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Analysis and optimisation of a M/M/1/WV queue with Bernoulli schedule vacation interruption and customer's impatience

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Abstract. In this investigation, we establish a steady-state solution of an infinite-space single-server Markovian queueing system with working vacation (WV), Bernoulli schedule vacation interruption, and impatient customers. Once the system becomes empty, the server leaves the system and takes a vacation with probability p or a working vacation with probability $1-p$, where $0 \leq p \leq 1$. The working vacation period is interrupted if the system is non empty at a service completion epoch and the server resumes its regular service period with probability $1-q$ or carries on with the working vacation with probability q. During vacation and working vacation periods, the customers may be impatient and leave the system. We use a probability generating function technique to obtain the expected number of customers and other system characteristics. Stochastic decomposition of the queueing model is given. Then, a cost function is constructed by considering different cost elements of the system states, in order to determine the optimal values of the service rate during regular

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busy period, simultaneously, to minimize the total expected cost per unit time by using a quadratic fit search method (QFSM). Further, by taking illustration, numerical experiment is performed to validate the analytical results and to examine the impact of different parameters on the system characteristics.

1 Introduction

Queueing modeling is being employed in a large variety of day-to-day congestion problems as well as in industrial scenarios, such as computer systems, call centers, web services, communication/telecommunication systems, etc. For nearly a century, many queueing models have been developed to analyze the characteristics of many systems and recommendations have been issued to suggest how to deal with congestion situations. In many queueing scenarios, when there is no job present in the system, the server may take a vacation (V) or may provide a service for a secondary job, known as working vacation (WV). Queueing systems with vacation and working vacation have been the subject of interest for the queueing theorists. A detailed surveys of the literature devoted to vacation queues are found in [9], [26], [27], and the references therein. Working vacation queue was first introduced by [24] in an $M/M/1$ queueing system. [17] analyzed a single server queue with batch arrivals and general service time distribution. [28] provided the analysis for an $M/G/1$ queueing model with multiple vacations and exhaustive service discipline at which the server works with different rate rather than completely stopping the service during vacation. [15] provided performance analysis of $GI/M/1$ queue with working vacations. Then, [23] analyzed the $M/M/1$ queue with single and multiple working vacation and impatient customers. They computed closed form solution and various performance measures with stochastic decomposition for both the working vacation policies. After that, a Markovian queueing system with two-stage working vacations has been considered by [25]. Recently, [18] examined an infinite-buffer multiserver queue with single and multiple synchronous working vacations.

In this investigation, we considered vacation interruption policy at which during working vacation period, the server may come back to the regular working period without completing the ongoing working vacation. The concept of vacation interruptions was introduced by [13]. After that, [14], [16], and [31] generated the vacation interruption model for $GI/Geo/1$, $GI/M/1$, and $M/G/1$ queueing models, respectively. Working vacation queueing system with service interruption and multi-optional repair was considered by [11]. Then,

[10] examined system performance measures for an $M/G/1$ queueing model with single working vacations and a Bernoulli interruption schedule. [29] studied the strategic behaviour in a discrete-time working vacation queue with a Bernoulli interruption schedule. [22] investigated a single server queueing model with multiple working vacations and vacation interruption where an arriving customer can balk the system at some particular times. Recently, a study of an infinite-space single server Markovian queue with working vacation and vacation interruption was established by [20].

Over recent decade, customer's impatience becomes the burning issue of private and government sector businesses. Thus, an increasing attention has been seen in queueing systems with impatient customers due to the absence (vacation) of the server. [1] gave the analysis of customers' impatience in different queues with server vacation. Then, vacation queueing models with impatient customers and a waiting server have been examined by [21]. [30] analyzed an $M/M/1$ queue with vacations and impatience timers which depends on the server's states. [8] examined a queueing model with feedback, reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption. Further, performance and economic analyzes of different queueing models with vacation/working vacation and customer's impatience have been treated by [5, 6], [2, 3], [4], [19, 7], and the references therein.

