreover, recall that $\min\{a, b\} = 2^{-1}\big(a + b - |a - b|\big)$ for all $a, b \in \mathbb{R}$, and thus in particular $\min\{x - y,\ y - x\} = -|x - y|$. Therefore,

$$r \leq \min\{x - y,\ y - x\} \iff r \leq -|x - y| \iff r = 0, \quad x = y.$$

Acknowledgement

The authors are indepted to the referee for suggesting grammatical corrections and stylistic changes.

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

[2] M. Alimohammady, S. Jafari, S. P. Moshokoa and M. K. Kalleji, A note on properties of hypermetric spaces, *J. Hyperstructures*, **3** (2014), 89–100.

[3] P. Augustová and L. Klapka, *Atlas as solutions of Sincov's inequality*, arXiv: 1612.00355v1 [math.DS] 1 Dec 2016, 7 pp.

[4] J. A. Baker, Solution of problem E 2607, *Amer. Math. Monthly*, **84** (1977), 824–825.

[5] G. Birkhoff, Lattice Theory, *Amer. Math. Soc. Colloq. Publ.*, 25, Providence, Rhode Island, 1967.

[6] H. Brandt, Über eine Verallgemeinerung der Gruppenbregriffes, *Math. Ann.*, **96** (1926), 360–366.

[7] M. J. Campión, R. G. Catalán, E. Induráin and G. Ochoa, Reinterpreting a fuzzy subset by means of a Sincov's functional equation, *J. Intelligent Fuzzy Systems*, **27** (2014), 367–375.

[8] M. J. Campión, E. Induráin, G. Ochoa and O. Valero, Functional equations related to weightable quasi-metrics, *Hacet. J. Math. Stat.*, **44** (2015), 775–787.

[9] M. Cantor, Funktionalgleichungen mit drei von einander unabhängigen Veränderlichen, *Zeitschrift Mat. Physik.*, **41** (1896), 161–163.

[10] E. Castillo-Ron and R. Ruiz-Cobo, *Functional Equations in Science and Engineering*, Marcel Decker, New York, 1992.

[11] R. Croisot, Une interprétation des relations d'équivalence dans un ensemble, *C. R. Acad. Sci. Paris*, **226** (1948), 616–617.

[12] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.

[13] W. Fechner, Richard's inequality, Cauchy-Schwarz's inequality and approximate solutions of Sincov's equation, *Proc. Amer. Math. Soc.*, **147** (2019), 3955–3960.

[14] W. Fechner, Sincov's inequalities on topological spaces, *Publ. Math. Debrecen*, **96** (2020), 63–76.

[15] G. Frege, *Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen*, Dissertation for the Venia docendi, Verlag Friedrich Frommann, Jena, 1874.

[16] G. Frege, *Methods of calculation based on an extension of the concept of quantity*, In: G. Frege, Collected Papers of on Mathematics, Logic, and Philosophy, Basil Blackwell, Oxford, 1984, 56–92.

[17] Z. Gajda, *Invariant means and representations of semigroups in the theory of functional equations*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach, **1273** (1992), 1–81.

[18] T. Glavosits, Generated preorders and equivalences, *Acta Acad. Paed. Agriensis, Sect. Math.*, **29** (2002), 95–103.

[19] T. Glavosits and Á. Száz, *Constructions and extensions of free and controlled additive relations*, In: Th.M. Rassias (Ed.), Handbook of Functional Equations: Functional Inequalities, Springer Optim. Appl. **95** (2014), 161–208.

[20] D. Gronau, *Gottlob Frege, a pioneer in iteration theory*, In: L. Reich, J. Smítal and Gy. Targonski (Eds.), Proceedings of the European Conference on Iteration Theory, ECIT94, Gracer Math. Ber. **334** (1997), 105–119.

[21] D. Gronau, *Gottlob Fregees Beiträge zur Iteratiostheorie und zur Theorie der Functionalgleichungen*, In: G. Gabriel (Ed.), Gottlob Frege – Werk und Wirkung, Mentis Verlag, Paderborn, 200, 151–169.

[22] D. Gronau, *A remark on Sincov's functional equation*, Notices South African Math. Soc. Esaim: Proceeding and Surveys, **31** (2000), 1–8.

[23] D. Gronau, *Translation equation and Sincov's equation – A historical remark*, Esaim: Proceeding and Surveys **46** (2014), 43–46.

[24] J. Mala and Á. Száz, Modifications of relators, *Acta Math. Hungar.*, **77** (1997), 69–81.

[25] Z. Moszner, Solution générale de l'équation $F(x,y)F(y,z) = F(x,z)$ pour $x \leq y \leq z$, *C. R. Acad. Sci. Paris*, **261** (1965), 28.

[26] Z. Moszner, L'équation de translation et l1équation de Sincov généralisée, *Rocznik Nauk.-Dydakt. Prace Mat.*, **16** (1999), 53–71.

[27] Z. Moszner, On the stability of functional equations, *Aequationes Math.*, **77** (2009), 33–88.

[28] K. Nikodem, Additive selections of additive set-valued functions, *Zb. Rad. Prirod.-Mat. Fak.*, **18** (1988), 143–148.

[29] K. Nikodem and D. Popa, On single-valuedness of set-valued maps satisfying linear inclusions, *Banach J. Math. Anal.*, **3** (2009), 44–51.

[30] G. Pataki and Á. Száz, A unified treatment of well-chainedness and connectedness properties, *Acta Math. Acad. Paedagog. Nyházi.* (N.S.), **19** (2003), 101–165.

[31] J. Pepis, Sur une famille d'ensembles plans et les solutions de l'équation fonctionnelle $F(x,z) = F(x,y) \cdot F(y,z)$ pour $0 \leq x \leq y \leq z$. Application á la théorie générale des intéréts, *Ann. Soc. Polon. Math.*, **17** (1937), 113.

[32] W. J. Pervin, Quasi-uniformization of topological spaces, *Math. Ann.*, **147** (1962), 316–317.

[33] B. Piatek, On the Sincov functional equation, *Demonstratio Math.*, **38** (2005), 875–881.

[34] P. K. Sahoo, Stability of a Sincov type functional equation, *J. Inf. Math. Sci.*, **1** (2009), 81–90.

[35] P. K. Sahoo, *On a Sincov type functional equation*, In: Th.M. Rassias and J. Brzdek (Eds.), Functional Equations in Mathematical Analysis, Springer Optimizations and Its Applications, **52**, Chapter 43, Springer, New York, 2012, 697–7008.

[36] D. M. Sincov, Über eine Funktionalgleichung, *Arch. Math. Phys.*, **6** (1904), 216–217.

[37] A. Smajdor, *Iterations of multi-valued functions*, Prace Naukowe Universytetu Ślaskiego w Katowicach 759, Universitet Ślaski, Katowice 1985.

[38] W. Smajdor, Set-valued version of Sincov's functional equation, *Demonstration Math.*, **39** (2006), 101–105.

[39] Á. Száz, Galois type connections and closure operations on preordered sets, *Acta Math. Univ.Comen.*, **78** (2009), 1–21.

[40] Á. Száz, *Corelations are more powerful tools than relations*, In: Th. M. Rassias (Ed.), Applications of Nonlinear Analysis, Springer Optimization and Its Applications, **134** (2018), 711–779.

[41] A. Weil, *Sur les espaces á structure uniforme at sur la topologie générale*, Actual. Sci. Ind., **551** Herman and Cie, Paris, 1937.

Received: January 17, 2019