



Common fixed point theorems for single and set-valued maps in non-Archimedean fuzzy metric spaces

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Abstract. The intent of this paper is to establish a common fixed point theorem for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a non-Archimedean fuzzy metric space.

1 Introduction

The concept of fuzzy sets was first coined by Zadeh [9] in 1965 to describe the situation in which data are imprecise or vague or uncertain. Consequently, the last three decades remained productive for various authors [1, 11, 13] etc. and they have extensively developed the theory of fuzzy sets due to a wide range of application in the field of population dynamics, chaos control, computer programming, medicine, etc. Kramosil and Michalek [10] introduced the concept of fuzzy metric spaces (briefly, FM-spaces) in 1975, which opened a new avenue for further development of analysis in such spaces. Later on the concept of FM-space is modified and a few concepts of mathematical analysis have been developed in fuzzy metric space by George and Veeramani [1, 2]. In fact, the concept of fixed point theorem have been developed in fuzzy metric space in the paper [12].

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In recent years several fixed point theorems for single and set valued maps were proved and have numerous applications and by now there exists a considerable rich literature in this domain [4, 7].

Various authors [3, 7, 8] have discussed and studied extensively various results on coincidence, existence and uniqueness of fixed and common fixed points by using the concept of weak commutativity, compatibility, non-compatibility and weak compatibility for single and set valued maps satisfying certain contractive conditions in different spaces and they have been applied to diverse problems.

The intent of this paper is to establish a common fixed point theorem for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a non-Archimedean fuzzy metric space.

2 Preliminaries

We quote some definitions and a few theorems which will be needed in the sequel.

Definition 1 [5] *A binary operation $*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is continuous t-norm if it satisfies the following conditions:*

- (i) $*$ is commutative and associative,
- (ii) $*$ is continuous,
- (iii) $a * 1 = a \quad \forall a \in [0, 1]$,
- (iv) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Result 1 [6] (a) *For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$, there exist $r_3 \in (0, 1)$ such that $r_1 * r_3 > r_2$,*

(b) *For any $r_5 \in (0, 1)$, there exist $r_6 \in (0, 1)$ such that $r_6 * r_6 \geq r_5$.*

Definition 2 [1] *The 3-tuple $(X, \mu, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and μ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:*

- (i) $\mu(x, y, t) > 0$;
- (ii) $\mu(x, y, t) = 1$ if and only if $x = y$
- (iii) $\mu(x, y, t) = \mu(y, x, t)$;

- (iv) $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, s + t)$;
 (v) $\mu(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous;
 for all $x, y, z \in X$ and $t, s > 0$.

Note that $\mu(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t .

Example 1 Let $X = [0, \infty)$, $a * b = ab$ for every $a, b \in [0, 1]$ and d be the usual metric defined on X . Define $\mu(x, y, t) = e^{-\frac{d(x, y)}{t}}$ for all $x, y, t \in X$. Then clearly $(X, \mu, *)$ is a fuzzy metric space.

Example 2 Let (X, d) be a metric space, and let $a * b = ab$ or $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $\mu(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, \mu, *)$ is a fuzzy metric space and this fuzzy metric μ induced by d is called the standard fuzzy metric [1].

Note 1 George and Veeramani [1] proved that every fuzzy metric space is a metrizable topological space. In this paper, they also have proved, if (X, d) is a metric space, then the topology generated by d coincides with the topology generated by the fuzzy metric μ of Example 2. As a result, we can say that an ordinary metric space is a special case of a fuzzy metric space.

Note 2 Consider the following condition:

$$(iv') \quad \mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, \max\{s, t\}).$$

If the condition (iv) in Definition 2 is replaced by the condition (iv'), the fuzzy metric space $(X, \mu, *)$ is said to be a non-Archimedean fuzzy metric space.

Remark 1 In fuzzy metric space X , for all $x, y \in X$, $\mu(x, y, \cdot)$ is non-decreasing with respect to the variable t . In fact, in a non-Archimedean fuzzy metric space, $\mu(x, y, t) \geq \mu(x, z, t) * \mu(z, y, t)$ for $x, y, z \in X$, $t > 0$.

Every non-Archimedean fuzzy metric space is also a fuzzy metric space.