In this work, the main objective is to analyze the queueing performance of an infinite-space single-server working vacation queueing system with Bernoulli schedule vacation interruption at which whenever the system becomes empty, the server switches to the vacation period with a certain probability p and to the working vacation with a complementary probability $1 - p$. During the vacation period, the customers are served at a lower service rate. During this period, at each service completion instant, if there are customers in the queue, the server either remains in the working vacation status with probability q, or switches to the regular service status with probability $1 - q$. During vacation and working vacation periods, the customers may get impatient with different rates and leave the system . In this study, the probability generating function (PGF) is used to determine the stationary system and queue lengths. The stochastic decomposition of the queueing model is also provided. Further, the cost optimization analysis of the system is carried out using quadratic fit search method (QFSM) in order to minimize the total expected cost of the system with respect to the service rate during normal busy period.

The rest of the paper is organized as follows. Section 2 describes the queueing system by stating the requisite hypotheses and notations which are needed to develop the model. Section 3 is devoted to a practical application of the proposed queueing model. In Section 4, the steady-state equations governing the queueing model are constructed and the steady-state solution of the considered queueing system is obtained, using the probability generating function technique. In the Section 5, we focus on useful system characteristics in terms of state probabilities. Section 6 is devoted to the stochastic decomposition of the queueing system. In Section 7, we construct a cost function. Numerical analysis has been carried out in Section 8. Finally, we ended the paper with a conclusion in Section 9.

2 Model description

Consider an infinite-buffer single server Markovian queueing system where the arriving customers follow Poisson process with rate λ. During the regular service period, the customers are served with an exponential rate μ_b. The server begins a vacation with probability p or a working vacation with probability $1-p$, where $0 \leq p \leq 1$, at the instant when he finds the system empty. During the working vacation period, the server renders service to the customers with a lower rate $\mu_v(\mu_v < \mu_b)$. A new busy period starts if the system is non empty after the end of vacation period or working vacation period. Further, it is assumed that the working vacation period is interrupted if the system is non empty at a service completion instant and the server resumes the regular service period with probability $1-q$ or carries on with the working vacation with probability q. Vacation and working vacation periods are assumed to be exponentially distributed with rates θ and ϕ respectively.

Whenever a customer arrives to the system and realizes that the server is on vacation (resp. working vacation) he activates an exponentially distributed impatience timer T_1 (resp. T_2) with parameter ξ (resp. α), where $\alpha < \xi$. If the server comes back from his vacation or working vacation before the timer T_1 or T_2 expires, the customer remains in the system till the completion of his service. The customer leaves the system and never returns if T_1 or T_2 expires while the server is still on vacation or working vacation.

At time t, let $L(t)$ denote the total number of customers in the system and $J(t)$ denotes the state of the server with

$$J(t) = \begin{cases} 0, & \text{when the server is in working vacation period,} \\ 1, & \text{when the server is in vacation period,} \\ 2, & \text{when the server is in regular service period.} \end{cases}$$

Then, the pair $\{L(t), J(t), t \geq 0\}$ defines a two dimensional continuous time discrete state Markov chain with state space $E = \{((0,0) \cup (0,1)) \cup (i,j), i = 1,2,\ldots, j = 0,1,2\}$. Let $P_{ij} = \lim_{t\to\infty} P\{L(t) = i, J(t) = j\}$ denote the stationary probabilities of the Markov process $\{L(t), J(t), t \geq 0\}$.

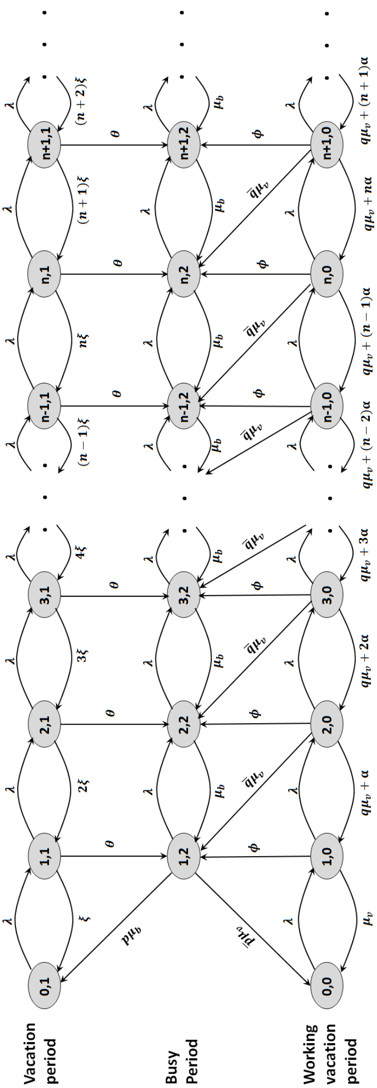

Figure 1: State-transition-rate diagram.

