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Some new results on colour-induced signed graphs

Niranjana SUDHEER
Department of Mathematics
CHRIST (Deemed to be University)
Bangalore-560029, Karnataka, INDIA.
email: niranjana.sudheer@maths.christuniversity.in

Ann Cherian GEORGE
Department of Mathematics
CHRIST (Deemed to be University)
Bangalore-560029, Karnataka, INDIA.
email: ann.george@maths.christuniversity.in

Achu ANIYAN
Department of Mathematics
CHRIST (Deemed to be University)
Bangalore-560029, Karnataka, INDIA.
email: achu.aniyan@res.christuniversity.in

Sudev NADUVATH
Department of Mathematics
CHRIST (Deemed to be University)
Bangalore-560029, Karnataka, INDIA.
email: sudev.nk@christuniversity.in

Abstract. A signed graph is a graph in which positive or negative signs are assigned to its edges. We consider equitable colouring and Hamiltonian colouring to obtain induced signed graphs. An equitable colour-induced signed graph is a signed graph constructed from a given graph in which each edge uv receives a sign $(-1)^{|c(v)-c(u)|}$, where c is an equitable colouring of vertex v. A Hamiltonian colour-induced signed graph is a signed graph obtained from a graph G in which for each edge $e = uv$, the signature function $\sigma(uv) = (-1)^{|c(v)-c(u)|}$, gives a sign such that, $|c(u) - c(v)| \geq n - 1 - D(u, v)$ where c is a function that assigns a colour to each vertex satisfying the given condition. This paper discusses the properties and characteristics of signed graphs induced by the equitable and Hamiltonian colouring of graphs.

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1 Introduction

A *signed graph* is a graph S obtained from a graph G where each edge is assigned a sign, either '+' or '−'. The function $\sigma : E(S) \rightarrow \{+, -\}$ which assigns the signs to edges is known as the *sign function* or *signature* of the signed graph (see [13]). The signature σ is defined on all edges and the sign of free loop is considered to be positive always. The *underlying graph* of a signed graph S is an ordinary undirected graph obtained by removing the signs of all edges of S. If the context is clear, the underlying graph of S may be denoted by G.

A cycle in a signed graph S is said to be *balanced* if the number of negative edges is even and the signed graph S is said to be a *balanced signed graph* if every cycle in it is balanced (see [5]). A signed graph S is said to be k-*clusterable* if its vertex set can be partitioned into k sets such that the edges between the points of the same part have the positive sign and the edges between the different parts get the negative sign. The sign of a free loop is always considered positive. Suppose every edge in a signed graph is positive. In that case, the graph is said to be *positive homogeneous*, and similarly, if it is negative, it is called *negative homogeneous*. Otherwise, a signed graph is called *heterogeneous* or *non-homogeneous* (see [14]).

A *marking* of a signed graph S is a function $\mu : V(S) \rightarrow \{+, -\}$, which assigns a sign to each vertex of S according to some conditions. A *marked signed graph* is a signed graph in which every vertex has a sign. A graph is called *marked* if each vertex is assigned either a positive or negative sign. We call a marked signed graph *coherent* if all its vertices have the same sign. A signed graph is said to be *positive coherent* (resp. *negative coherent*) if all its vertices have the positive sign (resp. negative sign). A signed graph is *incoherent* if it is not coherent. A signed graph S is said to be *sign-compatible* if there exists a marking μ of its vertices such that the end vertices of every negative edge receive the sign $-$ in μ and no positive edge holds this property (see [10]).

In this paper, we consider two types of markings: the canonical marking μ and the natural marking ν. In *canonical marking* $\mu : V(S) \rightarrow \{+, -\}$, each vertex v receives the sign which is the product of signs of the edges incident at v, whereas in *natural marking* $\nu : V(S) \rightarrow \{+, -\}$, the vertices are assigned signs as $\nu(u) = (-1)^{|\phi(u)|}$ where the function ϕ denotes a vertex labelling (see [6]). A signed graph is said to be *consistent* if for every cycle in the signed graph, have even number of negative vertices.

A switching function for a signed graph S is a function $\zeta : V \to \{+, -\}$. The switched signature is $\sigma^{\zeta}(e) = \zeta(v)\sigma(e)\zeta(w)$, where e has endpoints v, w. The *switched signed graph* is $S^{\zeta} := (G, \sigma^{\zeta})$. We say $\sum$ is switched by ζ (see [14]).

