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Closedness of the solution mapping to parametric vector equilibrium problems

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Abstract. The goal of this paper is to study the parametric vector equilibrium problems governed by vector topologically pseudomonotone maps. The main result gives sufficient conditions for closedness of the solution map defined on the set of parameters.

1 Introduction

M. Bogdan and J. Kolumbán [3] gave sufficient conditions for closedness of the solution map defined on the set of parameters. They considered the parametric equilibrium problems governed by topological pseudomonotone maps depending on a parameter. In this paper we extend this result for parametric vector equilibrium problems.

Let X be a Hausdorff topological space and let P (the set of parameters) be another Hausdorff topological space. Let $\mathcal{Z}$ be a real topological vector space with an ordering cone C, where C is a closed convex cone in $\mathcal{Z}$ with $\operatorname{Int} C \neq \emptyset$ and $C \neq \mathcal{Z}$.

We consider the following parametric vector equilibrium problem, in short $(VEP)_p$:

Find $a_p \in D_p$, such that

$$f_p(a_p, b) \notin -C \setminus \{0\}, \ \forall b \in D_p,$$

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where D_p is a nonempty subset of X and $f_p : X \times X \to \mathcal{Z}$ is a given function.

It is well-known that VEP contains several problems as special cases, namely, vector optimization problem, vector saddle point problem, vector variational inequality problem, vector complementarity problem, etc.

Denote by $S(p)$ the set of the solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P$. For sufficient conditions for the existence of solutions see [8], [13].

The paper is organized as follows. In Section 2, we introduce a new notion of the vector topological pseudomonotonicity and we recall the notion of the Mosco convergence of the sets. Section 3 is devoted to the closedness of the solution map for parametric vector equilibrium problems.

2 Preliminaries

In this section, we will introduce a new definition of the vector topologically pseudomonotone bifunctions with values in $\mathcal{Z}$. First, the definition of the suprema and the infima of subsets of $\mathcal{Z}$ are given. Following [1], for a subset A of $\mathcal{Z}$ the suprema of A with respect to C is defined by:

$$\mathrm{Sup}\, A = \left\{ z \in \bar{A} : A \cap (z + \mathrm{Int}\, C) = \emptyset \right\},$$

and the infima of A with respect to C is defined by:

$$\mathrm{Inf}\, A = \left\{ z \in \bar{A} : A \cap (z - \mathrm{Int}\, C) = \emptyset \right\}.$$

Let $(z_i)_{i \in I}$ be a net in $\mathcal{Z}$. Let $A_i = \{z_j : j \geq i\}$ for every i in the index set I. The limit inferior of (z_i) is given by:

$$\mathrm{Liminf}\, z_i = \mathrm{Sup} \left(\bigcup_{i \in I} \mathrm{Inf}\, A_i \right).$$

Similarly, the limit superior of (z_i) can be defined as

$$\mathrm{Limsup}\, z_i = \mathrm{Inf} \left(\bigcup_{i \in I} \mathrm{Sup}\, A_i \right).$$

Theorem 1 ([7], Theorem 2.1) *Let $(z_i)_{i \in I}$ be a net in $\mathcal{Z}$ convergent to z, and let $A_i = \{z_j : j \geq i\}$.*

i) *If there is an i_0 such that, for every $i \geq i_0$, there exists $j \geq i$ with $\mathrm{Inf}\, A_j \neq \emptyset$, then $z \in \mathrm{Liminf}\, z_i$.*

ii) *If there is an* i_0 *such that, for every* $i \geq i_0$*, there exists* $j \geq i$ *with* $\operatorname{Sup} A_j \neq \emptyset$*, then* $z \in \operatorname{Limsup} z_i$.

We introduce the definition of vector topologically pseudomonotonicity, which plays a central role in our main results.