3 Practical application of the queueing model

Reducing energy costs is a major problem in modern information and communication technology (ICT) systems, as the inactive devices in modern ICT systems consume a significant amount of energy. We consider a ICT system with a single device, wherein jobs arrive according to a Poisson process with rate λ. The job processing time is exponentially distributed with rate μ_b. When the system work has been done, to reduce energy costs, the device switches either to off state with probability p or to a lower energy state with a complimentary probability $1-p$ wherein it keeps part of its capacity and processes the incoming jobs with a lower rate μ_v ($\mu_b > \mu_v$), which is also exponentially distributed. The lower energy state can be considered as the working vacation status of the device. In order to avoid the increasing workload and the prolonged job sojourn time, once a job arrives at an empty device, the device processes the job with the rate μ_v, and begins to move to the regular service period. The switching process takes time and the processing of the current job can not be interrupted. Then, at each time of service completion during the working vacation period, the device can remain in the working vacation period with probability q or switch to the regular service period with probability$1-q$.

If the device successfully switches to the regular service period and finds jobs online, it will process them with rate μ_b (the working vacation period is interrupted).

Moreover, we suppose that whenever a customer arrives to the system and finds that the device is on vacation (resp. working vacation) he activates an impatience timer T_1, (resp. T_1) exponentially distributed with parameter ξ (resp. α). If the device returns from its vacation/working vacation before the time expires, the customer stays in the system until his service is completed. However, if impatience timer expires while the server is still on vacation/working vacation, the customer abandons the queue, never to return.

4 Stationary Solution of the Model

Using the theory of Markov process, the stationary equations governing the system are as follows

$$\lambda P_{01} = \xi P_{11} + p\mu_b P_{12}, \tag{1}$$

$$(\lambda + \theta + n\xi)P_{n,1} = \lambda P_{n-1,1} + (n+1)\xi P_{n+1,1}, \qquad n \geq 1, \tag{2}$$

$$\lambda P_{00} = \mu_v P_{10} + (1-p)\mu_b P_{1,2}, \tag{3}$$

$$(\lambda + \mu_v + \phi + (n-1)\alpha)P_{n,0} = \lambda P_{n-1,0} + (q\mu_v + n\alpha)P_{n+1,0}, \qquad n \geq 1, \quad (4)$$

$$(\lambda + \mu_b)P_{12} = \mu_b P_{2,2} + \theta P_{1,1} + \phi P_{1,0} + (1-q)\mu_v P_{2,0}, \quad (5)$$

$$(\lambda+\mu_b)P_{n2} = \lambda P_{n-1,2} + \theta P_{n,1} + \phi P_{n,0} + \mu_b P_{n+1,2} + (1-q)\mu_v P_{n+1,0}, \ n \geq 2. \quad (6)$$

Define the Probability generating functions (PGFs) as

$$P_0(z) = \sum_{n=0}^{\infty} P_{n,0} z^n,$$
$$P_1(z) = \sum_{n=0}^{\infty} P_{n,1} z^n,$$
$$P_2(z) = \sum_{n=1}^{\infty} P_{n,2} z^n,$$

with $P_0(1) + P_1(1) + P_2(1) = 1$, $P_0'(z) = \sum_{n=1}^{\infty} nz^{n-1}P_{n,0}$, and $P_1'(z) = \sum_{n=1}^{\infty} n$ $z^{n-1}P_{n,1}$.

