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On graphs associated to ring of Guassian integers and ring of integers modulo n

S. PIRZADA
Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India
email: pirzadasd@kashmiruniversity.ac.in

M. Imran BHAT
Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India
email: imran.uok.maths@gmail.com

Abstract. For a commutative ring R with identity 1, the zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set is the set of non-zero zero divisors $Z^*(R)$ and the two vertices x and $y \in Z^*(R)$ are adjacent if and only if $xy = 0$. In this paper, we compute the values of some graph parameters of the zero-divisor graph associated to the ring of Gaussian integers modulo n, $\mathbb{Z}_n[i]$ and the ring of integers modulo n, $\mathbb{Z}_n$.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with unity unless explicitly stated otherwise. Given a commutative ring with identity R, the zero-divisor graph of R, denoted by $\Gamma(R)$, is the graph where the vertices are the nonzero zero-divisors $(Z^*(R))$ of R, and there is an undirected edge between two distinct vertices x and y if and only if $xy = 0$. An annihilator of an element x of a ring R, denoted by $ann(x)$, is the set$ann(x) = \{r \in R : \ rx = 0\}$. The zero-divisor graph that Anderson and Livingston [1] introduced allows us to

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visually represent algebraic properties of a commutative, unital ring through graph theoretic properties. This ability to use graph-theoretic properties to visualize underlying algebraic properties is applicable to many different types of rings.

The set of Gaussian integers $\mathbb{Z}[i]$ is a subset of $\mathbb{C}$ defined as $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1}\}$. Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the ring of integers modulo n. Then, the quotient ring $\mathbb{Z}[i]/\langle n \rangle$ is isomorphic to $\mathbb{Z}_n[i] = \{a + ib \mid a, b \in \mathbb{Z}_n\}$, where $\langle n \rangle$ is a principal ideal generated by n for some positive integer larger than 1 in $\mathbb{Z}[i]$. Several results on zero-divisor graphs of the ring of integers modulo n and the ring of Gaussian integers modulo n can be found in [5, 6, 7, 8, 11].

All graphs G in this article will be simple. The vertex set of G will be denoted by $V(G)$. In G, the distance between two vertices x and y, denoted $d(x, y)$, is the length of the shortest path. A maximal connected subgraph of a graph G is called a *component* of G, and the number of components of a graph G is denoted by $k(G)$. A vertex v of G is called a cut vertex of G if $k(G - v) > k(G)$. The *vertex-connectivity* of G, denoted by $\kappa_v(G)$, is the smallest number of vertices whose removal from the graph G results in either a disconnected graph or a single vertex graph. The eccentricity of x, denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G. The minimum eccentricity value is the radius of G. Note that any graph G with radius 1 necessarily has at least one vertex adjacent to all other vertices of G. We denote the *minimum* and *maximum* degree of a graph G by $d_\delta(G)$ and $d_\Delta(G)$ respectively. For $n \geq 1$, K_n will denote a *complete graph* on n vertices containing all $\binom{n}{2}$ possible edges. In a graph G, if no two vertices of a subset $\mathcal{A}$ of the vertex set V are adjacent, then $\mathcal{A}$ is said to be an *independent* set. A maximal complete subgraph of G is a *clique* of G and the order of a clique of G is the *clique number*, denoted by $clq(G)$. More on graph theory definitions, the reader is referred to [9].

In Section 2, we obtain the values of some parameters, like clique number $\omega(G)$, chromatic number $\chi G)$ and radius $rad(G)$ of the zero-divisor graph associated to the ring of integers modulo n, denoted by $\mathbb{Z}_n$, and the ring Gaussian integers modulo n, denoted by $\mathbb{Z}_n[i]$.