Throughout the paper X will represent the non-Archimedean fuzzy metric space $(X, \mu, *)$ and $CB(X)$, the set of all non-empty closed and bounded subset of X . We recall these usual notations: for $x \in X$, $A \subseteq X$ and for every $t > 0$,

$$\mu(x, A, t) = \max\{\mu(x, y, t) : y \in A\}$$

and let H be the associated Hausdorff fuzzy metric on $CB(X)$: for every A, B in $CB(X)$,

$$H(A, B, t) = \min \left\{ \min_{x \in A} \mu(x, B, t), \min_{y \in B} \mu(A, y, t) \right\}.$$

Definition 3 [4] A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

- (i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, \dots$, and $\{a_n\}$ converges to a .
- (ii) given $\epsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ .

Definition 4 A point $x \in X$ is called a coincidence point (resp. fixed point) of $A : X \rightarrow X$, $B : X \rightarrow CB(X)$ if $Ax \in Bx$ (resp. $x = Ax \in Bx$).

Definition 5 Maps $A : X \rightarrow X$ and $B : X \rightarrow CB(X)$ are said to be compatible if $ABx \in CB(X)$ for all $x \in X$ and

$$\lim_{n \rightarrow \infty} H(ABx_n, BAx_n, t) = 1$$

whenever $\{x_n\}$ is a sequence in X such that $Bx_n \rightarrow M \in CB(X)$ and $Ax_n \rightarrow x \in M$.

Definition 6 Maps $A : X \rightarrow X$ and $B : X \rightarrow CB(X)$ are said to be weakly compatible if they commute at coincidence points ie., if $ABx = BAx$ whenever $Ax \in Bx$.

Definition 7 Maps $A : X \rightarrow X$ and $B : X \rightarrow CB(X)$ are said to be occasionally weakly compatible (**owc**) if there exists some point $x \in X$ such that $Ax \in Bx$ and $ABx \subseteq BAx$.

Example 3 Let $X = [1, \infty[$ with the usual metric. Define $f : X \rightarrow X$ and $F : X \rightarrow CB(X)$, for all $x \in X$ by

$$fx = x + 1, Fx = [1, x + 1],$$

$$fx = x + 1 \in Fx \text{ and } fFx = [2, x + 2] \subset Ffx = [1, x + 2].$$

Hence, f and F are occasionally weakly compatible but not weakly compatible.

Definition 8 Let $F : X \rightarrow 2^X$ be a set-valued map on X . $x \in X$ is a fixed point of F if $x \in Fx$ and is a strict fixed point of F if $Fx = \{x\}$.

Property 1 Let A and $B \in CB(X)$, then for any $a \in A$, we have

$$\mu(a, B, t) \geq qH(A, B, t).$$

Proof. Obvious. □

3 A strict fixed point theorem

Theorem 1 *Let $f, g : X \longrightarrow X$ be mappings and $F, G : X \longrightarrow CB(X)$ be set-valued mappings such that the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**. Let $\varphi : \mathbb{R}^6 \longrightarrow \mathbb{R}$ be a real valued map satisfying the following conditions*

$$\begin{aligned} (\varphi_1) : & \varphi \text{ is increasing in variables } t_2, t_5 \text{ and } t_6; \\ (\varphi_2) : & \varphi(u(t), u(t), 1, 1, u(t), u(t)) > 1 \quad \forall u(t) \in [0, 1]. \end{aligned}$$

If for all x and $y \in X$ for which

$$\begin{aligned} (\star) \quad & \varphi(H(Fx, Gy, t), \mu(fx, gy, t), \mu(fx, Fx, t), \mu(gy, Gy, t), \\ & \mu(fx, Gy, t), \mu(gy, Fx, t)) < 1 \end{aligned}$$

then f, g, F and G have a unique fixed point which is a strict fixed point for F and G .

Proof. (i) We begin to show the existence of a common fixed point. Since the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**, there exist u, v in X such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$ and $gGv \subseteq Ggv$. Also, using the triangle inequality and Property 1, we obtain

$$\mu(fu, gv, t) \geq H(Fu, Gv, t) \quad (1)$$

and

$$\mu(f^2u, gv, t) \geq H(Ffu, Gv, t). \quad (2)$$

First we show that $gv = fu$. The condition (\star) implies that

$$\begin{aligned} & \varphi(H(Fu, Gv, t), \mu(fu, gv, t), \mu(fu, Fu, t), \mu(gv, Gv, t), \\ & \mu(fu, Gv, t), \mu(gv, Fu, t)) < 1 \\ \implies & \varphi(H(Fu, Gv, t), \mu(fu, gv, t), 1, 1, \mu(fu, Gv, t), \mu(gv, Fu, t)) < 1. \end{aligned}$$

By (φ_1) we have

$$\varphi(H(Fu, Gv, t), H(Fu, Gv, t), 1, 1, H(Fu, Gv, t), H(Fu, Gv, t)) < 1$$

which from (φ_2) gives $H(Fu, Gv, t) = 1$.

