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Some properties of the closed global shadow graphs and their zero forcing number

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Abstract. Zero forcing is one of the dynamic vertex coloring problem. Zero forcing number is the minimum cardinality of the zero forcing sets. This parameter is the upper bound for the maximum nullity. A new class of graph where the maximum nullity is equal to the zero forcing number of the graph is defined as closed global shadow graph. Basic properties and zero forcing number of this graph class is analysed.

1 Introduction

All graphs considered in this article are finite, undirected and simple. A graph is a pair $G = (V, E)$. The set V or $V(G)$ is called the vertex set and E or $E(G)$ is called the edge set. $E(G) = \{(u, v) \mid u, v \in V(G) \text{ and } u \neq v\}$. Two vertices are said to be adjacent to each other if there exists an edge between them. If u and v are adjacent vertices in G then we represent this as $u \sim v$.

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A vertex is said to be a neighbour of other if they are adjacent to each other. Open neighbourhood of an arbitrary vertex v in the graph G is the set $N(v)$ containing all the vertices that are adjacent to v. Closed neighbourhood of an arbitrary vertex v is the set $N[v]$ containing the vertices in the open neighbour set $N(v)$ and the vertex v. Degree of the vertex v in the graph G is the number of edges incident to v. The minimum degree among the vertices of a graph G is represented by $\delta(G)$ and the maximum degree among the vertices of a graph G is represented by $\Delta(G)$.

A shadow graph of G is obtained by taking a graph and a copy of it say G and G'. Then making all the neighbouring vertices of u' in G' adjacent to the vertex u in G [17]. Motivated by the definition of shadow graphs the concept of open global shadow graphs were introduced in [14]. In this paper, we introduce a class of graphs which is closely related to the open global shadow graph and is known as the closed global shadow graph. Let G be a graph and G' be a copy of G such that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(G') = \{v'_1, v'_2, \ldots, v'_n\}$. The closed global shadow graph denoted by $GS[G]$ is obtained by taking two copies of G say, G and G' and joining the vertex v_i to each of the vertex in $\{V(G') \setminus N(v'_i)\}$, where $1 \leq i \leq n$.

The closed global shadow graph of the cycle C_5 is depicted in the figure 1. It is evident that C_5 is a graph having vertex set $\{v_1, v_2, \ldots, v_5\}$ and the copy of the graph C_5 that is C'_5 has the vertex set $\{v'_1, v'_2, \ldots, v'_5\}$. The vertex $v_i \in V(C_5)$ is adjacent to each of the vertex in $\{C'_5 \setminus N(v'_i)\}$, where i takes value between 1 to 5.

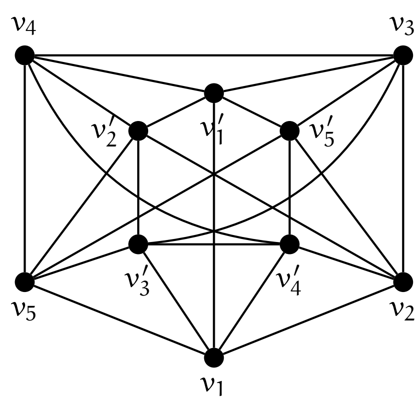


Figure 1: Closed global shadow graph of C_5: $GS[C_5]$

We use the following definitions for the further development of this article.

- Zero forcing set $S \subseteq V(G)$, is a set of black vertices which forces the entire graph black based on the following color change rule.
- Color change rule: A black vertex can force at most one white vertex black, provided it is the only white neighbour of it.
- The derived coloring of a graph G is the result of applying the color-change rule until no more changes are possible.
- Zero forcing number is a minimization problem. The zero forcing number of a graph is the minimum cardinality of the zero forcing set.
- For n number of vertices, the total number of maximum possible edges are $\frac{n(n-1)}{2}$. Let G be a graph, and let u, v be any two vertices which are not adjacent in G. Then we call uv as the missing edge in G.

For basic definitions related to graphs we refer to [18]. The zero forcing was initially introduced independently by AIM work group to bound the minimum rank [1]. Burgarth and Giovannetti introduced the zero forcing to understand the controlability of quantum system [6]. Zero forcing number of different type of graphs are studied in [8, 9, 13, 11]. Since the introduction of zero forcing it has been used in many areas like physics, disease and information spreading model in social network,logic circuit, coding theory and power network monitoring [4, 6, 7, 19, 12, 16].

The following section is intended to discuss about some basic properties of the closed global shadow graph of a given graph G.

2 Results on the closed global shadow graph of a graph

The closed global shadow graph is obtained by taking two copies of the given graph G with n vertices each. Therefore the following observation is obvious.

Observation 1 *Let G be a simple graph of order n. Then the total number of vertices in a closed global shadow graph is twice the number of vertices of the graph G.*

The next theorem gives the total number of edges of the closed global shadow graph of the given graph G with t number edges.