The term *induced signed graph*, denoted by S_G, was introduced in the paper [1]. It is a signed graph obtained from a graph G, where the signature function is defined for each edge $e = (uv)$ as $\sigma(uv) = (-1)^{|\phi(u)-\phi(v)|}$. The function ϕ maps from $V(G)$ to a set of labels or weights associated with each vertex. The authors considered the degree and eccentricity of vertices to sign a graph. Similarly, in the paper [2], an edge in a graph is signed considering the product of the degree of its end vertices.

Though the concept of a signed graph arose from its application, its has been extended. For a very long time, opposite relationships in social networks have been modelled using signed graphs. In social psychology, signed graphs with positive edges denoting friendships and negative edges denoting enmities between nodes, which represent persons, have been used to describe social situations. While negative cycles are unbalanced and are supposed to be unstable social conditions, positive cycles are balanced and are supposed to be stable social situations. By balance theory, an unbalanced graph can depict an unstable state. As a consequence, every cycle in a balanced signed graph has an even number of negative edges.

Meyer proposed an equitable colouring scheme that addressed the scheduling problem. Hamiltonian colouring also has numerous uses in various branches of science and technology. In this paper, we introduce an induced signed graph based on the equitable and Hamiltonian colouring of graphs, and we study the properties of this induced signed graph.

For the terminology of graphs, we refer to [12], and for signed graphs, we refer to [13, 8, 9]. The terminology of graph colouring follows from [3]. Unless otherwise mentioned, the graphs considered for the study are simple, connected and undirected graphs. Throughout the discussion, we use minimum colours available for the colouring.

2 Equitable colour-induced signed graphs

Equitable colouring of a graph is a proper colouring which assigns a colour to the vertices of a graph G such that the number of vertices in any two colour classes differ by at most one [11]. The notion of equitable colour-induced signed graphs is defined and explored in this section.

Definition 1 *The equitable colour-induced signed graph of a graph* G *is a signed graph, denoted by* S_E*, which is induced by equitably colouring the graph* G *whose signature function is defined by*

$$\sigma(uv) = (-1)^{|\phi(v)-\phi(u)|},$$

where $\sigma : E(G) \to \{+,-\}$ *and* ϕ *is a mapping defined by* $\phi : V(G) \to \mathbb{Z}$*,* $\mathbb{Z}$ *is the set of weights (or labels associated with each vertex in the signed graph* S*).*

Figure 1 is an equitable colour-induced graph of a wheel $W_{1,7}$, where the graph is equitably coloured with five colours and assigned signs to the edges using the Definition 1. The dashed line in the graph represents the negative edge. The vertices are marked using canonical marking.

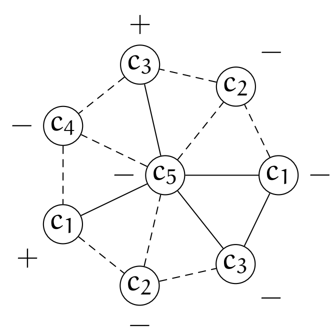

Figure 1: Equitably coloured induced signed graph of $W_{1,7}$

Theorem 2 (Harary's balance theorem) [5] *A signed graph* S *is balanced if and only if there is a bipartition of its vertex set,* $V = X \cup Y$*, such that every positive edge is induced by* X *or* Y *while every negative edge has one endpoint in* X *and one in* Y*.*

Theorem 3 *An equitable colour-induced signed graph* S_E *of a graph* G *is balanced.*

Proof. Consider an equitable colour-induced signed graph S_E with vertex set V_E. If all vertices of the induced signed graph are of the same parity, then it is positive homogeneous and hence is balanced. Hence, assume the contrary. Let V_1 and V_2 be two partitions of V_E such that vertices with even parity belongs to V_1 and vertices with odd parity comes in partition V_2. Therefore by Theorem 2, the graph is balanced. □

A marked signed graph S is said to be *coherent* if all the vertices are of the same sign. A signed graph S is called *positive coherent* if $\mu(v) = +$ for all $v \in V(S)$ and is called *negative coherent* if $\mu(v) = -$ for all $v \in V(S)$. An *incoherent signed graph* is one which is not coherent.