2 Zero-divisor graph of $\mathbb{Z}_{q^m}[i]$, $q \cong 3 \pmod 4$ and $\mathbb{Z}_n$

Theorem 1 *Let $\Gamma(\mathbb{Z}_{q^m}[i])$ be the zero-divisor graph of $\mathbb{Z}_{q^m}[i]$, where $q \cong 3 \pmod 4$. Then the clique number of $\Gamma(\mathbb{Z}_{q^m}[i])$ is given by* $\begin{cases} q^{2\lceil \frac{m}{2} \rceil} - 1 & \textit{if } m \text{ is even} \\ q^{2\lceil \frac{m}{2} \rceil} & \textit{if } m \text{ is odd} \end{cases}$

Proof. If α and β are two nonzero zero-divisors in $\mathbb{Z}_{q^m}[i]$ such that $q^{\lceil \frac{m}{2} \rceil} \mid \alpha$ and $q^{\lceil \frac{m}{2} \rceil} \mid \beta$, $\alpha \neq \beta$, then α and β are adjacent in $\Gamma(\mathbb{Z}_{q^m}[i])$. Thus, all such zero-divisors form a clique in $\Gamma(\mathbb{Z}_{q^m}[i])$ and there are $\dfrac{q^{2m}}{q^{2\lceil \frac{m}{2} \rceil}} - 1 = q^{2\lfloor \frac{m}{2} \rfloor} - 1$ such zero-divisors. Also notice that the vertices of $\Gamma(\mathbb{Z}_{q^m}[i])$ that are not in the clique form an independent set of vertices. Moreover, if m is odd then each vertex of the subring generated by $< q^{\lceil \frac{m}{2} \rceil} >$ is adjacent to α, where $q^{\lceil \frac{m}{2} \rceil} \mid \alpha$. Therefore, the clique number is given by $\begin{cases} q^{2\lceil \frac{m}{2} \rceil} - 1 & \text{if } m \text{ is even} \\ q^{2\lceil \frac{m}{2} \rceil} & \text{if } m \text{ is odd} \end{cases}$ □

Notice that for a Gaussian prime q and $m > 1$, $\mathbb{Z}_{q^m}[i] \cong \mathbb{Z}[i]/\langle q^m \rangle$ is a local ring with unique maximal ideal $\langle q \rangle$. Also, $|U(\mathbb{Z}_{q^m}[i])| = q^{2m} - q^{2m-2}$ [4], so that $|Z^*(\mathbb{Z}_{q^m}[i])| = q^{2m} - (q^{2m} - q^{2m-2}) - 1 = q^{2m-2} - 1$.

Theorem 2 *If m is a positive integer and q is a Gaussian prime, then*
(i) $d_\delta(\Gamma(\mathbb{Z}_{q^m}[i])) = q^2 - 1$, $d_\Delta(\Gamma(\mathbb{Z}_{q^m}[i])) = q^{2m-2} - 2$, $\kappa_v(\Gamma(\mathbb{Z}_{q^m}[i])) = q^2 - 1$, where κ_v denotes the vertex connectivity.

Proof. To find the minimum and maximum degree of $\Gamma(\mathbb{Z}_{q^m}[i])$, let $\lambda \in V(\mathbb{Z}_{q^m}[i])$. Then λq is the vertex with least degree because no vertex λq is adjacent to any other vertex except the vertices obtained as multiples of q^{m-1}. Thus, $d_\delta(\Gamma(\mathbb{Z}_{q^m}[i])) = q^2 - 1$.

Also, every vertex of the clique induced by vertices from the subring $< q^{m-1} >$ is adjacent to every vertex of $\Gamma(\mathbb{Z}_{q^m}[i])$. Therefore, $d_\Delta(\Gamma(\mathbb{Z}_{q^m}[i])) = q^{2m-2} - 2 = |Z^*(\mathbb{Z}_{q^m}[i])| - 1$. This implies that $rad(\Gamma(\mathbb{Z}_{q^m}[i])) = 1$ and the vertex connectivity of $\Gamma(\mathbb{Z}_{q^m}[i])$ can be obtained by removing the vertices associated to the subring generated by $< q^{m-1} >$, since their removal leaves each vertex of the form λq as isolated, where $\lambda \in V(\mathbb{Z}_{q^m}[i])$. Thus, the vertex connectivity is $q^2 - 1$. □

The following lemma gives a formula for calculating the clique number of zero-divisor graph $\Gamma(\mathbb{Z}_n)$ of $\mathbb{Z}_n$, for $n \geq 1$.