Theorem 2 *Let G be a simple graph of order n. Then $|E(GS[G])| = n^2$.*

Proof. Let t be the total edges present in G, and since G' is a copy of G $|E(G')| = t$. However, the total possible edges for an n vertex simple graph is

$\frac{n(n-1)}{2}$. Let v be a vertex in G and v' be its corresponding vertex in G'. By the definition of closed global shadow graph we know that each vertex v in G is adjacent to the corresponding vertex of its non-neighbours and the vertex v'. Similarly each vertex u' in G' is adjacent to the corresponding vertex of its non-neighbours and the vertex u'.

$$|E(GS[G])| = t + t + n + 2(\frac{n(n-1)}{2} - t) = n(n-1) + n = n^2$$

□

Theorem 3 *Let* G *be a simple graph of order* n*. Then the closed global shadow graph of* G*,* $GS[G]$ *is a connected* n *regular graph.*

Proof. To prove that $GS[G]$ **is a connected graph:** Let G be a connected graph. The graph $GS[G]$ will have the graph G and copy of G that is G' and the edges connecting each vertex in G to its corresponding vertices in G'. since G and G' are connected, the closed global shadow graph $GS[G]$ is connected.

Let G be a disconnected graph. Let u and v be the vertices in two different components of G. Clearly v is not adjacent to u in G, then v is made adjacent to u' and u is made adjacent to v' in $GS[G]$. Hence for each missing edge uv in G, there exists a path u, u', v or v, v', u between u, v in $GS[G]$. Due to which $GS[G]$ cannot be disconnected.

To prove that $GS[G]$ **is an** n **regular graph:** By the definition of closed global shadow graph each of the vertices v will be adjacent to $N(v)$ and the corresponding vertices of $V(G) \setminus N(v)$ that is $V(G') \setminus N(v')$. Which make the total degree of v as n. Similarly each vertex u' will be adjacent to $N(u')$ and the corresponding vertices of $V(G') \setminus N(u')$ that is $V(G) \setminus N(u)$. Making the total degree of u' as n. Therefore it is clear that in $GS[G]$, degree of each of the vertices is n. Hence $GS[G]$ is an n-regular graph. □

Theorem 4 *Let* G *be a simple graph of order* $n > 1$*. Then the closed global shadow graph of graph* G *has no cut edge or cut vertex.*

Proof. Case 1 Let us first prove that $GS[G]$ has no cut edge. On contrary let us assume that $GS[G]$ has a cut edge for $n > 1$.

Subcase 1.1 Assume that G has no cut edge. Clearly removal of any edge $E(G)$ or $E(G')$ cannot disconnected the graph $GS[G]$. According to the definition of closed global shadow graph, $GS[G]$ will have all the vertices in G

adjacent to their corresponding vertices in G'. Hence removal of any edge between $V(G)$ and $V(G')$ still keeps the graph $GS[G]$ connected. Therefore a contradiction.

Subcase 1.2 Assume that G has a cut edge let uv be the cut edge. Now removal of the edge uv from the graph G will lead to at least two disconnected components. In the graph $GS[G]$, the vertex u is adjacent to the vertex u', similarly the vertex v is adjacent to the vertex v'. Since there exists edge uv in $GS[G]$, u' and v' will also have an edge between them. Clearly removal of any edge uv will not disconnected the graph $GS[G]$. The only possibility for a cut edge to exists is in between G and G' but since all the vertices of G are adjacent to its corresponding vertices in G', the graph is connected even after the removal of the edge vv'. Hence a contradiction.

Case 2 Let us prove that $GS[G]$ has no cut vertex.

Subcase 2.1 Suppose G is a graph with no cut vertex. This implies that G' has no cut vertex. Clearly for each missing vertex in the graph G a pair of edges are added in $GS[G]$, hence the graph $GS[G]$ doesn't have cut vertex.

Subcase 2.2 Suppose G is a graph with a cut vertex. Let v be a cut vertex in G and K and H be the components of the graph $G - v$. Let K' and H' be the graphs corresponding to K and H respectively in $GS[G]$. Clearly in $GS[G]$ all the vertices in the component H will be adjacent to all the vertices in K' similarly all the vertices in the component K are made adjacent to all the vertices in H'. Therefore we cannot find any cut vertex. □

Theorem 5 *Every pair of vertices in a simple graph G, induces a cycle C_4 in $GS[G]$ with their corresponding vertices in G'.*

Proof. Let G be any simple graph, v and u be any two vertices in G. Then for v and u there are two possibilities that is $v \sim u$ or $v \nsim u$.

Case 1 Assume that v and u are adjacent in G ($v \sim u$). In the graph $GS[G]$, v is adjacent to v', u is adjacent to u'. Since G' is a copy of G, v' and u' are also adjacent in $GS[G]$. Therefore, the vertices v, v', u' and u forms a cycle C_4 in $GS[G]$.

Case 2 Assume that v and u are not adjacent in G ($v \nsim u$). Clearly by the definition of closed global shadow graph, v is adjacent to u' in $GS[G]$ as v and u are not adjacent in G. Similarly u is adjacent to v'. The vertices u, u' and v, v' are adjacent in $GS[G]$, since u' and v' are the vertices corresponding to u and v in $GS[G]$. Hence the vertices v, u', u and v' forms a cycle C_4 in $GS[G]$. □

Theorem 6 [15] *Subgraph of a bipartite graph is bipartite.*

Theorem 7 *Let* G *be a simple graph of order* n*. Then* $GS[G]$ *is a complete bipartite graph* $K_{n,n}$ *if and only if either* G *is a null graph of order* n *or* G *is a complete bipartite graph of order* n*.*

Proof. Let G be a null graph of order n. All the vertices of graph G forms an independent set. In the closed global shadow graph of G, $\forall\ v \in V(G)$, v is adjacent to all the vertices in $V(G')$ and $\forall\ u' \in V(G')$, the vertex u' is adjacent to all the vertices in the set $V(G)$. This makes the graph $GS[G]$ as a complete bipartite graph.