A necessary and sufficient condition for the equitable colour-induced signed graphs to be sign-invariant is discussed in the following theorem.

Proposition 4 *Consider a cycle C_n where n is even. Then, the corresponding equitable colour-induced signed graph is negative homogeneous, and it is positive coherent with respect to canonical marking.*

Proof. Let C_n be an even cycle. The equitable colouring of C_n is done with two colours so that adjacent vertices receive different colours. Then, the edges of the colour-induced signed graph receive the negative sign, and hence it is negative homogeneous. Since every edge is negative, the vertices will be signed positive with respect to the canonical marking. □

Theorem 5 *The equitable colour-induced signed graph of a complete graph K_n is positive coherent with respect to canonical marking if and only if $n \equiv 0, 2 \pmod 4$.*

Proof. Consider a complete graph K_n equitably coloured with n distinct colours. For every vertex, $\frac{n}{2}$ or $\frac{n}{2} - 1$ negative edges are incident to it. Since $n \equiv 0, 2 \pmod 4$, we have $\frac{n}{2}$ or $\frac{n}{2} - 1$ is even and hence the vertices receive the positive sign by canonical marking. Hence, the signed graph is positive coherent.

Now, suppose that the graph is positive coherent. Thus, there is an even number of negative edges incident with every vertex of K_n. Since every vertex in K_n has degree $n - 1$, either $\frac{n}{2}$ or $\frac{n}{2} - 1$ negative edges should be incident to each vertex. Since the graph is positive coherent, the number of negative edges incident should be even. Hence, $\frac{n}{2}$ or $\frac{n}{2} - 1$ is even. □

The *ladder graph* L_n (see [7]) is a planar undirected graph with $2n$ vertices and $3n - 2$ edges. The ladder graph can be obtained as the Cartesian product of two path graphs, one of which has only one edge: $L_{n,1} = P_n \square P_2$.

Proposition 6 *The equitable colour-induced signed graph $S_E(L_n)$ of a ladder graph is negative homogeneous.*

Proof. The equitable chromatic number of a ladder graph L_n is 2, and any two adjacent vertices receive different colours. Since for each edge, the end vertices are of opposite parity; every edge becomes negative after signing. Hence, $S_E(L_n)$ is negative homogeneous. □

Proposition 7 *The equitable colour-induced signed graph of a ladder graph* $S_E(L_n)$ *is consistent with respect to canonical marking.*

Proof. Let $S_E(L_n)$ be the equitable colour-induced graph of a ladder graph. The vertices with degree 2 receive the positive sign, and the remaining vertices receive the negative sign with respect to the canonical marking. Since the graph contains only even cycles, there can be cycles with an even number of positive and negative vertices or cycles with an even number of negative vertices alone. In both cases, the cycles are consistent. Hence, the graph $S_E(L_n)$ is consistent. □

The *circular ladder graph* CL_n (see [4]) is a connected planar graph with $2n$ vertices and $3n$ edges. The circular ladder graph can be obtained as the Cartesian product of a cycle of length $n \geq 3$ and an edge: $CL_n = C_n \square P_2$.

Proposition 8 *The equitable colour-induced signed graph of a circular ladder graph* $S_E(CL_n)$ *is negative homogeneous if and only if* n *is even.*

Proof. Consider a colour-induced signed graph of a circular ladder graph $S_E(CL_n)$. Since n is even, the equitable chromatic number is 2. By Proposition 4, the equitable colour-induced signed graph of C_n is negative homogeneous. Thus, the Cartesian product of C_n with P_2 gives negative sign to every edge. Hence, $S_E(CL_n)$ with n even is negative homogeneous. Suppose that $S_E(CL_n)$ is negative homogeneous. If n is odd, the CL_n can be equitably coloured with 3 colours. Thus by the Definition 1, there will be at least one positive edge, which contradicts that $S_E(CL_n)$ is negative homogeneous. Hence n is even. □

A necessary and sufficient condition for the equitable colour-induced signed graphs to be sign-invariant is discussed in the following theorem.

Theorem 9 *An equitable colour-induced signed graph* S_E *is sign-invariant if and only if the corresponding marked graph is coherent (both positive and negative) with respect to canonical marking.*

Proof. Assume that the equitable colour-induced signed graph S_E is sign-invariant. That is, the equitable colour-induced signed graph and its switched graph are sign-isomorphic. Consider an edge $e = uv$ in the signed graph S_E. The sign of the edge is preserved only if the product of the sign of the vertices is positive. That is, u and v are of the same parity. This implies that the marked graph is coherent.