Lemma 3 [2] *If* $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ *is the canonical representation of* n*, then* $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{\alpha_k}\mathbb{Z}$ *as rings. If each* α_i $(1 \leq i \leq r)$ *is even, then* $\omega(\Gamma(\mathbb{Z}_n) = p_1^{\frac{\alpha_1}{2}} p_2^{\frac{\alpha_2}{2}} \ldots p_r^{\frac{\alpha_r}{2}} - 1$ *and if*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} q_1^{\beta_1} q_2^{\beta_2} \ldots p_s^{\beta_s}$$

such that $\alpha_i{}^s$ *are even and* $\beta_i{}^s$ *are odd, then*

$$\omega(\Gamma(\mathbb{Z}_n) = p_1^{\frac{\alpha_1}{2}} p_2^{\frac{\alpha_2}{2}} \ldots p_r^{\frac{\alpha_r}{2}} q^{\frac{\beta_1 - 1}{2}} q^{\frac{\beta_2 - 1}{2}} \ldots q^{\frac{\beta_s - 1}{2}} + s - 1,$$

where s *is the number of odd primes.*

From Lemma 3, the following observation is immediate.

Theorem 4 *If* p *is a prime number and* $n \in \mathbb{N}$*, then the clique number of* $\Gamma(\mathbb{Z}_{p^n})$ *is given by*

$$\omega(\Gamma(\mathbb{Z}_{p^n})) = \begin{cases} p^{\frac{n}{2}-1} & \text{if } n \text{ is even} \\ p^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

Theorem 5 *If* $\Gamma(\mathbb{Z}_{2^m}[i])$ *be a zero-divisor graph of the ring* $\mathbb{Z}_{2^m}[i]$*, where* $m \geq 1$ *is an integer, then* $d_\Delta(\Gamma(\mathbb{Z}_{2^m}[i])) = 2^{2m-1} - 2$*,* $\omega(\Gamma(\mathbb{Z}_{2^m}[i])) = 2^m - 1$*,* $d_\delta(\Gamma(\mathbb{Z}_{2^m}[i])) = 1$*,* $k_\nu(\Gamma(\mathbb{Z}_{2^m}[i])) = 1$ *and* $rad(\Gamma(\mathbb{Z}_{2^m}[i])) = 1$*, where* $m \geq 2$.

Proof. For $m = 1$, the case is trivial. For $m > 1$, one can see that $\mathbb{Z}_{2^m}[i] \cong \mathbb{Z}[i]/\langle 2^m \rangle = \mathbb{Z}[i]/\langle (1+i)^{2m} \rangle$. Clearly, $Z^*(\mathbb{Z}_{2^m}[i]) = \langle 1+i \rangle - \{0\}$ is an annihilator ideal, that is, there exists a vertex, say $\alpha \in Z^*(\mathbb{Z}_{2^m}[i])$, adjacent to every other vertex. Also, by Proposition 2.4 in [3], $\Gamma(\mathbb{Z}_{2^m}[i]) \cong \Gamma(\mathbb{Z}_{2^{2m}})$. With this property, we have $d_\Delta(\Gamma(\mathbb{Z}_{2^m}[i])) = 2^{2m-1} - 2$, $\omega(\Gamma(\mathbb{Z}_{2^m}[i])) = 2^m - 1$, $d_\delta(\Gamma(\mathbb{Z}_{2^m}[i])) = 1$, $k_\nu(\Gamma(\mathbb{Z}_{2^m}[i])) = 1$, where $m \geq 2$. Also, the degree of the vertices are given as $deg(\nu_i) = 2^i - 1$, if $1 \leq i < m$ and $deg(\nu_i) = 2^i - 2$, for $m \leq i \leq 2m - 1$.

Now, we claim that there exists a pendent vertex $1+i$ in $\Gamma(\mathbb{Z}_{2^m}[i])$. Assume that $(1+i)(a+ib) = 0$, which implies that $(a-b) + i(a+b) = 0$, that is, $a = b$ or $a + b = 0$, that is $a + b = 2^m$. Thus, $(1+i) \sim (\frac{2^m}{2} + i\frac{2^m}{2})$. Hence, $1+i$ is a pendent vertex. □

