

Logarithmic mean inequality for generalized trigonometric and hyperbolic functions

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Abstract. In this paper we study the convexity and concavity properties of generalized trigonometric and hyperbolic functions in case of Logarithmic mean.

1 Introduction

Recently, the study of the generalized trigonometric and generalized hyperbolic functions has got huge attention of numerous authors, and has appeared the huge number of papers involving the equalities and inequalities and basis properties of these function, e.g. see [7, 8, 9, 6, 10, 13, 14, 18, 23] and the references therein. These generalized trigonometric and generalized hyperbolic functions p -functions depending on the parameter $p > 1$ were introduced by Lindqvist [19] in 1995. These functions coincides with the usual functions for $p = 2$. Thereafter Takeshu took one further step and generalized these function for two parameters $p, q > 1$, so-called (p, q) -functions. In [8], some convexity and concavity properties of p -functions were studied. Thereafter those results were

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extended in [5] for two parameters in the sense of Power mean inequality. In this paper we study the convexity and concavity property of p -function with respect Logarithmic mean. Before we formulate our main result we will define generalized trigonometric and hyperbolic functions customarily.

The eigenfunction $\sin_p$ of the so-called one-dimensional p -Laplacian problem [12]

$$-\Delta_p u = - \left(|u'|^{p-2} u' \right)' = \lambda |u|^{p-2} u, \quad u(0) = u(1) = 0, \quad p > 1,$$

is the inverse function of $F: (0, 1) \rightarrow (0, \frac{\pi_p}{2})$, defined as

$$F(x) = \arcsin_p(x) = \int_0^x (1 - t^p)^{-\frac{1}{p}} dt,$$

where

$$\pi_p = 2\arcsin_p(1) = \frac{2}{p} \int_0^1 (1-s)^{-1/p} s^{1/p-1} ds = \frac{2}{p} B\left(1 - \frac{1}{p}, \frac{1}{p}\right) = \frac{2\pi}{p \sin\left(\frac{\pi}{p}\right)},$$

here $B(.,.)$ denotes the classical beta function.

The function $\arcsin_p$ is called the generalized inverse sine function, and coincides with usual inverse sine function for $p = 2$. Similarly, the other generalized inverse trigonometric and hyperbolic functions $\arccos_p: (0, 1) \rightarrow (0, \pi_p/2)$, $\arctan_p: (0, 1) \rightarrow (0, b_p)$, $\operatorname{arcsinh}_p: (0, 1) \rightarrow (0, c_p)$, $\operatorname{arctanh}_p: (0, 1) \rightarrow (0, \infty)$, where

$$b_p = \frac{1}{2p} \left(\psi\left(\frac{1+p}{2p}\right) - \psi\left(\frac{1}{2p}\right) \right) = 2^{-\frac{1}{p}} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right),$$

$$c_p = \left(\frac{1}{2}\right)^{\frac{1}{p}} F\left(1, \frac{1}{p}; 1 + \frac{1}{p}, \frac{1}{2}\right),$$

are defined as follows

$$\arccos_p(x) = \int_0^{(1-x^p)^{\frac{1}{p}}} (1-t^p)^{-\frac{1}{p}} dt, \quad \arctan_p(x) = \int_0^x (1+t^p)^{-1} dt,$$

$$\operatorname{arcsinh}_p(x) = \int_0^x (1+t^p)^{-\frac{1}{p}} dt, \quad \operatorname{arctanh}_p(x) = \int_0^x (1-t^p)^{-1} dt,$$

where $F(a, b; c; z)$ is *Gaussian hypergeometric function* [1].

The generalized cosine function is defined by

$$\frac{d}{dx} \sin_p(x) = \cos_p(x), \quad x \in [0, \pi_p/2].$$

It follows from the definition that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p},$$

and

$$|\cos_p(x)|^p + |\sin_p(x)|^p = 1, \quad x \in \mathbb{R}. \quad (1)$$

Clearly we get

$$\frac{d}{dx} \cos_p(x) = -\cos_p(x)^{2-p} \sin_p(x)^{p-1}.$$

The generalized tangent function $\tan_p$ is defined by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)},$$

and applying (1) we get

$$\frac{d}{dx} \tan_p(x) = 1 + \tan_p(x)^p.$$

For $x \in (0, \infty)$, the inverse of generalized hyperbolic sine function $\sinh_p(x)$ is defined by

$$\operatorname{arcsinh}_p(x) = \int_0^x (1 + t^p)^{-1/p} dt,$$

and generalized hyperbolic cosine and tangent functions are defined by

$$\cosh_p(x) = \frac{d}{dx} \sinh_p(x), \quad \tanh_p(x) = \frac{\sinh_p(x)}{\cosh_p(x)},$$

respectively. It follows from the definitions that

$$|\cosh_p(x)|^p - |\sinh_p(x)|^p = 1. \quad (2)$$

From above definition and (2) we get the following derivative formulas,

$$\frac{d}{dx} \cosh_p(x) = \cosh_p(x)^{2-p} \sinh_p(x)^{p-1}, \quad \frac{d}{dx} \tanh_p(x) = 1 - |\tanh_p(x)|^p.$$

Note that these generalized trigonometric and hyperbolic functions coincide with usual functions for $p = 2$.