Let graph G is a complete bipartite graph $K_{p,q}$ such that $p + q = n$. Let P and Q be the partite sets having p and q number of vertices respectively. Clearly the subgraph induced by the vertices in set P and Q independently forms a null subgraphs. In the closed global shadow graph of the graph $K_{p,q}$, each of the vertices in the set P are adjacent to all the vertices in the set P'. Similarly each of the vertices in the set Q are adjacent to all the vertices in the set Q'. Where P' and Q' are the vertex set of the partite set of G'. Clearly the set P and Q' forms one of the partite set and the set Q and P' forms another partite set of the complete bipartite graph $K_{n,n}$.

To prove the converse part let us assume the contrary, suppose there is a graph G other than the null graph and complete bipartite graph for which $GS[G]$ forms a complete bipartite graph. It can be seen that the graph G is a subgraph of the graph $GS[G]$ with $V(G)$ as the vertex set. The theorem 6 shows that the subgraph induced by a complete bipartite graph is either a null graph or complete bipartite graph. Hence a contradiction. □

Definition 8 *A graph is said to be hamiltonian, if there exist a closed walk such that all the vertices are in the walk and an edge is visited only once.*

Theorem 9 [10] *If* G *is a simple graph of order* $n \geq 3$ *and the degree of every vertex in* G *is greater than or equal to* $\frac{n}{2}$*, then* G *is Hamiltonian.*

Theorem 10 *Every closed global shadow graph of a graph is Hamiltonian.*

Proof. Total number of vertices in the graph $GS[G]$ is $2n$, where n is the order of graph G. Further from theorem 3, $GS[G]$ is a n regular graph. Meaning every vertex in $GS[G]$ has a degree $d(v) = d(v') = n \geq \frac{2n}{2}$, $\forall v \in V(G)$ and $v' \in V(G')$. From dirac's theorem (theorem 9) it can be seen that $GS[G]$ is Hamiltonian. □

Definition 11 *A dominating set is a set of vertices* $\mathsf{D} \in \mathsf{V}(\mathsf{G})$ *such that a vertex not in the set* D *is adjacent to at least one vertex in the set* D.
The minimum cardinality among the dominating set is called the domination number and is denoted by $\gamma(\mathsf{G})$.

Definition 12 *If subgraph formed by the dominating set is connected, then such a set is called the connected dominating set.*
The minimum cardinality among the connected dominating set is called the connected domination number and is denoted by $\gamma_c \mathsf{G}$.

Theorem 13 *Let* $\mathsf{GS}[\mathsf{G}]$ *be a simple graph of order n. Then the domination and connected domination number is respectively given by* $\gamma(\mathsf{GS}[\mathsf{G}]) = \gamma_c(\mathsf{GS}[\mathsf{G}]) = 2$.

Proof. Any two vertex ν and ν', ($\nu \in \mathsf{V}(\mathsf{G})$ and $\nu' \in \mathsf{V}(\mathsf{G}')$) can dominate the entire graph. That is ν can dominate $\mathsf{N}(\nu)$ and $\mathsf{V}(\mathsf{G}') \setminus \mathsf{N}(\nu')$. Similarly ν' can dominated $\mathsf{N}(\nu')$ and $\mathsf{V}(\mathsf{G}) \setminus \mathsf{N}(\nu)$. Also it is clear that any one vertex in $\mathsf{GS}[\mathsf{G}]$ is not sufficient to force the entire graph $\mathsf{GS}[\mathsf{G}]$. Hence $\gamma(\mathsf{G}) = 2$. Also the subgraph induced by ν and ν' are connected $\gamma(\mathsf{G}) = \gamma_c(\mathsf{G}) = 2$. □

Definition 14 *Matching in a graph is a set of edges such that no two edges in the set are incident to the same vertex.*
Perfect matching in a graph is a matching that matches all the vertices in the graph.

Theorem 15 *The closed global shadow graph has perfect matching.*

Proof. In the closed global shadow graph all the vertices are adjacent to their corresponding vertices. Hence each of the edge $\nu\nu'$ ($\forall \nu \in \mathsf{V}(\mathsf{G})$ and $\forall \nu' \in \mathsf{V}(\mathsf{G}')$) form a perfect matching. □

3 Zero forcing number of closed global shadow graph

In this section we find the zero forcing number of closed global shadow graph and give some upper bounds. Also we provide the relation between the chromatic number and zero forcing number of the closed global shadow graph.

Theorem 16 [5] *The zero forcing number of any graph* G *is given by* $\mathsf{Z}(\mathsf{G}) \geq \delta(\mathsf{G})$.