By Theorem 3 in [10], if we partition the vertex set of $\Gamma(\mathbb{Z}_{p^m})$ into the sets $S_1, S_2, \ldots, S_{m-1}$, where $S_i = \{k_i p^i : p \nmid k_i\}$, $1 \leq i \leq m-1$, then it is easy to see that $|V_i| = (p-1)p^{m-i-1}$, $1 \leq i \leq m-1$ and therefore $|\Gamma(\mathbb{Z}_{p^m})| = \sum_{i=1}^{m-1} (p-1)p^{m-i-1} = p^{m-1} - 1$. Also, for a positive integer k, $1 \leq k \leq m-1$, the degrees of the vertices in $\Gamma(\mathbb{Z}_{p^m})$ are given by

$deg(V_k) = \begin{cases} p^k - 1 & \text{if } 1 \le k < \lceil \frac{m}{2} \rceil \\ p^k - 2 & \text{if } \lceil \frac{m}{2} \rceil \le k \le m-1 \end{cases}$ where $\lceil x \rceil$ denotes the smallest integer function.

A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

Theorem 6 *For each $m \ge 1$, the graph $\Gamma(\mathbb{Z}_{p^m})$, where p is a prime, is not Eulerian.*

Proof. We divide the vertex set of $\Gamma(\mathbb{Z}_{p^m})$ into the sets $S_1, S_2, \dots, S_{m-1}$, where $S_i = \{k_i p^i : p \nmid k_i\}$, $1 \le i \le m-1$. Clearly, a vertex of S_i is adjacent to a vertex of S_j if and only if $i + j \ge m$. This implies that a vertex $v \in S_1$ is adjacent to a vertex $u \in V(\Gamma(\mathbb{Z}_{p^m}))$ if and only if $u \in S_{m-1}$. Now, for each $v_1 \in S_1$ and $v_{m-1} \in S_{m-1}$, we have $deg(v_1) = p-1$ and $deg(v_{m-1}) = p^{m-1} - 2$. So, for each prime p and a positive integer m, it follows that either $deg(v_1)$ or $deg(v_{m-1})$ is odd, where $v_1 \in S_1$, $v_{m-1} \in S_{m-1}$. □

Now, we obtain the values of some graph parameters of the zero-divisor graph associated to the ring $\mathbb{Z}_{pq^2}$, where p and q are distinct prime integers.

Theorem 7 *Let $\Gamma(\mathbb{Z}_{pq^2})$ be the zero-divisor graph of the ring $\mathbb{Z}_{pq^2}$, where p and q are distinct prime integers. If*
(i) p is odd prime and $q = 2$, then $diam(\Gamma(\mathbb{Z}_{pq^2})) = 3$, $d_\delta(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$, $d_\Delta(\Gamma(\mathbb{Z}_{pq^2})) = pq - q$, $rad(\Gamma(\mathbb{Z}_{pq^2})) = 1$, $\kappa_v(\Gamma(\mathbb{Z}_{pq^2})) = 1$ and $\omega(\Gamma(\mathbb{Z}_{pq^2}) = q - 1$.
(ii) $p = 2$ and q is odd, then $diam(\Gamma(\mathbb{Z}_{pq^2})) = 3$, $rad(\Gamma(\mathbb{Z}_{pq^2}) = 1$, $d_\Delta(\Gamma(\mathbb{Z}_{pq^2}) = q^2 - 1$, $d_\delta(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$ and $\omega(\Gamma(\mathbb{Z}_{pq^2}) = q - 1$.
(iii) $diam(\Gamma(\mathbb{Z}_{pq^2})) = 3$, $rad(\Gamma(\mathbb{Z}_{pq^2}) = 1$, $d_\Delta(\Gamma(\mathbb{Z}_{pq^2}) = q^2 + p - 1$, $d_\delta(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$ and $\omega(\Gamma(\mathbb{Z}_{pq^2}) = q - 1$.