So $Fu = Gv$ and by (1), $fu = gv$. Again by (2), we have

$$\mu(f^2u, fu, t) \geq H(Ffu, Gv, t).$$

Next, we claim that $f^2u = fu$. The condition (\star) implies that

$$\begin{aligned} & \varphi(H(Ffu, Gv, t), \mu(f^2u, gv, t), \mu(f^2u, Ffu, t), \mu(gv, Gv, t), \\ & \qquad \qquad \qquad \mu(f^2u, Gv, t), \mu(gv, Ffu, t)) < 1 \\ \implies & \varphi(H(Ffu, Gv, t), \mu(f^2u, fu, t), 1, 1, \mu(f^2u, Gv, t), \mu(fu, Ffu, t)) < 1. \end{aligned}$$

By (φ_1) we have

$$\implies \varphi(H(Ffu, Gv, t), H(Ffu, Gv, t), 1, 1, H(Ffu, Gv, t), H(Ffu, Gv, t)) < 1$$

which, from (φ_2) , gives $H(Ffu, Gv, t) = 1$.

By (2), we obtain $f^2u = fu$. Since (f, F) and (g, G) have the same role, we have $gv = g^2v$. Therefore,

$$ffu = fu = gv = ggv = gfu$$

and

$$fu = f^2u \in fFu \subset Ffu$$

So $fu \in Ffu$ and $fu = gfu \in Gfu$. Then fu is common fixed point of f, g, F and G .

(ii) Now, we show uniqueness of the common fixed point.

Put $fu = w$ and let w' be another common fixed point of the four maps, then we have

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw', t) \quad (3)$$

by (\star) , we get

$$\begin{aligned} & \varphi(H(Fw, Gw', t), \mu(fw, gw', t), \mu(fw, Fw, t), \mu(gw', Gw', t), \\ & \qquad \qquad \qquad \mu(fw, Gw', t), \mu(gw', Fw, t)) < 1 \\ \implies & \varphi(H(Fw, Gw', t), \mu(fw, gw', t), 1, 1, \mu(fw, Gw', t), \\ & \qquad \qquad \qquad \mu(gw', Fw, t)) < 1 \end{aligned}$$

By (φ_1) we get

$$\varphi(H(Fw, Gw', t), H(Fw, Gw', t), 1, 1, H(Fw, Gw', t), H(Fw, Gw', t)) < 1$$

So, by (φ_2) , $H(Fw, Gw', t) = 1$ and from (3), we have

$$\mu(fw, gw', t) = \mu(w, w', t) = 1 \implies w = w'.$$

(iii) Let $w \in Ffu$. Using the triangle inequality and Property (1), we have

$$\mu(fu, w, t) \geq \mu(fu, Ffu, t) * H(Ffu, Gv, t) * \mu(w, Gv, t).$$

Since $fu \in Ffu$ and $H(Ffu, Gv, t) = 1$,

$$\mu(w, fu, t) \geq \mu(w, Gv, t) \geq H(Ffu, Gv, t) = 1.$$

So $w = fu$ and $Ffu = \{fu\} = \{gv\} = Ggv$.

This completes the proof. \square

4 A Gregus type fixed point theorem

Theorem 2 Let $f, g : X \longrightarrow X$ be mappings and $F, G : X \longrightarrow CB(X)$ be set-valued mappings such that the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing map such that, for every $0 \leq l < 1$, $\psi(l) > l$ and satisfies the following condition:

$$(\star) \quad H^p(Fx, Gy, t) \geq \psi \left[\alpha \mu^p(fx, gy, t) + (1 - \alpha) \mu^{\frac{p}{2}}(gy, Fx, t) \mu^{\frac{p}{2}}(fx, Gy, t) \right]$$

for all x and $y \in X$, where $0 < \alpha \leq 1$ and $p \geq 1$.

Then f, g, F and G have a unique fixed point which is a strict fixed point for F and G .

Proof. Since $\{f, F\}$ and $\{g, G\}$ are **owc**, as in proof of Theorem 1, there exist $u, v \in X$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$ and (1), (2) holds.