Definition 1 *Let* (X, σ) *be a Hausdorff topological space, and let* D *be a nonempty subset of* X*. A function* $f : D \times D \to \mathcal{Z}$ *is called vector topologically pseudomonotone if for every* $b \in D$*,* $\nu \in C$ *and for each net* $(a_i)_{i \in I}$ *in* D *satisfying* $a_i \xrightarrow{\sigma} a \in D$ *and*

$$\operatorname{Liminf} f(a_i, a) \cap (-\operatorname{Int} C) = \emptyset, \tag{1}$$

then for every i *in the index set* I

$$\overline{\{f(a_j, b) : j \geq i\}} \cap [f(a, b) + \nu - C] \neq \emptyset.$$

In Definition 1, if $\mathcal{Z} = \mathbb{R}$, and if C is the set of all non-negative real numbers, then we get back the well-known topological pseudomonotonicity introduced by Brézis [4].

Let us consider σ and τ two topologies on X. Suppose that τ is stronger than σ on X.

For the parametric domains in $(VEP)_p$, we shall use a slight generalization of Mosco's convergence [14].

Definition 2 ([3], Definition 2.2.) *Let* D_p *be subsets of* X *for all* $p \in P$*. The sets* D_p *converge to* D_{p_0} *in the Mosco sense* ($D_p \xrightarrow{M} D_{p_0}$) *as* $p \to p_0$ *if:*

a) for every subnet $(a_{p_i})_{i \in I}$ *with* $a_{p_i} \in D_{p_i}$*,* $p_i \to p_0$ *and* $a_{p_i} \xrightarrow{\sigma} a$ *implies* $a \in D_{p_0}$*;*

b) for every $a \in D_{p_0}$*, there exists* $a_p \in D_p$ *such that* $a_p \xrightarrow{\tau} a$ *as* $p \to p_0$*.*

3 Closedness of the solution map

This section is devoted to prove the closedness of the solution map for parametric vector equilibrium problems.

Theorem 2 *Let* X *be a Hausdorff topological space with* σ *and* τ *two topologies, where* τ *is stronger than* σ*. Let* D_p *be nonempty sets of* X*, and let* $p_0 \in P$ *be fixed. Suppose that* $S(p) \neq \emptyset$ *for each* $p \in P$ *and the following conditions hold:*

i) $D_p \overset{M}{\to} D_{p_0}$;

ii) For each net of elements $(p_i, a_{p_i}) \in \mathrm{Graph}S$*, if* $p_i \to p_0$*,* $a_{p_i} \overset{\sigma}{\to} a$*,* $b_{p_i} \in D_{p_i}$*,* $b \in D_{p_0}$*, and* $b_{p_i} \overset{\tau}{\to} b$*, then*

$$\mathrm{Liminf}\,(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)) \cap (-\mathrm{Int}\, C) \neq \emptyset.$$

iii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0*, i.e. for each net of elements* $(p_i, a_{p_i}) \in \mathrm{Graph}S$*,* $p_i \to p_0$ *and* $a_{p_i} \overset{\sigma}{\to} a$ *imply* $(p_0, a) \in \mathrm{Graph}S$*.*

Proof. Let $(p_i, a_{p_i})_{i\in I}$ be a net of elements $(p_i, a_{p_i}) \in \mathrm{Graph}S$, i.e.

$$f_{p_i}(a_{p_i}, b) \notin -C\backslash\{0\},\ \forall b \in D_{p_i}, \tag{2}$$

with $p_i \to p_0$ and $a_{p_i} \overset{\sigma}{\to} a$. By the Mosco convergence of the sets D_p, we get $a \in D_{p_0}$. Moreover, there exists a net $(b_{p_i})_{i\in I}$, $b_{p_i} \in D_{p_i}$ such that $b_{p_i} \overset{\tau}{\to} a$. From the assumption ii) we obtain that

$$\mathrm{Liminf}\,(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a)) \cap (-\mathrm{Int}\, C) \neq \emptyset. \tag{3}$$

Since $-\mathrm{Int}\, C$ is an open cone, it follows that there exists a subnet (a_{p_i}) denoted by the same indexes such that

$$f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a) \in -\mathrm{Int}\, C \text{ for all } i \in I. \tag{4}$$