Conversely, assume that all the vertices are coherent. Then, each edge of the switched graph of S_E receives the sign which is the sign of the product of two vertices and the sign of the edge between them. Since the graph is coherent, the switched graph will be the same as the signed graph. Hence, the graph is sign-invariant. □

Proposition 10 *If the equitable colour-induced signed graph S_E is positive coherent or negative coherent without odd cycles, then S_E is consistent.*

Proof. Consider an equitable colour-induced signed graph S_E. The case is trivial if the graph is positive coherent. Suppose that the graph S_E is negative coherent without odd cycles. Then every cycle in S_E has an even number of negative vertices. Hence the equitable colour-induced signed graph S_E is consistent. □

Theorem 11 *The equitable colour-induced signed graph of a complete bipartite $K_{m,n}$, where $m = n$ is negative homogeneous, consistent and sign-invariant.*

Proof. Consider an equitable colour-induced signed graph of a complete bipartite $K_{m,n}$, where $m = n$. The equitable chromatic number of a complete bipartite $K_{m,m}$ is 2 as vertices in each partite can be labelled with each one of the two colours. Now we have the following cases.

Case 1: When m is even, by Definition 1, each edge receives negative sign. Hence, the graph is negative homogeneous. Also, by canonical marking, every vertex receives a positive sign as the degree of each vertex is even. As a result, the graph is positive coherent and hence consistent. Thus, by Theorem 9, equitable colour-induced signed graph of a complete bipartite $K_{m,m}$ is sign-invariant.

Case 2: When m is odd, the graph is similar to case 1 and is negative homogeneous. The canonical marking makes the graph negative coherent as the degree of each vertex is odd. Since a complete bipartite graph has no odd cycles, each cycle will contain an even number of negative vertices. Hence, the graph is consistent. Then, by Theorem 9, equitable colour-induced signed graph of a complete bipartite $K_{m,m}$ is sign-invariant. □

Proposition 12 *Let C_E be an equitable colour-induced cycle with n vertices. If n is even, then C_E is atmost n clusterable and if n is odd, then C_E is at* *most* $n - |E^+(G)|$ *clusterable where* $|E^+(G)|$ *denotes the cardinality of edge set with positive sign.*

Proof. Consider C_E, an equitable colour-induced cycle with n vertices. If n is even, then C_E is negative homogeneous. Thus, each vertex can be occupied in each cluster. Hence, it is at most n clusterable. If n is odd, then the clusterability can be done in the following manner:

The end vertices of each edge with the positive sign can be partitioned in at most $|E^+(G)|$ clusters and each of the remaining vertices can be occupied in one cluster. Thus, odd cycle is $n - |E^+(G)|$ clusterable. □

Theorem 13 *The equitable colour-induced signed graph of a star graph* S_E *is at most* $|E^-(S_E)| + 1$ *clusterable where* $|E^-(G)|$ *denotes the number of negative edges in the signed graph.*

Proof. Consider an equitable colour-induced signed graph of a star graph S_E. The clusterability of $V(S_E)$ can be done in such a way that the end vertices of positive edges along with the central vertex can be occupied in one cluster. The remaining vertices are adjacent to the central vertex by negative edges and hence can be partitioned so that each vertex is in one cluster. The number of such negative edges is nothing but $|E^-(S_E)|$. Hence, the maximum clusterability of the signed graph S_E is $|E^-(S_E)| + 1$. □

Theorem 14 *The equitable colour-induced signed graph of a complete graph* K_n *is atmost 2-clusterable.*

Proof. Consider an equitable colour-induced signed graph $S_E(K_n)$ whose vertices are coloured with n distinct colours. For a complete graph, the vertex set can be partitioned into two sets V_1 and V_2 such that the vertices with even parity are in V_1 and vertices with odd parity are in V_2. Here the condition for clusterability is satisfied as edges with end vertices in different clusters have the negative sign and edges with end vertices within the cluster have the positive sign by Definition 1. Hence it is 2-clusterable.

Suppose there exist a third cluster V_3 which contains vertices from either of the two clusters or from both.