Theorem 17 *The zero forcing number of closed global shadow graph is bound by the order of graph* G*,* $Z(GS[G]) \geq n$*.*

Proof. From theorem 16 we know that $Z(G) \geq \delta$. It can be seen from theorem 3 that $\delta(GS[G]) = n$. Hence $Z(GS[G]) \geq n$. □

Theorem 18 [3] *The zero forcing number of a graph* G *of order n is bound by* Δ *as,* $Z(G) \leq \frac{\Delta}{\Delta+1}n$*.*

Theorem 19 *Let* G *be a simple graph of order* $n \geq 2$ *and* $GS[G]$ *be its closed global shadow graph of order* $2n$*. Then the zero forcing number of* $GS[G]$ *is given by* $2 \leq Z(GS[G]) \leq 2n-2$*.*

Proof. From theorem 17, $Z(GS[G]) \geq n$. When the order of G is 2, $Z(GS[G]) \geq 2$. On the other hand the upper bound can be found by using theorem 18.

$$Z(G) \leq \frac{\Delta}{\Delta+1}n$$

$$Z(GS[G]) \leq \frac{n}{n+1}2n$$

$$Z(GS[G]) \leq \frac{2n^2}{n+1}$$

The above equation can be factorised as

$$\frac{2n^2}{n+1} = 2n - 2 + \frac{2}{n+1}.$$

Since $n \geq 2$, $\frac{2}{n+1}$ is never a whole number. Hence $2 \leq Z(GS[G]) \leq 2n-2$. □

Theorem 20 *Let* G *be a simple graph with two connected components* K_m, K_n*, that is* G *is isomorphic to* $K_m \cup K_n$*. Then the zero forcing number* $Z(GS[G]) = m+n$*.*

Proof. From theorem 17 it is known that $Z(GS[G]) \geq m+n$. It is left to show that $Z(GS[G]) \leq m+n$. Let G be the graph with two components K_m and K_n as cliques. Similarly let G' be the graph with two components K'_m and K'_n as cliques. In $GS[G]$, all the vertices of K_m are adjacent to all the vertices of K'_n and all the vertices of K_n are adjacent to all the vertices in K'_m. By taking $V(K_m)$ and $V(K'_n)$ as the black vertices each vertex in K_m can force its corresponding vertex in K'_m. Similarly all vertices in K'_n can force its corresponding vertex in K_n as black. Thereby forcing the entire graph black. This implies that $Z(GS[G]) \leq m+n$. □

Theorem 21 *Let* G *be the complete graph* K_n *of order* $n \geq 2$*. Then* $Z(GS[G]) = n$*.*

Proof. If G is the complete graph, then closed global shadow graph of G, $GS[G]$ contains two copies of K_n and the corresponding vertices in each copy are adjacent. By taking all the n vertices of G in $GS[G]$ as black we can force the entire graph $GS[G]$ as black. Once all the n vertices of G are taken black, each black vertex is left with exactly one corresponding white vertex which can be forced black. Hence $Z(GS[G]) \leq n$. From theorem 17 we know that $Z(GS[G]) \geq n$. Hence the proof. □

Theorem 22 [13] *The zero forcing number of complete bipartite graph is given as* $Z(K_{m,n}) = n + m - 2$*.*

Theorem 23 *The zero forcing number* $Z(GS[G]) = 2n-2$ *if and only if* $GS[G]$ *is the complete bipartite graph* $K_{n,n}$*, where* n *is the number of vertices in* G*.*

Proof. If $GS[G]$ is a complete bipartite graph, then according to theorem 22 the zero forcing number, $Z(GS[G]) = n + n - 2 = 2n - 2$.

When $Z(GS[G]) = 2n - 2$, we need to show that the graph $GS[G]$ is the complete bipartite $K_{n,n}$. If there are just 2 vertices in G, it can be seen in figure 2 that $Z(GS[G]) = 2$ and both the graph are complete bipartite. Hence the theorem is true when $n = 2$.

For graph G with more than 2 vertices, let us assume that $Z(GS[G]) = 2n-2$ but $GS[G]$ is not a complete bipartite graph.

Claim *For a connected graph of order* $n \geq 2$*, the only bipartite closed global shadow graph of* G *is the complete bipartite graph.*

Proof of the Claim Assume that $GS[G]$ is a bipartite graph but not a complete bipartite graph. Since the subgraph of a bipartite graph is a bipartite graph, the graph G which is the subgraph of $GS[G]$ is also a bipartite graph. If G is a complete bipartite graph, then $GS[G]$ is also a complete bipartite graph from theorem 7, this is a contradiction.

Hence G is a bipartite graph but not a complete bipartite graph. If K and H are the two partite set of the graph G then there exist a vertex $v \in K$ and $u \in H$, such that $v \nsim u$. Let K' and H' be the partite set of G' and there exist a vertex $v' \in K'$ and $u' \in H'$ such that $v' \nsim u'$. Clearly in $GS[G]$, $v \sim u'$, $u \sim v'$. Since $v \sim v'$ and $u \sim u'$. The only possibility to divide these four vertices into two partite set is by taking G and G' as the two partite sets. However G and G' are not null graph. Hence a contradiction.

Now it is evident that $GS[G]$ is not a bipartite graph. Let v_i, v_j and v_k be three arbitrary vertices in $V(GS[G])$.

Case 1 If the vertex v_i is not adjacent to the vertex v_j, both the vertices v_i and v_j are adjacent to the vertex v_k. Clearly the induced graph formed by vertices $v_i, v_j, v_k, v_i', v_j'$ and v_k' forms a complete bipartite graph with v_i, v_j, v_k' in one partite set and v_i', v_j', v_k in other partite set. Since $GS[G]$ is not a complete bipartite graph, there exist v_t such that v_t is adjacent to v_i and not adjacent to v_j. Now by taking v_i as the initial black vertex along with $n-1$ of its neighbour v_i can force remaining one white neighbour black. From the above construction clearly v_t is adjacent to at least two black vertices (v_i and v_j') so clearly we need to choose at most $n-3$ of its neighbours to force the remaining white neighbour of v_t black. At this stage either the entire graph is forced black with at most $1+n-1+n-3=2n-3$ black vertices or the forcing process continues. If the forcing process continues, then there will be at least one more white vertex which gets forced hence there will be at most $2n-3$ black vertices, so the zero forcing number will be at most $2n-3$. Hence a contradiction.