Multiplying equation (2) by z^n and summing over n, we get after using equation (1)

$$\xi(1-z)P_1'(z) - [\lambda(1-z) + \theta]P_1(z) + p\mu_b P_{12} + \theta P_{01} = 0. \quad (7)$$

Multiplying equation (4) by z^n and summing over n, we get after using equation (3)

$$\begin{aligned} &\alpha z(1-z)P_0'(z) - [(1-z)(\lambda z - \mu_v + \alpha) + \mu_v(1-q) + z\phi]P_0(z) \\ &+[z\phi - (1-z)(\mu_v - \alpha) + (1-q)(\lambda z + \mu_v)]P_{00} + q(1-p)z\mu_b P_{12} = 0. \end{aligned} \quad (8)$$

Remark 1 *If* $p = 1$, *equation* (7) *becomes*

$$\xi(1-z)P_1'(z) = [\lambda(1-z) + \theta]P_1(z) - (\mu_b P_{12} + \theta P_{01}),$$

which matches with the result given in [1].

Remark 2 *If* $q = 1$ *and* $p = 0$, *equation* (8) *becomes*

$$\alpha z(1-z)P_0'(z) - [(1-z)(\lambda z - \mu_v + \alpha) + z\phi]P_0(z) + [z\phi - (1-z)(\mu_v - \alpha)]P_{00} + \mu_b P_{12} z = 0.$$

This matches with the result done in [23].

Remark 3 *If* $q = 1$, $p = 0$ *and* $\alpha = 0$, *then equation* (8) *reduces to*

$$P_0(z) = \frac{\mu_v(1-z)P_{00} - z(\mu_b P_{12} + \phi P_{00})}{\lambda z^2 - z(\lambda + \mu_v + \phi) + \mu_v},$$

which is same as in [24].

Multiplying equation (6) by z^n and summing over n, we get after using equation (5)

$$\begin{aligned}(1-z)(\lambda z - \mu_b)P_2(z) &= (z\phi + (1-q)\mu_v)P_0(z) + \theta z P_1(z) \\ &\quad - [(\phi + (\lambda + \mu_v)(1-q))P_{00} + q(1-p)\mu_b P_{12}]z \\ &\quad - \mu_v(1-q)(1-z)P_{00} - (\theta P_{01} + p\mu_b P_{12}).\end{aligned} \tag{9}$$

Putting $z = 1$ into equations (7) and (8), we respectively get

$$\theta P_1(1) = p\mu_b P_{12} + \theta P_{01}, \tag{10}$$

and

$$[\phi + \mu_v(1-q)]P_0(1) = [\phi + (1-q)(\lambda + \mu_v)]P_{00} + (1-p)q\mu_b P_{12}. \tag{11}$$

4.1 Solution of differential equations

Equation (7) can be rewritten as

$$P_1'(z) - \left[\frac{\lambda}{\xi} + \frac{\theta}{\xi(1-z)}\right]P_1(z) + \frac{p\mu_b P_{12} + \theta P_{01}}{\xi(1-z)} = 0, \tag{12}$$

for $\xi \neq 0$ and $z \neq 1$.

To solve the linear differential equation (12), we multiple both sides of the equation by $I.F = e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\theta}{\xi}}$ and integrating from 0 to z, we have

$$P_1(z) = e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}}\left[P_1(0) - \left(\frac{p\mu_b P_{12} + \theta P_{01}}{\xi}\right)K(z)\right], \tag{13}$$

where

$$K(z) = \int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx.$$

Then, by letting $z \to 1$, we obtain

$$P_1(1) = e^{\frac{\lambda}{\xi}}\left[P_1(0) - \left(\frac{p\mu_b P_{12} + \theta P_{01}}{\xi}\right)K(1)\right]\lim_{z\to 1}(1-z)^{-\frac{\theta}{\xi}}.$$

Since $0 \leq P_1(1) = \sum_{n=0}^{\infty} P_{n,1} \leq 1$ and $\lim_{z\to 1}(1-z)^{-(\frac{\theta}{\xi})} \to \infty$,we must have

$$P_{01} = P_1(0) = \left(\frac{p\mu_b P_{12} + \theta P_{01}}{\xi}\right) K(1), \tag{14}$$

which gives

$$P_{12} = T_0 P_{01}, \tag{15}$$

where $T_0 = \frac{\xi - \theta K(1)}{p\mu_b K(1)}$.