Case 1: Consider a vertex $v_i \in V_3$ of even (odd) parity. Then, the edges with end vertices v_i and vertex from V_1(or V_2) should be negative, which is a contradiction, as an edge with end vertices having the same parity receives positive sign.

Case 2: Suppose $v_i, v_j \in V_3$ are of opposite parity. Since the graph is complete and hence by Definition 1 the edge (v_i, v_j) is negative which again contradicts the definition of clusterability. □

Proposition 15 *The switched graph of an equitable colour-induced signed graph of a complete bipartite $K_{m,m+1}$ is positive homogeneous.*

Proof. Consider an equitable colour-induced signed graph of a complete bipartite $K_{m,m+1}$. Since the equitable chromatic number is 2, by Definition 1, each edge receives negative sign. Hence, the graph is negative homogeneous. while marking, the vertices in the partite with an even degree get the negative sign and the vertices with an odd degree get the positive sign. Thus, while switching, we get a positive homogeneous graph. □

Definition 16 *A natural marking is the assignment of signs of vertices v of S_E where each vertex receives a sign $(-1)^{|\phi(v)|}$, where $\phi(v)$ is a parameter under consideration that is assigned to the vertex v.*

Theorem 17 *The switched graph of any colour-induced signed graph S_E with respect to natural marking is positive homogeneous.*

Proof. Let S_E be an equitable colour-induced signed graph. By Definition 1 an edge $e = uv$ in S_E receives the sign by the sigma function $\sigma(uv) = (-1)^{|c(v)-c(u)|}$, where $c(v)$ is the colour of the vertex v and also by Definition 16, each vertex receives a sign $(-1)^{|c(v)|}$. Now we have the following cases.

Case 1: Let $c(v)$ and $c(u)$ have colours of same parity. Since both the colours are of the same parity, the difference gives the positive sign to the edge uv. Also, since the end vertices of the edge uv are of the same parity, the product of the sign of end vertices will be positive. Thus, the switched graph gets a positive sign.

Case 2: Let $c(v)$ and $c(u)$ have colours of opposite parity then, by Definition 1, the get the uv receives negative sign. Also since the end vertices of the edges are of opposite parity, the product of signs becomes negative. Hence while switching, the graph will be positive homogeneous. □

Corollary 18 *The switched graph of any equitable colour-induced signed graph S_E with respect to natural marking is positive homogeneous.*

3 Hamiltonian colour-induced signed graphs

Recall that the *detour distance* between two vertices u and v in a connected graph G is the length of the longest uv path in G. A connected graph G of order n has a vertex colouring c such that $D(u,v) + |c(u) - c(v)| \geq n - 1$ for every two distinct vertices u and v of G is called as *Hamiltonian colouring*, where $D(u,v)$ is called as the detour distance.

In this section, we define a new induced signed graph considering the Hamiltonian colouring of graphs. The Hamiltonian colour-induced signed graph is defined as follows:

Definition 19 *The Hamiltonian colour signed graph, denoted by* S_H*, is the graph obtained from a graph* G *assigning each edge a sign either* $+$ *or* $-$ *with respect to the signature function* σ *defined as,*

$$\sigma(uv) = (-1)^{|c(u)-c(v)|}$$

where c *is the function which assigns a colour to each vertex, such that,*

$$|c(u) - c(v)| \geq n - 1 - D(u,v),$$

where $D(u,v)$ *is the length of the longest* $u - v$ *path in* G*.*

Figure 2, illustrates a Hamiltonian colour-induced signed graph of paw graph. The negative edges of the signed graph are represented by the dashed edges.

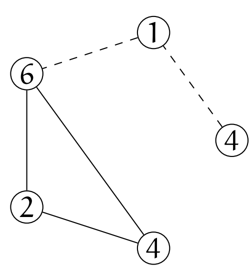

Figure 2: Example of Hamiltonian colour-induced signed graph of paw graph.