Case 2 If the vertex v_i is not adjacent to the vertex v_j, the vertex v_i is adjacent to the vertex v_k and the vertex v_j is not adjacent to the vertex v_k. Now consider v_i and $n-1$ neighbours of v_i to be black, so that v_i can force the remaining one white neighbour of v_i black. From the construction v_k is adjacent to at least 2 black vertices v_i and v_j'. By taking at most $n-3$ of its neighbours black the remaining white neighbour of v_k can be forced. At this stage either the entire graph is forced black with at most $1+n-1+n-3=2n-3$ black vertices or the forcing process continues. If the forcing process continues there there will be at least one more white vertex which gets forced hence there will be at most $2n-3$ black vertices, so the zero forcing number will be at most $2n-3$. Hence a contradiction.

Case 3 If the vertex v_i is adjacent to the vertex v_j, both the vertices v_i and v_j are adjacent to the vertex v_k. Now consider v_i and $n-1$ neighbours of v_i to be black so that v_i can force the remaining one white vertex black. From the construction v_k is adjacent to at least 2 black vertices v_i and v_j. By taking at most $n-3$ of its neighbours black the remaining white neighbour of v_k can be forced. At this stage either the entire graph is forced black with at most $1+n-1+n-3=2n-3$ black vertices or the forcing process continues. If the forcing process continues there there will be at least one more white vertex which gets forced hence there will be at most $2n-3$ black vertices, so the zero forcing number will be at most $2n-3$. Hence a contradiction.

Case 4 If the vertex ν_i is adjacent to the vertex ν_j, both the vertices ν_i and ν_j are not adjacent to the vertex ν_k. Clearly the induced graph formed by vertices $\nu_i, \nu_j, \nu_k, \nu_i', \nu_j'$ and ν_k' forms a complete bipartite graph with the vertices ν_i, ν_j', ν_k' in one partite set and the vertices ν_i', ν_j, ν_k in other partite set. Since $GS[G]$ is not a complete bipartite graph, there exist the vertex ν_t such that ν_t is adjacent to ν_i and not adjacent to ν_j'. Now by taking ν_i as the initial black vertex along with $n-1$ of its neighbour ν_i can force remaining one white neighbour black. From the above construction clearly ν_t is adjacent to at least two black vertices (ν_i and ν_j) so clearly we need to choose at most $n-3$ of its neighbours to force the remaining white neighbour of ν_t. At this stage either the entire graph is forced black with at most $1+n-1+n-3=2n-3$ black vertices or the forcing process continues. If the forcing process continues there there will be at least one more white vertex which gets forced hence there will be at most $2n-3$ black vertices, so the zero forcing number will be at most $2n-3$. Hence a contradiction. □

Theorem 24 *If G is a path P_n where $n > 2$, then the zero forcing number of closed global shadow graph $Z(GS[P_n]) = n+1$.*

Proof. From theorem 17 we know that $Z(GS[G]) \geq n$. We need to show that n initially colored black vertices are not sufficient to force the whole graph black. Let G be a path P_n with vertex set $\{\nu_1, \nu_2, \ldots, \nu_n\}$. Without loss of generality let vertex ν_1 and ν_n be the two vertices in the graph P_n having degree one and remaining ν_i, $2 \leq i \leq n-1$ be $n-2$ vertices in the graph P_n having degree 2.

Let G' be the path P_n' with vertex set $\{\nu_1', \nu_2', \ldots, \nu_n'\}$. Without loss of generality let ν_1' and ν_n' be two vertices in the graph P_n' having degree one and remaining ν_j', $2 \leq j \leq n-1$ be $n-2$ vertices in the graph P_n' having degree 2.

Case 1 When a degree 1 vertex in P_n is taken initially black:

Let ν_1 be the initially colored black vertex. In $GS[P_n]$, vertex ν_1 is adjacent to ν_2, ν_1', ν_j', where $3 \leq j \leq n$. That is $\mid N(\nu_1) \mid = n$, by taking $n-1$ of its neighbours to be initially black ν_1 can force the remaining one white neighbour black. The forcing process stops as the black vertices ν_1', ν_j', where $3 \leq j \leq n$ have more than two white neighbours and vertex ν_2 has exactly two white neighbours. Therefore it is not possible to force the entire graph black by taking a degree 1 vertex in P_n as initially black.

Case 2 When a degree 1 vertex in P_n' is taken initially black. Same as Case 1 is followed.

Case 3 When a degree 2 vertex in P_n is taken initially black:

Let ν_2 be the initially colored black vertex. In $GS[P_n]$, ν_2 is adjacent to $\nu_1, \nu_3, \nu_2', \nu_j'$ where where $4 \leq j \leq n$. Totally ν_2 has n neighbours, by taking

$n-1$ of its neighbour to be initially black v_2 can force the remaining one white neighbour black. The forcing process stops as v_1 has two white neighbours, v_3 has three white neighbour (in case when G is P_3, v_3 will have 2 white neighbours) and v_2', v_j', where $4 \leq j \leq n$ have two or more white neighbours. Therefore it is not possible to force the entire graph black by taking a degree 2 vertex in P_n as initially black.