(i) As in proof of Theorem 1, we begin to show the existence of a common fixed point. We have,

$$H^p(Fu, Gv, t) \geq \psi \left[\alpha \mu^p(fu, gv, t) + (1 - \alpha) \mu^{\frac{p}{2}}(gv, Fu, t) \mu^{\frac{p}{2}}(fu, Gv, t) \right]$$

and by (1) and Property 1,

$$\begin{aligned} H^p(Fu, Gv, t) &\geq \psi [\alpha H^p(Fu, Gv, t) + (1 - \alpha) H^p(Gv, Fu, t)] \\ &= \psi (H^p(Fu, Gv, t)) \end{aligned}$$

So, if $0 \leq H(Fu, Gv, t) < 1$, $\psi(l) > l$ for $0 \leq l < 1$, we obtain

$$H^p(Fu, Gv, t) \geq \psi[H^p(Fu, Gv, t)] > H^p(Fu, Gv, t)$$

which is a contradiction, thus we have $H(Fu, Gv, t) = 1$ and hence $fu = gv$. Again, if $0 \leq H(Ffu, Gv, t) < 1$ then by (2) and (\star) , we have

$$\begin{aligned} H^p(Ffu, Gv, t) &\geq \psi[a\mu^p(f^2u, gv, t) + (1-a)\mu^{\frac{p}{2}}(gv, Ffu, t)\mu^{\frac{p}{2}}(f^2u, Gv, t)] \\ &\geq \psi[aH^p(Ffu, Gv, t) + (1-a)H^p(Ffu, Gv, t)] \\ &= \psi(H^p(Ffu, Gv, t)) \end{aligned}$$

If $0 \leq H(Ffu, Gv, t) < 1$, we obtain

$$H^p(Ffu, Gv, t) \geq \psi[H^p(Ffu, Gv, t)] > H^p(Ffu, Gv, t)$$

which is a contradiction, thus we have $H(Ffu, Gv, t) = 1$,

$$\implies Ffu = Gv \implies f^2u = fu$$

Similarly, we can prove that $g^2v = gv$.

Let $fu = w$ then $fw = w = gw$, $w \in Fw$ and $w \in Gw$, this completes the proof of the existence.

(ii) For the uniqueness, let w' be a second common fixed point of f, g, F and G . Then

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw', t)$$

and by assumption (\star) , we obtain

$$\begin{aligned} H^p(Fw, Gw', t) &\geq \psi \left[a\mu^p(fw, gw', t) + (1-a)\mu^{\frac{p}{2}}(fw, Gw', t)\mu^{\frac{p}{2}}(gw', Fw, t) \right] \\ &\geq \psi(H^p(Fw, Gw', t)) > H^p(Fw, Gw', t) \text{ if } 0 \\ &\leq H(Fw, Gw', t) < 1 \end{aligned}$$

which is a contradiction. So, $Fw = Gw'$. Since w and w' are common fixed point of f, g, F and G , we have

$$\mu(fw, gw', t) \geq \mu(fw, Fw, t) * H(Fw, Gw', t) * \mu(gw', Gw', t) \geq H(Fw, Gw', t)$$

So, $w = fw = gw' = w'$ and there exists a unique common fixed point of f, g, F , and G .

(iii) The proof that the fixed point of F and G is a strict fixed point is identical of that of theorem (1). \square

Theorem 3 Let $f, g : X \longrightarrow X$ and $F, G : X \longrightarrow CB(X)$ be single and set-valued maps respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are *owc* and satisfy inequality

$$(\star) \quad H^p(Fx, Gy, t) \geq \alpha(\mu(fx, gy, t))[\min\{\mu(fx, gy, t)\mu^{p-1}(fx, Fx, t), \\ \mu(fx, gy, t)\mu^{p-1}(gy, Gy, t), \mu(fx, Fx, t)\mu^{p-1}(gy, Gy, t), \mu^{p-1}(fx, Gy, t)\mu(gy, Fx, t)\}]$$

for all $x, y \in X$, where $p \geq 2$ and $\alpha : [0, 1] \longrightarrow [0, \infty)$ is decreasing and satisfies the condition

$$\alpha(t) > 1 \quad \forall 0 \leq t < 1 \quad \text{and} \quad \alpha(t) = 1 \quad \text{if } f \cdot t = 1$$

Then f, g, F and G have a unique fixed point which is a strict fixed point for F and G .

Proof. Since the pairs $\{f, F\}$ and $\{g, G\}$ are *owc*, then there exist two elements u and v in X such that $fu \in Fu$, $fFu \subseteq Ffu$ and $gv \in Gv$, $gGv \subseteq Ggv$.