For two distinct positive real numbers x and y , the Arithmetic mean, Geometric mean, Logarithmic mean, Harmonic mean and the Power mean of order $p \in \mathbb{R}$ are respectively defined by

$$A(x, y) = \frac{x + y}{2}, \quad G(x, y) = \sqrt{xy},$$

$$L(x, y) = \frac{x - y}{\log(x) - \log(y)}, \quad x \neq y,$$

$$H(x, y) = \frac{1}{A(1/x, 1/y)},$$

and

$$M_t = \begin{cases} \left(\frac{x^t + y^t}{2} \right)^{1/t}, & t \neq 0, \\ \sqrt{x y}, & t = 0. \end{cases}$$

Let $f : I \rightarrow (0, \infty)$ be continuous, where I is a sub-interval of $(0, \infty)$. Let M and N be the means defined above, then we call that the function f is MN -convex (concave) if

$$f(M(x, y)) \leq (\geq) N(f(x), f(y)) \quad \text{for all } x, y \in I.$$

Recently, Generalized convexity/concavity with respect to general mean values has been studied by Anderson et al. in [2]. We recall one of their results as follows

Lemma 1 [2, Theorem 2.4] *Let I be an open sub-interval of $(0, \infty)$ and let $f : I \rightarrow (0, \infty)$ be differentiable. Then f is HH -convex (concave) on I if and only if $x^2 f'(x)/f(x)^2$ is increasing (decreasing).*

In [4], Baricz studied that if the functions f is differentiable, then it is (a, b) -convex (concave) on I if and only if $x^{1-a} f'(x)/f(x)^{1-b}$ is increasing (decreasing).

It is important to mention that $(1, 1)$ -convexity means the AA -convexity, $(1, 0)$ -convexity means the AG -convexity, and $(0, 0)$ -convexity means GG -convexity.

Motivated by the results given in [2, 4], we contribute to the topic by giving the following result.

Theorem 1 *Let $f : I \rightarrow (0, \infty)$ be a continuous and $I \subseteq (0, \infty)$, then*

$$1. L(f(x), f(y)) \geq (\leq) f(L(x, y)),$$

$$2. L(f(x), f(y)) \geq (\leq) f(A(x, y)),$$

if f is increasing and log-convex (concave).

Theorem 2 For $x, y \in (0, \pi_p/2)$, the following inequalities

$$1. L(\sin_p(x), \sin_p(y)) \leq \sin_p(L(x, y)), \quad p > 1,$$

$$2. L(\cos_p(x), \cos_p(y)) \leq \cos_p(L(x, y)), \quad p \geq 2.$$

Theorem 3 For $p > 1$, we have

$$1. L(1/\sin_p(x), 1/\sin_p(y)) \geq 1/\sin_p(A(x, y)), \quad x, y \in (0, \pi_p/2),$$

$$2. L(1/\cos_p(x), 1/\cos_p(y)) \geq 1/\cos_p(L(x, y)), \quad x, y \in (0, \pi_p/2),$$

$$3. L(\tanh_p(x), \tanh_p(y)) \leq \tanh_p(A(x, y)), \quad x, y \in (0, \infty),$$

$$4. L(\operatorname{arcsinh}_p(x), \operatorname{arcsinh}_p(y)) \leq \operatorname{arcsinh}_p(A(x, y)), \quad x, y \in (0, 1),$$

$$5. L(\operatorname{arctan}_p(x), \operatorname{arctan}_p(y)) \leq \operatorname{arctan}_p(A(x, y)), \quad x, y \in (0, 1).$$

2 Preliminaries and Proofs

We give the following lemmas which will be used in the proof of our main result.

Lemma 2 [22] Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \rightarrow \mathbb{R}$ be a positive, integrable function. Then

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx. \quad (3)$$

If one of the functions f or g is non-increasing and the other non-decreasing, then the inequality in (3) is reversed.

Lemma 3 [17] If $f(x)$ is continuous and convex function on $[a, b]$, and $\varphi(x)$ is continuous on $[a, b]$, then

$$f\left(\frac{1}{b-a} \int_a^b \varphi(x)dx\right) \leq \frac{1}{b-a} \int_a^b f(\varphi(x))dx. \quad (4)$$

If function $f(x)$ is continuous and concave on $[a, b]$, then the inequality in (4) reverses.

Lemma 4 [3] For two distinct positive real numbers a, b , we have $L < A$.