Case 4 When a degree 2 vertex in P_n' is taken initially black. Same as Case 3 is followed.

Clearly from the above cases we can conclude that $Z(GS[G]) > n$. Now we are left to show that $Z(GS[G]) \leq n+1$.

Consider all the vertices in G' and one of the end vertex in G (v_1 or v_n) say v_1 to be black. Then v_1 can force v_2 black, v_2 can force v_3 black, ... this process continues till all the vertices are forced black. Hence $Z(GS[G]) \leq n+1$. □

In the above theorem when $n = 2$, that is when graph G is P_2. $GS[G]$ is a cycle C_4 and $Z(GS[G]) = Z(C_4) = 2$.

Theorem 25 *If G is a cycle C_n, then the zero forcing number of closed global shadow graph of C_n is*

$$Z(GS[G]) = \begin{cases} 3 & \text{if } n = 3 \\ 6 & \text{if } n = 5 \\ n+2 & \text{if } n = 4 \text{ and } n > 5 \end{cases}$$

.

Proof. Let the graph G be a cycle C_n with n number of vertices. When $n = 3$, C_3 is same as the complete graph K_3. Hence from theorem 21, 3 black vertices are enough to force the entire graph black. When $n = 4$, C_4 forms a complete bipartite graph according to theorem 7 and theorem 23 $Z(GS[C_4]) = 4+2 = 6$. When $n = 5$, let the graph $GS[C_5]$ have vertex set $V(GS[G]) = \{V(G), V(G')\}$ ($V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V(G') = \{v_1', v_2', v_3', v_4', v_5'\}$). By taking v_1 and 4 of its neighbours say v_2, v_5, v_3', v_4' as black v_1 can force v_1' black. Clearly, all the black vertices have 2 white neighbours. we know from theorem 17 $Z(GS[G]) \geq n$, but here with 5 black vertices it is not possible to force the entire graph black. By taking one more black vertex say v_5', then v_5' can force v_3 and v_3' can force v_2' and further v_3 can force v_4 black. There by forcing the entire graph black.

It is clear from the theorem 17 that $Z(GS[G]) \geq n$. Now we need to show that with $n+1$ black vertices it is not possible to force the entire graph black when $n > 5$. On contrary let us assume that $n+1$ vertices are enough to force the whole graph $GS[G]$ black. Since G or G' is a regular graph(cycle C_n where $n \geq 6$), choosing any vertex as the initial black vertex makes no difference. Let v_1 be the initial black vertex. Now, v_1 is adjacent to n other vertices of $GS[G]$ (v_2, v_n, v_1', v_j' where $3 \leq j \leq n-1$). By choosing any of the $n-1$ neighbours black the remaining white neighbour of v_1 can be forced black.

Clearly v_2 and v_n have three white neighbours (v_2', v_n' and v_3 or v_{n-1} respectively). Hence we need to select at least two of them as black in order for v_2 or v_n to continue forcing, a contradiction.

v_1' has $n-1$ black neighbours this implies that $n-2$ of its neighbours should be taken as black, contradiction.

Finally for the black vertices v_j',$3 \leq j \leq n-1$ either $n-3$ or $n-5$ white vertices are left.

Case 1 If $n-3$ white vertices are left it is evident that $n-3 > 2$ for all $n \geq 6$. Hence the forcing process stops.

Case 2 If $n-5$ white vertices are left, then the following subcases follows

Subcase 2.1 When $n = 6$, $n-5$ is one hence this vertex v_j' (in particular v_4') can force its only white neighbour (v_4) black. However at this stage the black vertex has two or more white neighbour. That is v_4 has v_3, v_5, v_2' and v_n' as its white neighbours Hence the process stops.

Subcase 2.2 When $n = 7$, $n-5$ is two this vertex v_j' (in particular v_4' or v_5') having $n-5 = 7-5 = 2$ white vertex, can force one of the white neighbour black by taking the other white neighbour as initially black. Say v_4 and v_3 are the white neighbours of v_4', by taking one of them black other vertex can be forced black. Now clearly any black vertices has either no white neighbour or two or more white neighbour. Hence the process stops.

Subcase 2.3 It can be observed that for $n > 7$, $n-5$ take values more than 2. Hence it becomes a contradiction to our assumptions.

Form the above cases we can conclude that we need at least $n+2$ initial black vertices in order to force the entire graph black.

In the graph $GS[G]$, by taking all the n vertex of G' to be black, the graph $GS[G]$ reduces to C_n. In other words by taking all the n vertices of G' to be black, the only white vertices left is from G. Hence by taking two out of n vertices in G whole graph $GS[G]$ can be forced black. This forcing process shows that $Z(GS[G]) \leq n+2$. □

Definition 26 *Join of two graphs* K *and* H *is the graph obtained by taking a copy of* K *and a copy of* H *and making each of the vertices in the graph* K *to be adjacent to each of the vertices in the graph* H. *The join of* K *and* H *is denoted by* $K+H$.

Theorem 27 *Let* G *be an* n *order simple graph and* G^1 *be the join of* G *and* K_1 *that is* $G^1 = G + K_1$. *Then* $Z(GS[G^1]) \geq Z(GS[G]) + 1$.