First we prove that $fu = gv$. By property (1) and the triangle inequality we have $\mu(fu, gv, t) \geq H(Fu, Gv, t)$, $\mu(fu, Gv, t) \geq H(Fu, Gv, t)$ and $\mu(Fu, gv, t) \geq H(Fu, Gv, t)$.

Suppose that $H(Fu, Gv, t) < 1$. Then by inequality (\star) we get

$$\begin{aligned} (\star) \quad H^p(Fu, Gv, t) &\geq \alpha(\mu(fu, gv, t))[\min\{\mu(fu, gv, t)\mu^{p-1}(fu, Fu, t), \\ &\quad \mu(fu, gv, t)\mu^{p-1}(gv, Gv, t), \mu(fu, Fu, t)\mu^{p-1}(gv, Gv, t), \\ &\quad \mu^{p-1}(fu, Gv, t)\mu(gv, Fu, t)\}] \\ &= \alpha(\mu(fu, gv, t))[\min\{\mu(fu, gv, t), \mu(fu, gv, t), 1, \\ &\quad \mu^{p-1}(fu, Gv, t)\mu(gv, Fu, t)\}] \\ &\geq \alpha(H(Fu, Gv, t))[\min\{H(Fu, Gv, t), 1, H^p(Fu, Gv, t)\}] \\ &> H^p(Fu, Gv, t) \end{aligned}$$

which is a contradiction. Hence $H(Fu, Gv, t) = 1$ which implies that $fu = gv$. Again by property (1) and the triangle inequality we have

$$\mu(f^2u, fu, t) = \mu(f^2u, gv, t) \geq H(Ffu, Gv, t)$$

We prove that $f^2u = fu$. Suppose $H(Ffu, Gv, t) < 1$ and by (\star) , property (1) we obtain

$$\begin{aligned}
 H^p(Ffu, Gv, t) &\geq qa(\mu(f^2u, gv, t))[\min\{\mu(f^2u, gv, t)\mu^{p-1}(f^2u, Ffu, t), \\
 &\quad \mu(f^2u, gv, t)\mu^{p-1}(gv, Gv, t), \mu(f^2u, Ffu, t)\mu^{p-1}(gv, Gv, t), \\
 &\quad \mu^{p-1}(f^2u, Gv, t)\mu(gv, Ffu, t)\}] \\
 &= a(\mu(f^2u, gv, t))[\min\{\mu(f^2u, gv, t), \mu(f^2u, gv, t), 1, \\
 &\quad \mu^{p-1}(f^2u, Gv, t)\mu(gv, Ffu, t)\}] \\
 &\geq qa(H(Ffu, Gv, t))[\min\{H(Ffu, Gv, t), H^p(Ffu, Gv, t)\}] \\
 &> H^p(Ffu, Gv, t)
 \end{aligned}$$

which is a contradiction. Hence $H(Ffu, Gv, t) = 1$ which implies that $f^2u = gv = fu$.

Similarly, we can prove that $g^2v = gv$. Putting $fu = gv = z$, then $fz = gz = z$, $z \in Fz$ and $z \in Gz$. Therefore z is a common fixed point of maps f, g, F and G . Now, suppose that f, g, F and G have another common fixed point $z' \neq qz$. Then, by property (1) and the triangle inequality we have

$$\mu(z, z', t) = \mu(fz, gz', t) \geq H(Fz, Gz', t)$$

Assume that $H(Fz, Gz', t) < 1$. Then the use of inequality (\star) gives

$$\begin{aligned}
 H^p(Fz, Gz', t) &\geq qa(\mu(fz, gz', t))[\min\{\mu(fz, gz', t)\mu^{p-1}(fz, Fz, t), \mu(fz, gz', t) \\
 &\quad \mu^{p-1}(gz', Gz', t), \mu(fz, Fz, t)\mu^{p-1}(gz', Gz', t), \\
 &\quad \mu^{p-1}(fz, Gz', t)\mu(gz', Fz, t)\}] \\
 &= a(\mu(fz, gz', t))[\min\{\mu(fz, gz', t), \mu(f^2z, gz', t), 1, \\
 &\quad \mu^{p-1}(f^2z, Gz', t)\mu(gz', Ffz, t)\}] \\
 &\geq qa(H(Fz, Gz', t))[\min\{H(Fz, Gz', t), H^p(Fz, Gz', t)\}] \\
 &> H^p(Fz, Gz', t)
 \end{aligned}$$

which is a contradiction. Hence $H(Fz, Gz', t) = 1$ which implies that $z' = z$.