By replacing b with b_{p_i} in (2), we get

$$f_{p_i}(a_{p_i}, b_{p_i}) \notin -C\backslash\{0\}. \tag{5}$$

From (5) and (4) we obtain that

$$f_{p_0}(a_{p_i}, a) \in (-C)^c \subset (-\mathrm{Int}\, C)^c,\ \text{for all } i \in I,$$

since $(-\mathrm{Int}\, C)^c$ is closed, it follows

$$\mathrm{Liminf}\, f_{p_0}(a_{p_i}, a) \cap (-\mathrm{Int}\, C) = \emptyset.$$

Now, we can apply iii) and we obtain that for every $b \in D_{p_0}$, $\nu \in C$, and for every $i \in I$ we have

$$\overline{\left\{f_{p_0}\left(a_{p_j}, b\right) : j \geq i\right\}} \cap \left[f_{p_0}(a, b) + \nu - C\right] \neq \emptyset. \tag{6}$$

We have to prove that

$$f_{p_0}(a,b) \notin -C\backslash\{0\},\ \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\overline{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a,\overline{b}\right) \in -C\backslash\{0\}.$$

Let be $f_{p_0}\left(a,\overline{b}\right) = -\nu$, where $\nu \in C\backslash\{0\}$. From (6) we obtain that for every $i \in I$ we have

$$\overline{\left\{f_{p_0}\left(a_{p_j},\overline{b}\right) : j \geq i\right\}} \cap (-C) \neq \emptyset, \tag{7}$$

i.e. there exists a subnet (a_{p_i}) denoted by the same indexes such that

$$f_{p_0}\left(a_{p_i},\overline{b}\right) \in -C \text{ for all } i \in I, \tag{8}$$

or

$$f_{p_0}\left(a_{p_i},\overline{b}\right) \text{ converges to a point in } -\partial C. \tag{9}$$

Since $\overline{b} \in D_{p_0}$ from the Mosco convergence of the sets D_p, we have that there exists $\left(\overline{b}_{p_i}\right)_{i\in I} \subset D_{p_i}$ such that $\overline{b}_{p_i} \xrightarrow{\tau} \overline{b}$. By using again the assumption ii), it follows that there exists a subnet (a_{p_i}) denoted by the same indexes, for which

$$f_{p_i}\left(a_{p_i},\overline{b}_{p_i}\right) - f_{p_0}\left(a_{p_i},\overline{b}\right) \in -\operatorname{Int} C,\ \text{for all } i \in I. \tag{10}$$

From (8), (9) and (10) it follows that there exists an index $i_0 \in I$ such that

$$f_{p_i}\left(a_{p_i},\overline{b}_{p_i}\right) \in -\operatorname{Int} C,\ i \geq i_0, \tag{11}$$

but on the other side $(p_i, a_{p_i}) \in \mathsf{GraphS}$, and

$$f_{p_i}\left(a_{p_i},\overline{b}_{p_i}\right) \notin -C\backslash\{0\},$$

which is a contradiction. Hence $(p_0, a) \in \mathsf{GraphS}$. ■

M. Bogdan and J. Kolumbán [3] showed that the topological pseudomonotonicity and the assumption ii) are essential in scalar case.

Remark 1 *The assigment* ii) *can not be replaced by*

ii') *For each net of elements* $(p_i, a_{p_i}) \in \mathsf{GraphS}$, *if* $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, *and* $b_{p_i} \xrightarrow{\tau} b$, *then*

$$\operatorname{Liminf}\left(f_{p_i}\left(a_{p_i},b_{p_i}\right) - f_{p_0}\left(a_{p_i},b\right)\right) \cap \left(-\operatorname{Int} C \cup \{0\}\right) \neq \emptyset.$$

Therefore Theorem 2 does not imply Theorem 1 in [3].

The following example confirms this statement.

Example 1 *Let* $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$ *(*∞ *means* $+\infty$ *from real analysis), where we consider the topology induced by the metric given by* $d(m,n) = |1/m - 1/n|$, $d(n,\infty) = d(\infty,n) = 1/n$, *for* $m,n \in \mathbb{N}$, *and* $d(\infty,\infty) = 0$. *Let* $X = [0,1]$ *where* σ, τ *are natural topologies,* $\mathcal{Z} = \mathbb{R}^2$, $D_p = [0,1]$, $p \in P$, *the real vector functions* $f_n : [0,1] \times [0,1] \to \mathbb{R}^2$. *The ordering cone* C *is the third quadrant, i.e.* $C = \left\{(a,b) \in \mathbb{R}^2 : a \leq 0, b \leq 0\right\}$.