Proof. The number of zero-divisors in $\mathbb{Z}_m$ is given by $m - \phi(m) - 1 = pq^2 - \phi(pq^2) - 1 = pq^2 - (p-1)(q^2 - q) - 1 = q(p + q - 1) - 1$.
(i). When p is odd prime and $q = 2$. In this case, we partition the vertex set as $V_1 = \{q^2 k : 1 \le k < p, (k, p) = 1\}$, $V_2 = \{kp : 1 \le k < q^2\}$ and $V_3 = \{2k : 1 \le k < pq, k \ne p, (2, k) = 1\}$. Now, in $\Gamma(\mathbb{Z}_{pq^2})$, we take pq as a center vertex. Clearly, no two vertices of V_1 are adjacent. However, for each $u \in V_1$ and $v \in V_2$, $uv = 0$. Thus, V_1 and V_2 form a complete bipartite graph. Furthermore, the vertices of V_3 are adjacent to the vertex pq, which form an independent set (tail vertices). In this way, we get $p - 1$ number of

pendent vertices. So, clearly $d_\delta(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$. Also, $\mathrm{diam}(\Gamma(\mathbb{Z}_{pq^2})) = 3$, $d_\Delta(\Gamma(\mathbb{Z}_{pq^2})) = 2$, $\kappa_v(\Gamma(\mathbb{Z}_{pq^2})) = 1$, $\mathrm{rad}(\Gamma(\mathbb{Z}_{pq^2})) = 1$ and $\omega(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$. For illustration, consider $\mathbb{Z}_{28}$ in Figure 1(a).

(ii). When $p = 2$ and q is odd prime, we partition the vertex set into the subsets: $V_1 = \{k(2q) : 1 \leq k < q, (2k, q) = 1\}$, $V_2 = \{kq : 1 \leq k \leq q, (k, q) = 1\}$ and $V_3 = \{2k : 1 \leq k \leq q^2 - q, (2, q) = 1\}$. Clearly, no two vertices of V_1 and V_2 are adjacent. However, for each $u \in V_1$ and $v \in V_2$, $uv = 0$. Also, there exists a vertex $q^2 \in V_2$ such that each vertex $v_i \in V_3$ is adjacent to q^2. So, we observe that $\mathrm{diam}(\Gamma(\mathbb{Z}_{pq^2})) = 3$, $\mathrm{rad}(\Gamma(\mathbb{Z}_{pq^2})) = 1$, $d_\Delta(\Gamma(\mathbb{Z}_{pq^2})) = q^2 - 1$, $d_\delta(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$, the set of vertices $\{2q, 2.2q, 3.2q, \ldots q - 1.2q\}$ form a clique, that is, $\omega(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$. For example, consider $\mathbb{Z}_{2.5^2}$, see Figure 1(b).

(iii). When both p and q are odd primes. In this case, we partition the vertex set as: $V_1 = \{kq : 1 \leq k < q^2, (k, p) = 1\}$, $V_2 = \{kp : 1 \leq k < q^2 - q, (k, q) = 1\}$, $V_3 = \{kq^2 : 1 \leq k < p, (k, p) = 1\}$ and $V_4 = \{kpq : 1 \leq k < pq - p, (k, p) = 1, (k, q) = 1\}$. It is clear that $\mathrm{diam}(\Gamma(\mathbb{Z}_{pq^2})) = 3$, $\mathrm{rad} = 1$, $d_\Delta(\Gamma(\mathbb{Z}_{pq^2})) = q^2 + p - 1$, $d_\delta(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$ and $\omega(\Gamma(\mathbb{Z}_{pq^2})) = q - 1$. For example, consider $\mathbb{Z}_{3^2.5}$, see Figure 2. □