(iii) The proof that the fixed point of F and G is a strict fixed point is identical of that of theorem (1) \square

5 Another type fixed point theorem

Theorem 4 Let $f, g : X \longrightarrow X$ be mappings and $F, G : X \longrightarrow CB(X)$ be set-valued maps and ϕ be non-decreasing function of $[0, 1]$ into itself such that

$\phi(t) = 1$ iff $t = 1$ and for all $t \in [0, 1]$, ϕ satisfies the following inequality

$$(\star) \quad \begin{aligned} \phi(H(Fx, Gy, t)) &\geq a(\mu(fx, gy, t))\phi(\mu(fx, gy, t)) \\ &\quad + b(\mu(fx, gy, t)) \min\{\phi(\mu(fx, Gy, t)), \phi(\mu(gy, Fx, t))\} \end{aligned}$$

for all x and y in X , where $a, b : [0, 1] \longrightarrow [0, 1]$ are satisfying the conditions

$$a(t) + b(t) > 1 \quad \forall t > 0$$

and

$$a(t) + b(t) = 1 \quad \text{iff } t = 1$$

If the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**, then f, g, F and G have a unique common fixed point in X which is a strict fixed point for F and G .

Proof. Since $\{f, F\}$ and $\{g, G\}$ are **owc**, as in proof of theorem(1), there exist u, v in X such that $fu \in Fu, gv \in Gv, fFu \subseteq Ffu, gGv \subseteq Ggv$,

$$\mu(fu, gv, t) \geq H(Fu, Gv, t) \quad (1)$$

and

$$\mu(f^2u, gv, t) \geq H(Ffu, Gv, t) \quad (2)$$

(i) First we prove that $fu = gv$. Suppose $H(Fu, Gv, t) < 1$. By (\star) , Property (1), we have

$$\begin{aligned} \phi(H(Fu, Gv, t)) &\geq a(\mu(fu, gv, t))\phi(\mu(fu, gv, t)) \\ &\quad + b(\mu(fu, gv, t)) \min\{\phi(\mu(fu, Gv, t)), \phi(\mu(gv, Fu, t))\} \\ &\geq [a(\mu(fu, gv, t)) + b(\mu(fu, gv, t))]\phi(H(Fu, Gv, t)) \\ &> \phi(H(Fu, Gv, t)) \end{aligned}$$

which is a contradiction. Hence $H(Fu, Gv, t) = 1$ and thus $fu = gv$. Now we prove that $f^2u = fu$. Suppose $H(Ffu, Gv, t) < 1$. By (\star) and Property 1, we have

$$\begin{aligned} \phi(H(Ffu, Gv, t)) &\geq qa(\mu(f^2u, gv, t))\phi(\mu(f^2u, gv, t)) \\ &\quad + b(\mu(f^2u, gv, t)) \min\{\phi(\mu(f^2u, Gv, t)), \phi(\mu(gv, Ffu, t))\} \\ &\geq [a(\mu(f^2u, fu, t)) + b(\mu(f^2u, fu, t))]\phi(H(Ffu, Gv, t)) \\ &> \phi(H(Ffu, Gv, t)) \end{aligned}$$

which is a contradiction. Hence $H(Fu, Gv, t) = 1$ and this implies that $f^2u = fu$. Similarly, we can prove that $g^2v = gv$. So, if $w = fu = gv$ then $fw = w = gw$, $w \in Fw$ and $w \in Gw$. Existence of a common fixed point is proved.

(ii) Assume that there exists a second common fixed point w' of f, g, F and G . We see that

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw', t)$$

If $H(Fw, Gw', t) < 1$, by inequality (\star) we obtain

$$\begin{aligned} \phi(H(Fw, Gw', t)) &\geq a(\mu(fw, gw', t))\phi(\mu(fw, gw', t)) \\ &\quad + b(\mu(fw, gw', t)) \min\{\phi(\mu(fw, Gw', t)), \phi(\mu(gw', Fw, t))\} \\ &\geq [a(\mu(w, w', t)) + b(\mu(w, w', t))]\phi(H(Fw, Gw', t)) \\ &> \phi(H(Fw, Gw', t)) \end{aligned}$$

this contradiction implies that $H(Fw, Gw', t) = 1$, hence $w' = w$

(iii) This part of the proof is analogous of that of Theorem 1. □

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