Lemma 5 For $p > 1$, the function $\sin_p(x)$ is HH-concave on $(0, \pi_p/2)$.

Proof. Let $f(x) = f_1(x)f_2(x)$, $x \in (0, \pi_p/2)$, where $f_1(x) = 1/\sin(x)$ and $f_2(x) = x^2 \cos_p(x)/\sin_p(x)$. Clearly, f_1 is decreasing, so it is enough to prove that f_2 is decreasing, then the proof follows from Lemma 1. We get

$$\begin{aligned} f'_2(x) &= \frac{\sin_p(x)(\cos_p(x) - x \cos_p(x)^{2-p} \sin_p(x)^{p-1}) - x \cos_p(x)^2}{\sin_p(x)^2} \\ &= \frac{\cos_p(x)^2((1 - x \tan_p(x)^{p-1}) \tan_p(x) - x)}{\sin_p(x)^2} = f_3(x) \frac{\cos_p(x)^2}{\sin_p(x)^2}, \end{aligned}$$

where $f_3(x) = \tan_p(x) - x \tan_p(x)^p - 1$. Again, one has

$$f'_3(x) = p \tan_p(x)^{p-1} (1 + \tan_p(x)^p) x < 0.$$

Thus, f_3 is decreasing and $g(x) < g(0) = 0$. This implies that $f'_2 < 0$, hence f_2 is strictly decreasing, the product of two decreasing functions is decreasing. This implies the proof. $\square$

Proof of Theorem 1. We get

$$L(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} 1 dt}{\int_{f(y)}^{f(x)} \frac{1}{t} dt} = \frac{\int_y^x f'(u) du}{\int_y^x \frac{f'(u)}{f(u)} du}. \quad (5)$$

It is assumed that the function $f(x)$ is increasing and $\log f$ is convex, this implies that $\frac{f'(x)}{f(x)}$ is increasing. Letting $p(x) = 1$, $f(x) = f(u)$ and $g(x) = f'(u)/f(u)$ in Lemma 2, we get

$$\int_y^x 1 du \int_y^x f'(u) du \geq \int_y^x \frac{f'(u)}{f(u)} du \int_y^x f(u) du.$$

This is equivalent to

$$L(f(x), f(y)) = \frac{\int_y^x f'(u) du}{\int_y^x \frac{f'(u)}{f(u)} du} \geq \frac{\int_y^x f(u) du}{\int_y^x 1 du}.$$

By Lemmas 3 and 4, and keeping in mind that log-convexity of f implies the convexity of f , we get

$$L(f(x), f(y)) \geq f\left(\frac{\int_y^x u du}{x - y}\right) = f\left(\frac{x + y}{2}\right) \geq f(L(x, y)).$$

The proof of converse follows similarly. If we repeat the lines of proof of part (1), and use the concavity of the function, and Lemmas 3 & 4 then we arrive at the proof of part (2).

Proof of Theorem 2. It is easy to see that the function $\sin_p(x)$ is increasing and log-concave. So the proof of part (1) follows easily from Theorem 1. We also offer another proof as follows:

It can be observed easily that

$$L(\sin_p(x), \sin_p(y)) = \frac{\int_y^x \cos_p(u) du}{\int_{\sin_p(y)}^{\sin_p(x)} \frac{1}{t} dt} = \frac{\int_y^x \cos_p u du}{\int_y^x \frac{\cos_p u}{\sin_p(u)} du},$$

and

$$\sin_p(L(x, y)) = \sin_p\left(\frac{x - y}{\log \frac{x}{y}}\right) = \sin_p\left(\frac{\int_y^x 1 du}{\int_y^x \frac{1}{u} du}\right).$$

Clearly, $\cos_p(u)$ and $\sin_p(1/u)$, utilizing Chebyshev inequality, we have

$$\int_y^x \cos_p(u) du \int_y^x \sin_p(1/u) du \leq \int_y^x 1 du \int_y^x \cos_p u \sin_p \frac{1}{u} du.$$

So, we get

$$\int_y^x \cos_p u du \int_y^x \sin_p(1/u) du < \int_y^x 1 du \int_y^x \frac{\cos_p(u)}{\sin_p(u)} du.$$

Where we apply simple inequality $\sin_p(\frac{1}{u}) < \frac{1}{\sin_p(u)}$. In order to prove inequality (1), we only prove

$$\frac{\int_y^x 1 du}{\int_y^x \sin_p(1/u) du} \leq \sin_p\left(\frac{\int_y^x 1 du}{\int_y^x \sin_p(1/u) du}\right).$$