Theorem 20 *The Hamiltonian colour-induced signed graphs are balanced.*

Proof. Let S_H be a Hamiltonian colour-induced signed graph of a graph G. If S_H is positive homogeneous, then the proof is complete. Hence, assume that

S_H is non-homogeneous with the vertex set $V(S_H)$. Then, we can partition the vertex set $V(S_H)$ into two sets, say X and Y such that all vertices of even parity (for which $c(v)$ is even) belong to X, and all vertices of odd parity (for which $c(v)$ is odd) belong to Y. Clearly, the edges having both end vertices in the same partition receives the sign $+$, while the edges with one end vertex in X and the end vertex in Y will get the sign $-$. Then, by Theorem 2 the signed graph S_H is balanced. □

Proposition 21 *The Hamiltonian colour-induced signed graph of path graph P_n is negative homogeneous when n is odd.*

Proof. Consider a path graph $S_H(P_n)$ with n odd which is coloured with Hamiltonian colouring. The adjacent vertices are received colours with opposite parity, and hence every edge of the Hamiltonian colour-induced signed graph of P_n receives a negative sign. □

The following proposition gives the number of negative edges in a Hamiltonian colour-induced graph of cycles.

Proposition 22 *Consider the Hamiltonian colour-induced signed graph of cycles C_n. Then,*

$$|E^-(C_n)| = \begin{cases} n-2 & \text{when } n \text{ is even,} \\ n-3 & \text{when } n \text{ is odd.} \end{cases}$$

Proof. Let C_n be a Hamiltonian colour-induced signed cycle. Here, the following cases are to be considered:

Case 1: When n is even, two pairs of adjacent vertices receive the same odd colour and the edges with end vertices having the same colours receives a positive sign. Since there are only two such pairs, the number of positive edges is 2.

Case 2: When n is odd, along with the two pairs of adjacent vertices having the same odd colour, one vertex of either of these pairs is adjacent to a vertex with an odd colour. Hence there are 3 edges with a positive sign. □

Theorem 23 *The Hamiltonian colour-induced signed graphs of cycles are consistent with respect to canonical marking. Furthermore, their switched signed graphs are balanced.*

Proof. Consider the Hamiltonian colour-induced signed graph C_n and its canonically marked graphs. We have the following cases:

Case 1: Let n be even. Then, by Proposition 22, there are only two positive edges. These edges can either be adjacent or non-adjacent. Since we consider the Hamiltonian colouring, the two positive edges are non-adjacent. The $(n-2)$ edges of C_n are negative, and hence the end vertices of the two positive edges receive a negative sign in the canonical marking. Therefore, the number of negative vertices of C_n is even and hence the graph is consistent.

Case 2: Let n be odd. By Proposition 22, the graph consists of three positive edges. Then, we have the following three subcases:
Subcase 2.1: The three positive edges induce a path graph of length three. The end vertices of this path receive negative signs as the other edges incident to these vertices are negative. We know that there are $(n-3)$ negative edges for C_n and hence $(n-3)$ negative vertices in the canonically marked graph, making it consistent.

Subcase 2.2: In this case, the two positive edges are adjacent and one is non-adjacent, that is, we get a path graph of length 2 and 1 which are non-adjacent to each other. Then, the edges at the end of the path receives negative signs.
Subcase 2.3: In this case, the three positive edges are non-adjacent. That is, there exists three paths of length 1 which are non-adjacent to each other and having negative edges at the end vertices of all the three paths.

In all three cases mentioned above, the positive edges contribute to even number of negative vertices which are adjacent to them. Hence, the canonically marked graphs are consistent and switched graphs are balanced. (The pendant vertices of positive paths will receive negative signs.) □

Note that the naturally marked graph of cycles are inconsistent and since the switched graphs of Hamiltonian colour-induced signed graph is positive homogeneous with respect to natural marking, the switched graph is balanced. The Hamiltonian colour-induced signed graph and the switched graph (natural marking) does not preserve sign isomorphism.

Proposition 24 *The Hamiltonian colour-induced signed graph of complete graphs have the following properties:*

(i) $S_H(K_n)$ *is positive homogeneous.*

(ii) $S_H(K_n)$ *is consistent with respect to canonical marking.*

(iii) The switched graphs of $S_H(K_n)$ *are balanced.*

(iv) The $S_H(K_n)$ *and its switched graph are sign-isomorphic.*

Proof. We prove some properties of the Hamiltonian colour-induced signed graph of a complete graph $S_H(K_n)$.

(i) The Hamiltonian chromatic number of complete graphs K_n is 1. Therefore, each vertex of the complete graph receives the same colour. This implies that every edge of the Hamiltonian colour-induced signed graph receives a positive sign.

(ii) Since the graphs are positive homogeneous after canonical marking; the marked graph of $S_H(K_n)$ is positive coherent. Hence every cycle in it is positive, and the graph is consistent.