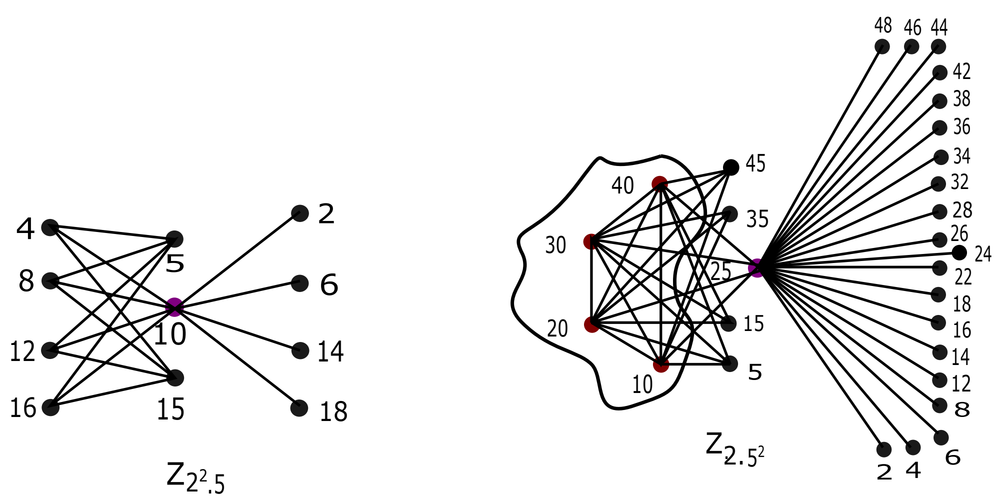

Figure 1: Example $\Gamma(\mathbb{Z}_{p^2q})$, p and q are odd

A ring R is said to be decomposable if it can be written as a direct product $R_1 \times R_2$, where R_1 and R_2 are nonzero rings, otherwise R is said to be indecomposable. In the next two results, we find the values of graph parameters of

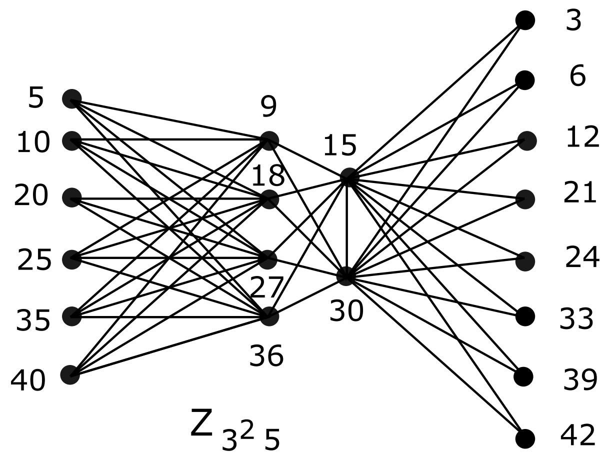

Figure 2: Example $\Gamma(\mathbb{Z}_{p^2q})$, p, q are odd primes

the zero divisor graph associated to the finite commutative ring which is the direct product of finite local rings.

Theorem 8 *Let* $R \cong R_1 \times R_2$ *be a commutative ring and* $ann(x)$ *be minimal non-trivial ideal.*
(i) If $R_1 = \mathbb{Z}_2$*, then* $k_v(\Gamma(R)) = 1$*,* $d_\Delta(\Gamma(R)) = |R_2| - 1$*,* $d_\delta(\Gamma(R)) = 1$*.*
(ii) If $R_2 = \mathbb{Z}_2$*, then* $k_v(\Gamma(R)) = 1$*,* $d_\Delta(\Gamma(R)) = |R_1| - 1$*,* $d_\delta(\Gamma(R)) = 1$*.*
(iii) $\kappa_v(\Gamma(R)) = d_\delta(\Gamma(R)) = min(|R_1|, |R_2|)$*,* $d_\Delta(\Gamma(R)) = max(|R_1|, |R_2|)$*, if* $R_1 = \mathcal{F}$*, where* $\mathcal{F}$ *is a field and* $R_1 \ncong \mathbb{Z}_2$*.*
(iv) $\kappa_v(\Gamma(R)) = |ann(x, 1)|$*.*

Proof. (i). As $\mathbb{Z}_2$ is a field, we partition the vertex set of $\mathbb{Z}_2 \times R_2$ as $V_1 = \{(0,1), (0,x_1), \ldots, (0,x_m)\}$ and $V_2 = \{(1,0)\}$. Now, there exists a vertex $(1,0)$ joined to every vertex of V_1. When the cut-vertex $(1,0)$ is removed from $\Gamma(R)$, the resulting graph is no longer connected leaving $(0,1)$ as an isolated vertex. Hence, $\kappa_v(\Gamma(R)) = 1$. Also, if $ann(x)$ is a minimal non-trivial annihilator ideal in R_2, where $x \in Z^*(R_2)$, then $ann(0,x_i) = \{(0,x_j)$ and $(1,x_j) \mid x_j \in ann(x_i)\}$. Thus, the graph $\Gamma(R)$ is incomplete. Clearly, $|ann(0,x_i)| < deg(1,0)$ implies that $d_\Delta(\Gamma(R)) = |R_2| - 1$ and $d_\delta(\Gamma(R)) = 1$.
(ii). This follows by using the argument similar to above.
(iii). Since $R \cong \mathcal{F} \times R_2$ is a commutative ring, where $\mathcal{F}$ is a field and $R_2 \ncong \mathbb{Z}_2$, so $S_1 = \{(u,0) \mid u \in \mathcal{F}^*\}$ is a cut-set of $\Gamma(R)$ if $|\mathcal{F}| < |R_2|$ and $S_2 = \{(0,a) \mid a \in R_2\}$ if $|\mathcal{F}| > |R_2|$. Thus, $\kappa_v(\Gamma(R)) = min(|R_1|, |R_2|)$, $d_\Delta = max(|R_1|, |R_2|)$. In case $R_2 \cong \mathbb{Z}_2$, then $(0,1)$ is the cut-vertex of $\Gamma(R)$.