Consider a partition T of the interval $[y, x]$ into n equal length sub-interval by means of points $y = x_0 < x_1 < \dots < x_n = x$ and $\Delta x_i = \frac{x-y}{n}$. Picking an arbitrary point $\xi_i \in [x_{i-1}, x_i]$ and using Lemma 1, we have

$$\frac{n}{\sum_{i=1}^n \sin_p \frac{1}{\xi_i}} \leq \sin_p\left(\frac{n}{\sum_{i=1}^n \frac{1}{\xi_i}}\right)$$

$\Leftrightarrow$

$$\lim_{n \rightarrow \infty} \left(\frac{x-y}{n} \sum_{i=1}^n \sin_p \frac{1}{\xi_i} \right) \leq \sin_p \left(\frac{x-y}{\lim_{n \rightarrow \infty} \left(\frac{x-y}{n} \sum_{i=1}^n \frac{1}{\xi_i} \right)} \right)$$

$\Leftrightarrow$

$$\frac{\int_y^x 1 \, du}{\int_y^x \sin_p(1/u) \, du} \leq \sin_p \left(\frac{\int_y^x 1 \, du}{\int_y^x \sin_p(1/u) \, du} \right).$$

This completes the proof.

For (2), clearly $\cos_p(x)$ is decreasing and $\tan_p(x)^{p-1}$ is increasing. One has

$$(\cos_p(x))'' = \cos_p(x) \tan_p(x)^{p-2} (1 - p + (2 - p) \tan_p(x)^p) < 0,$$

this implies that $\cos_p(x)$ is concave on $(0, \pi_p/2)$.

Using Tchebyshef inequality, we have

$$\int_y^x 1 \, du \int_y^x \cos_p(u) \tan_p(u)^{p-1} \, du \leq \int_y^x \cos_p(u) \, du \int_y^x \tan_p(u)^{p-1} \, du,$$

which is equivalent to

$$\frac{\int_y^x \cos_p(u) \tan_p(u)^{p-1} \, du}{\int_y^x \tan_p(u)^{p-1} \, du} \leq \frac{\int_y^x \cos_p(u) \, du}{\int_y^x 1 \, du}. \quad (6)$$

Substituting $t = \cos_p(u)$ in (6), we get

$$L(\cos_p(x), \cos_p(y)) = \frac{\int_{\cos_p(y)}^{\cos_p(x)} \frac{1}{t} \, dt}{\int_{\cos_p(y)}^{\cos_p(x)} \frac{1}{t} \, dt} = \frac{\int_y^x \cos_p(u) \tan_p(u)^{p-1} \, du}{\int_y^x \tan_p(u)^{p-1} \, du} \leq \frac{\int_y^x \cos_p(u) \, du}{\int_y^x 1 \, du}.$$

Using Lemma 3 and concavity of $\cos_p(x)$, we obtain

$$L(\cos_p(x), \cos_p(y)) \leq \cos_p \left(\frac{\int_y^x u \, du}{x - y} \right) = \cos_p \left(\frac{x + y}{2} \right) \leq \cos_p(L(x, y)).$$

Proof of Theorem 3. Let $g_1(x) = 1/\cos_p(x)$, $x \in (0, \pi_p/2)$ and $g_2(x) = \tanh_p(x)$, $x > 0$. We get

$$(\log(g_1(x)))'' = (p-1) \tan_p(x)^{p-2} (1 + \tan_p(x)^p) > 0,$$

and

$$(\log(g_2(x)))'' = \frac{1 - \tanh_p(x)^p}{\tanh_p(x)^2} ((1 - p) \tanh_p(x)^p - 1) < 0.$$

This implies that g_1 and g_2 are log-convex, clearly both functions are increasing, and log-convexity implies the convexity, so g_1 and g_2 are convex functions. Now the proof follows easily from Theorem 1. The rest of proof follows similarly.

Corollary 1 For $p > 1$, we have

1. $L(\tan_p(x), \tan_p(y)) \geq \tan_p(L(x, y))$, $x, y \in (s_p, \pi_p/2)$, where s_p is the unique root of the equation $\tan_p(x) = 1/(p-1)^{1/p}$,
2. $L(\operatorname{arctanh}_p(x), \operatorname{arctanh}_p(y)) \geq \operatorname{arctanh}_p(L(x, y))$, $x, y \in (r_p, 1)$, where r_p is the unique root of the equation $x^{p-1} \operatorname{arctanh}_p(y) = 1/p$.

Proof. Write $f_1(x) = \tan_p(x)$. We get

$$\left(\frac{f_1'(x)}{f_1(x)} \right)' = \left(\frac{1 + \tan_p^p(x)}{\tan_p(x)} \right)' = \frac{1 + \tan_p^p(x)}{\tan_p^2(x)} [(p-1) \tan_p^p(x) - 1] > 0$$

on $(s_p, \frac{\pi_p}{2})$. This implies that f_1 is log-convex, clearly f_1 is increasing, and the proof follows easily from Theorem 1. The proof of part (2) follows similarly. $\square$

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