(iii) $S_H(K_n)$ is positive homogeneous and positive coherent, then the switched graph of $S_H(K_n)$ is positive homogeneous and balanced.

(iv) Since the $S_H(K_n)$ and its switched graph is positive homogeneous; they are sign-isomorphic.

□

The *natural marking* of $\nu : V(S_H) \to \{+, -\}$ of a Hamiltonian colour-induced signed graph is a marking of the signed graph S_H defined by $\nu(v) = (-1)^{|c(v)|}$, for all $v \in V(S_H)$, where c is a Hamiltonian colouring of the underlying graph G.

Proposition 25 *Let* S *be the Hamiltonian colour-induced signed graph of the complete graph* K_n*. Then,*

(i) The naturally marked signed graph of S *is always negative coherent.*

(ii) The naturally marked signed graph of S *is always inconsistent.*

(iii) S *is isomorphic to its switched signed graph.*

Proof.

(i) Let S denote the Hamiltonian colour-induced signed graph of K_n. Each vertex of the naturally marked signed graph of S is negative since the Hamiltonian chromatic number of a complete graph is 1. Hence, the graph is negative coherent.

(ii) The naturally marked signed graph of S is inconsistent as any odd cycle in this graph is inconsistent with an odd number of negative vertices.

(iii) Since S is positive homogeneous and the corresponding naturally marked signed graph is negative coherent, each edge of the corresponding switched signed graph is positive (by taking the product of two negative vertices and a positive edge). Therefore, the switched signed graph is positive homogeneous and hence is isomorphic to S. This completes the proof.

□

Proposition 26 *The Hamiltonian colour-induced signed graph of a star graph $K_{1,(n-1)}$ is negative homogeneous when n is odd. Moreover, when n is even, the signed graph is non-homogeneous with $\frac{n}{2}$ positive edges.*

Proof. Let S denote a Hamiltonian colour-induced signed graph of a star graph $K_{1,n-1}$. Let v be the central vertex of the star graph. Then, for any other vertex u of $K_{1,n-1}$, we have $D(u,v) = 1$. We start the Hamiltonian colouring of the graph by giving the colour 1 to v. That is, $c(v) = 1$.

Let n be odd. As per colouring protocol, we have $|c(u) - c(v)| \geq n - 1 - D(u,v) = n - 2$, and hence $|c(u) - 1| \geq (2\ell - 1)$, where ℓ is a positive integer. Note that the smallest $c(u)$ should be obtained such that $c(u) - 1$ is odd. Therefore, $c(u) = 2\ell = n - 1$, which is even. Therefore, S is negative homogeneous.

Let n be even. Then, we have $|c(u)-1| \geq n-1-D(u,v) = n-2 = 2\ell$, which is also even for any given positive integer ℓ. As mentioned earlier, we have to choose the smallest $c(u)$; we have $c(u) = 2\ell - 1$, which is odd. Therefore, S is positive homogeneous. □

Proposition 27 *The Hamiltonian colour-induced signed graph of wheel graph $S_H(W_{1,n})$ have the following properties:*

(i) $S_H(W_{1,n})$ *is positive homogeneous.*

(ii) The canonically marked graph of $S_H(W_{1,n})$ *is positive coherent and consistent.*

(iii) Switched graph of $S_H(W_{1,n})$ *is positive homogeneous and balanced.*

(iv) The signed graph $S_H(W_{1,n})$ *and switched graph are sign-invariant.*

Proof.

(i) The Hamiltonian chromatic number of wheel graph $W_{1,n}$ is 1 and hence every edge of the Hamiltonian colour-induced signed graph $S_H(W_{1,n})$ receives positive sign. Thus the graph $S_H(W_{1,n})$ is positive homogeneous.

(ii) In canonical marking, each vertex received a sign which is the product of signs of the edges adjacent to it. Since $S_H(W_{1,n})$ is positive homogeneous, every vertex gets a positive sign after canonical marking, and the graph becomes positive coherent. Every cycle in $S_H(W_{1,n})$ is consistent since the graph is positive coherent, and hence the marked graph is consistent.