(iv). Let $x \in Z^*(R_1)$. If $ann(x) = \{y \in Z^*(R_1) \ : \ xy = 0\}$ and $ann(x)$ is the minimal non-trivial annihilator ideal, then $ann(x, 1) = \{(y, 0)\}$. Clearly, when $S = \{(y, 0) \ : \ y \in ann(x)\}$ is removed, $\Gamma(R)$ becomes disconnected. Thus, S is the minimal cut-set. Hence, $k_v(\Gamma(R)) = |S|$. □

Theorem 9 *Let* $R = R_1 \times R_2 \times \cdots \times R_n$, $n \geq 2$, $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and each* R_i *is a finite local ring, then*
(i) $\kappa_v(\Gamma(R)) = 1$, *if* $R_1 \cong \mathbb{Z}_2$
(ii) $\kappa_v(\Gamma(R)) = 1$, *if* $R_1 \cong \mathbb{Z}_4$ *or* $\dfrac{\mathbb{Z}_2[x]}{x^2}$.

Proof. (i). Let $R = R_1 \times R_2 \times \cdots \times R_n$, $n \geq 2$, $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ and each R_i be a finite local ring. When $R_1 \cong \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2 \times R_2 \times \cdots \times R_n$. Clearly, $ann((1, 0, \ldots, 0))$ consists of at least one pendent vertex $(0, 1, \ldots, 1)$. Deletion of the vertex $(1, 0, \ldots, 0)$ isolates $(0, 1, \ldots, 1)$ and hence disconnects the graph with $\kappa_v(\Gamma(R)) = 1$.
(ii). When $R_1 \cong \mathbb{Z}_4$, let $ann(2)$ be the minimal annihilator ideal of $\mathbb{Z}_4$. Then $ann((2, 0, \ldots, 0)) = \{(2, 1, \ldots, 1), \ldots \ : \ 2 \in ann(2)\}$. Clearly, the vertex $(2, 1, \ldots, 1)$ is only adjacent to $(2, 0, \ldots, 0)$. Hence, $\kappa_v(\Gamma(R)) = 1$, in this case also.

When $R_1 \cong \dfrac{\mathbb{Z}_2[x]}{x^2}$, there exists only one path from $(x, 0, \ldots, 0)$ to $(x, 1, \ldots, 1)$. Then the graph becomes disconnected on removing $(x, 0, \ldots, 0)$. □

Using Lemma 3, we can find the clique number of a zero-divisor graph $\Gamma(\mathbb{Z}_n)$ for any large n. To calculate this, we factorize the integers in different forms. For example, $\omega(\Gamma(\mathbb{Z}_{2000})) = 2^2 \cdot 5 + 1 - 1 = 20$. For distinct prime integers p, q, we have (i) $\omega(\Gamma(\mathbb{Z}_{pq})) = p^0q^0 + 2 - 1 = 2$. (ii) If $n = p^2q$, then $\omega = pq^0 + 1 - 1 = p$. (iii) $\omega(\Gamma(\mathbb{Z}_{p^3q}) = pq^0 + 2 - 1 = p + 1$ (iv) $\omega(\Gamma(\mathbb{Z}_{p^2q^2}))$ $= pq - 1$. (v) $\omega(\Gamma(\mathbb{Z}_{p^3}) = p$. (vi) $\omega(\Gamma(\mathbb{Z}_{p^4}) = p^2 - 1$.

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