Proof. Let G be a graph of order n with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $Z(GS[G]) = z$ and the zero forcing set be S. Clearly G^1 is a graph of order $n + 1$ such that v_{n+1} is adjacent to all the other n vertices. In $GS[G^1]$ all the vertices in S will be adjacent to either v_{n+1} or v'_{n+1}. That is degree of each of the vertices in $GS[G]$ is increased by one. $Z(GS[G^1])$ cannot be z as every vertex in the set S will have at least 2 white neighbours. Thereby increasing the zero forcing number of $GS[G^1]$, $Z(GS[G^1]) \geq Z(GS[G]) + 1$. □

Theorem 28 *Let* G *be a wheel graph* $W_{1,n}$. *Then the zero forcing number of closed global shadow graph of* $W_{1,n}$ *is*

$$Z(GS[G]) = \begin{cases} 4 & \textit{if } n = 3 \\ n+3 & \textit{if } n \geq 4. \end{cases}$$

Proof. Wheel graph $W_{1,n}$ is obtained by adding a central vertex to a cycle which is adjacent to all the other vertices of the graph C_n. When $n = 3$, the wheel graph is similar to that of a complete graph K_4 hence by theorem 21 the zero forcing number is 4. When $n \geq 4$, by taking all the vertices in the set $V(W_{1,n})$ and two of the vertices in the set $V(W'_{1,n})$ other than v'_{n+1} to be initially black, the graph $GS[W_{1,n}]$ can be completely forced black. Hence $Z(GS[G]) \leq n + 3$.
Clearly when $n = 4$ and $n > 5$, from the theorem 27, we know that

$$Z(GS[W_{1,n}]) \geq Z(GS[C_n]) + 1 = n + 2 + 1 = n + 3$$

For graph when $n = 5$, $Z(GS[W_{1,5}]) \geq Z(GS[C_5]) + 1 = 6 + 1 = 7$. But it can be shown that with 7 black vertices it is not sufficient to force the entire graph black. Let $v_1, v_2, \ldots, v_6$ be the vertices of the graph $W_{1,5}$ such that v_6 is the central vertex. In $GS[W_{1,5}]$, by taking v_1 and 5 of its neighbours black, the remaining one white neighbour of v_1 is forced black. Further the forcing stops as all the black vertices except v_1 that is $v'_1, v_2, v_5, v_6, v'_3, v'_4$ have three white neighbour. In order for any of $v'_1, v_2, v_5, v_6, v'_3$ or v'_4 vertices to force their neighbour, two of their neighbours must be taken initially black. There by forcing the entire graph black. It is clear that with 7 black vertices it is not possible to force the graph black. Hence $Z(GS[W_{1,5}]) = 8 = n + 3$. □

Theorem 29 *The zero forcing number* $Z(GS[G]) = 2$ *if and only if* G *is either* K_2 *or* $\bar{K}_2$.

Proof. If G is either K_2 or $\bar{K_2}$, then $GS[G]$ is cycle C_4.We know that the zero forcing number of cycle is 2. Hence $Z(GS[K_2]) = Z(GS[\bar{K_2}]) = Z(C_4) = 2$.

If $Z(GS[G]) = 2$, from theorem 17 it evident that $n = 1$ or 2. Clearly when $n = 1$, $GS[G]$ is P_2 implies $Z(GS[G]) = Z(P_2) = 1$. Hence G is either K_2 or $\bar{K_2}$. In both the cases $GS[G]$ is a cycle C_4 as shown in the figure 2. Hence $Z(GS[G]) = 2$. □

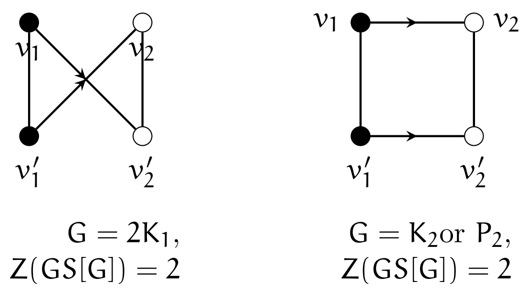


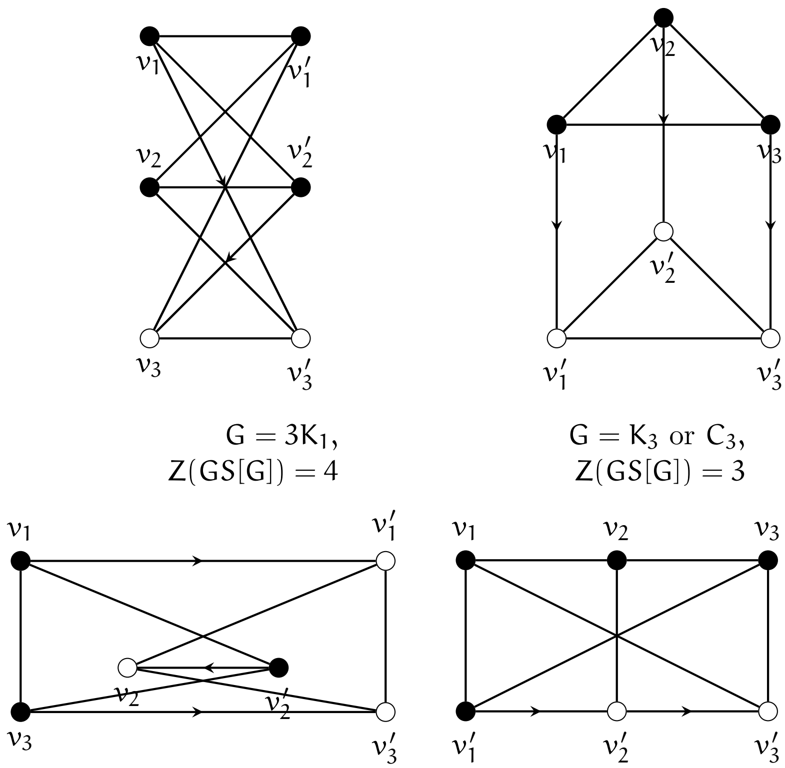
Figure 2: All the possible closed global shadow graph of graph when n=2.