Let $f_n(a,b) = (a - b - 2/n, 1 - 2a)$, $n \in \mathbb{N}$ *and the function* f_∞ *be defined by*

$$f_\infty(a,b) = \begin{cases} (a-b, 1-a) & \textit{if} \quad a > 0 \\ (b,1) & \textit{if} \quad a = 0 \end{cases}.$$

The f_∞ *is vector topologically pseudomonotone. Indeed, for* $a > 0$, f_∞ *is continuous, therefore it is vector topologically pseudomonotone. Let us study the case when* $a = 0$.

We have to prove that for every $b \in [0,1]$, $v \in C$ *for each* $(a_n)_n$, $a_n \in [0,1]$ *with* $a_n \to 0$ *satisfying*

$$\operatorname{Liminf} f_\infty(a_n, 0) \cap (-\operatorname{Int} C) = \emptyset,$$

then for every $m \in \mathbb{N}$ *we have*

$$\overline{\{f_\infty(a_n,b) : n \geq m\}} \cap [f_\infty(a,b) + v - C] \neq \emptyset.$$

If $a_n = 0$, *for all* $n \in \mathbb{N}$, *one has the obvious relation for every* $b \in [0,1]$, $v \in C$

$$\overline{\{f_\infty(0,b) : n \geq m\}} \cap [f_\infty(0,b) + v - C] \neq \emptyset, \ \forall m \in \mathbb{N}.$$

If there exists a $k \in \mathbb{N}$ *such that* $a_k \neq 0$, *then one has that*

$$f_\infty(a_k, 0) \in \operatorname{Liminf} f_\infty(a_n, 0). \tag{12}$$

Indeed, $f_\infty(a_k,0)$ *is an inferior point, because otherwise it has to exist an* $j > k$ *such that*

$$(a_j, 1 - a_j) \in (a_k, 1 - a_k) - \operatorname{Int} C.$$

This implies that

$$\begin{cases} a_j > a_k \\ 1 - a_j > 1 - a_k, \end{cases}$$

which is a contradiction. Similarly we can prove that $f_\infty(a_k,0)$ *is a superior point.*

Since $f_\infty(a_k, 0) \in (-\operatorname{Int} C)$, *it follows from (12), that*

$$\operatorname{Liminf} f_\infty(a_n, 0) \cap (-\operatorname{Int} C) \neq \emptyset,$$

so f_∞ *is vector topologically pseudomonotone.*

If $a_n = 1/n$ *for all* $n \in \mathbb{N}$, *the assumption* $ii')$ *holds. Indeed, from Theorem 1, it follows that*

$$(0, 0) \in \operatorname{Liminf}(f_n(a_n, b_n) - f_\infty(a_n, b)),$$

where $b_n \to b$. *We have* $(n, 1/n) \in \mathsf{GraphS}$ *for each* $n \in \mathbb{N}$, $S(\infty) = \{1\}$, *so* $0 \notin S(\infty)$. *Hence* S *is not closed at* ∞.

If the $(VEP)_p$ is defined on constant domains, $D_p = X$ for all $p \in P$, we can omit the Mosco convergence. In this case condition $ii)$ can be weakened.

Theorem 3 *Let* (X, σ) *be a Hausdorff topological space, and let* $p_0 \in P$ *be fixed. Suppose that* $S(p) \neq \emptyset$, *for each* $p \in P$, *and*

i) For each net of elements $(p_i, a_{p_i}) \in \mathsf{GraphS}$, *if* $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, *and* $b \in X$, *then*

$$\operatorname{Liminf}(f_{p_i}(a_{p_i}, b) - f_{p_0}(a_{p_i}, b)) \cap (-\operatorname{Int} C) \neq \emptyset.$$

ii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0.

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