(iii) The sign of each edge of the switched graph is assigned by the signature function, $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has endpoints v, w. By (i) and (ii), we know that the Hamiltonian colour-induced signed graph is positive homogeneous and the canonically marked graph is positive coherent, respectively. Hence each edge of the switched graph becomes positive. Since the switched graph is positive homogeneous, every cycle in it is positive, and therefore it is balanced.

(iv) From (i) and (iii), the signed graph of $S_H(W_{1,n})$ and its switched graph (with respect to canonical marking) are both positive homogeneous, and hence they are sign-invariant. □

Proposition 28 *Consider the Hamiltonian colour-induced signed graph of wheel graph* $S_H(W_{1,n})$*. The naturally marked graph of* $S_H(W_{1,n})$ *is negative coherent. Moreover, the switched graph of this graph is balanced and also the signed graph* $S_H(W_{1,n})$ *and switched graph are sign-invariant.*

Proof. The Hamiltonian chromatic number of wheel graph $W_{1,n}$ is 1 and according to the definition of the naturally marked graph, that is, the vertices are assigned signs as $(-1)^{\phi(u)}$ where the function ϕ is the vertex labelling, we obtain the vertex signs as all negative, making the graphs negative coherent. □

4 Conclusion

The major research areas in mathematics have gone wide, especially in the field of graph colouring and signed graphs. Signed graphs have many applications in social network analysis, data clustering, neuroscience and software engineering. In this paper, we have introduced the concept of equitable and Hamiltonian colour-induced signed graphs. So far, we have discussed the properties such as balance, consistency, clusterability, homogeneity, and sign compatibility of these colour-induced signed graphs for different graph classes. We have also found out some properties of switched graphs of these signed graphs with respect to canonical as well as natural marking.

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References

[1] A. Aniyan, S. Naduvath, Induced signed graph of some classes of graphs, *Proc. Jangjeon Math. Soc.*, **23**(2020), pp. 283-291, doi: 10.17777/pjms2020.23.2.283. ⇒340

[2] A. Aniyan, S. Naduvath, On degree product induced signed graphs of graphs, *AIP Conference Proceedings*, **1**(2020), pp. 030018, doi:10.1063/5.0019097 ⇒340

[3] G. Chartrand, L. Nebeskỳ, P. Zhang, Hamiltonian colorings of graphs, *Discrete Appl. Math.*, **146**(2005), pp. 257-272, doi:10.1016/j.dam.2004.08.007 ⇒340

[4] Y. Chen, J.L. Gross, T. Mansour, Total embedding distributions of circular ladders, *J. Graph Theory*, **74**(2013), pp. 32-57, doi:10.1016/j.disc.2011.07.020 ⇒343

[5] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, **2**(1953), pp. 143-146, doi: 10.1307/mmj/1028989917 ⇒339, 341

[6] S. Naduvath, Some new results on sumset-labelled signed graphs, *communicated.*, 2021. ⇒339

[7] M. Noy, A. Ribó, Recursively constructible families of graphs, *Adv. Appl. Math.*, **32**(2004), 350-363, doi:10.1016/S0196-8858(03)00088-5 ⇒342

[8] E. Sampathkumar, *Point signed and line signed graphs*, **7**(1984), pp. 91-93. ⇒340

[9] E. Sampathkumar, M. A. Sriraj, L. Pushpalatha, Notions of balance in signed and marked graphs, *Indian J. Discrete Math.*, **3** (2017) 25–32. ⇒340

[10] D. Sinha, A. Dhama, Sign-compatibility of some derived signed graphs, *Indian J. Math.*, **55**(2013), pp. 41-55, doi:10.12723/mjs.23.1 ⇒339

[11] N. K. Sudev, K. P. Chithra, S. Satheesh, J. Kok, On certain parameters of equitable coloring of graphs, *Discrete Math. Algorithms Appl.*, **9**(2017), pp.1750054, doi:10.1142/S1793830917500549 ⇒340

[12] D.B. West, *Introduction to graph theory*, Prentics Hall of India, New Delhi, 2001. ⇒340

[13] T. Zaslavsky, Signed graphs, *Discrete Appl. Math.*, **4**(1982), pp. 47-74, doi: 10.1016/0166-218X(82)90033-6 ⇒339, 340

[14] T. Zaslavsky, Signed graphs and geometry, *J. Combin. Inform. System Sci.*, **1**(2013), pp. 6-7, doi:10.48550/arXiv.1303.2770 ⇒339, 340

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