Theorem 30 *The zero forcing number* $Z(GS[G]) = 3$ *if and only if* G *is either* K_3 *or* $\bar{P_3}$.

Proof. When G is K_3, from theorem 21 it can be concluded that $Z(GS[K_3]) = 3$. When G is $\bar{P_3}$, the forcing process is depicted in the figure 3 (since $Z(GS[G]) \geq n = 3$, 3 black vertices are enough to force the entire graph).
Let the zero forcing number of closed global shadow graph is $Z(GS[G]) = 3$, this implies that $n = 1, 2$ or 3 from theorem 3. But from theorem 29 there exist no graph when $n = 1, 2$ that has the zero forcing number of its closed global shadow graph to be 3. Hence $Z(GS[G]) = 3$ is possible only when $n = 3$. There are 4 possible graphs G when $n = 3$. null graph $\bar{K_3}$, complete graph K_3, $\bar{P_3}$ and path P_3. When G is complete graph K_3 and $\bar{P_3}$ it can be seen in figure 3 that $Z(GS[G]) = 3$. Where as when G is null graph from theorem 23 we known that $Z(GS[G]) = 2n - 2 = 2 * 3 - 2 = 4$ and when G is path P_3 it can be seen from the theorem 24 that $Z(GS[G]) = n + 1 = 3 + 1 = 4$. □

Theorem 31 [1] *If* G *is a Hamiltonian graph and* $M(G)$ *is the maximum nullity of graph* G*, then the zero forcing number is related to maximum nullity of the graph as* $Z(G) = M(G)$.

Theorem 32 *Let* G *be any simple graph of order n, then* $Z(GS[G]) = M(GS[G])$.



$G = 3K_1$,
$Z(GS[G]) = 4$

$G = K_3$ or C_3,
$Z(GS[G]) = 3$

$G = \overline{P_3}$,
$Z(GS[G]) = 3$

$G = P_3$,
$Z(GS[G]) = 4$

Figure 3: All the possible closed global shadow graph of graph when n=3.

Proof. It is proved in theorem 10 that closed global shadow graph is Hamiltonian. From the above theorem 31 we conclude that $Z(GS[G]) = M(GS[G])$. □

Theorem 33 [2] *Let* G *be any graph, then* $\chi(G) \leq Z(G) + 1$.

Theorem 34 (Brook's) *For any connected undirected graph* G*, the chromatic number of* G *that is* $\chi(G) \leq \Delta$*. Where* Δ *is the maximum degree of graph* G*. Provided* G *is not a complete graph or odd cycle.*

Theorem 35 *Let* G *be simple graph, then* $\chi(GS[G]) \leq n$.

Proof. The graph $GS[G]$ is always connected n-regular graph. $GS[G]$ can never form a complete graph or odd cycle. Hence according to brook's theorem $\chi(GS[G]) \leq n$. □

Theorem 36 *Let* G *be any graph, then* $\chi(GS[G]) < Z(GS[G]) + 1$.

Proof. The graph $GS[G]$ is n-regular graph and clearly $Z(GS[G]) \geq n$. According to theorem 35, $\chi(GS[G]) \leq n$. Therefore the theorem 33 can be rewritten as $\chi(GS[G]) < Z(GS[G]) + 1$. □

Theorem 37 *If* G *is a complete graph, then* $\chi(GS[G]) = Z(GS[G])$.

Proof. According to theorem 21, $Z(GS[G]) = n$ if G is a complete graph. From theorem 35, $\chi(GS[G]) \leq n$ is known. It is left to show that $\chi(GS[G]) \geq n$. G being a complete graph on n vertices is a subgraph of $GS[G]$ and $\chi(G) = n$. The chromatic number of $GS[G]$ will be at least that of its subgraph (G). Therefore $\chi(GS[G]) \geq n$. □

4 Conclusion

The natural and intrinsic characterisation of the closed global shadow graph is provided. This includes some of the characterisation like hamiltonicity, perfect matching, regularity, etc., The zero forcing number of various classes of closed global shadow graph are studied. In few cases, the necessary and sufficient condition for equality of certain zero forcing number is analysed. The closed global shadow graph $GS[G]$ has a subgraph graph $G\square K_2$. It can be seen that $Z(GS[G]) = Z(G\square K_2)$ when G is a complete graph K_n. It is an open problem to solve when the zero forcing number of $GS[G]$ becomes equal to the zero forcing number of $G\square K_2$. The relation between chromatic number and zero forcing number of $GS[G]$ is understood. One may still find the characterisation for $\chi(GS[G])$ to be equal to $Z(GS[G])$.

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