Then, substituting equation (15) into equations (10) and (13), we respectively get

$$P_1(1) = \frac{\xi}{\theta K(1)} P_{01}, \tag{16}$$

and

$$P_1(z) = e^{\frac{\lambda}{\xi} z}(1-z)^{-\frac{\theta}{\xi}} \left[1 - \frac{K(z)}{K(1)}\right] P_{00}. \tag{17}$$

From equations (1) and (15), we get

$$P_{11} = U_1 P_{01}, \tag{18}$$

where $U_1 = \frac{\lambda - p\mu_b T_0}{\xi}$.

From equations (2) (for $n = 1$) and (18), we get

$$P_{21} = U_2 P_{01}, \tag{19}$$

where $U_2 = g_1 U_1 - \frac{\lambda}{2\xi} U_0$, $g_1 = \frac{\lambda + \phi + \xi}{2\xi}$ and $U_0 = 1$.
From equations (2) (for $n = 2$) and (18)-(19), we get

$$P_{31} = U_3 P_{01}, \tag{20}$$

where $U_3 = g_2 U_2 - \frac{\lambda}{3\xi} U_1$ and $g_2 = \frac{\lambda + \phi + 2\xi}{3\xi}$.
Then, recursively, it yields

$$P_{n1} = U_n P_{01},$$

where

$$U_n = \begin{cases} \frac{\lambda - p\mu_b T_0}{\xi}, & \text{if } n = 1, \\ g_{n-1} U_{n-1} - \frac{\lambda}{n\xi} U_{n-2}, & \text{if } n \geq 2, \end{cases}$$

with

$$g_{n-1} = \frac{\lambda + \theta + (n-1)\xi}{n\xi}.$$

Next, equation (8) can be expressed as

$$P_0'(z) - \left\{ \frac{\lambda z - \mu_v + \alpha}{z\alpha} + \frac{\phi}{\alpha(1-z)} + \frac{\mu_v(1-q)}{\alpha z(1-z)} \right\} P_0(z) + \left\{ \frac{\phi}{\alpha(1-z)} - \frac{\mu_v - \alpha}{z\alpha} + \frac{(1-q)(z\lambda + \mu_v)}{\alpha z(1-z)} \right\} P_{00}$$

$$+ \frac{q(1-p)\mu_b}{\alpha(1-z)} P_{12} = 0,$$

for $\alpha \neq 0$, $z \neq 0$, and $z \neq 1$.

Now, in order to solve the above differential equation we multiply it both sides by $I.F = e^{\frac{-\lambda}{\alpha} z} z^{\left(\frac{\mu_v q}{\alpha} - 1\right)} (1-z)^{\frac{\phi + \mu_v(1-q)}{\alpha}}$ and integrating from 0 to z, we get

$$\begin{aligned} P_0(z) = z^{-\left(\frac{\mu_v q}{\alpha} - 1\right)} (1-z)^{-\left(\frac{\phi + \mu_v(1-q)}{\alpha}\right)} \Bigg\{ \left(\frac{\mu_v}{\alpha} - 1\right) P_{00} A(z) \\ - \frac{\mu_v(1-q)}{\alpha} P_{00} B(z) - \left(\frac{\phi + (1-q)\lambda}{\alpha} P_{00} + \frac{q(1-p)\mu_b}{\alpha} P_{12} \right) C(z) \Bigg\}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} A(z) &= \int_0^z e^{\frac{\lambda}{\alpha}(z-x)} x^{\frac{\mu_v q}{\alpha} - 2} (1-x)^{\frac{\phi + \mu_v(1-q)}{\alpha}} dx, \\ B(z) &= \int_0^z e^{\frac{\lambda}{\alpha}(z-x)} x^{\frac{\mu_v q}{\alpha} - 2} (1-x)^{\frac{\phi + \mu_v(1-q)}{\alpha} - 1} dx, \\ C(z) &= \int_0^z e^{\frac{\lambda}{\alpha}(z-x)} x^{\frac{\mu_v q}{\alpha} - 1} (1-x)^{\frac{\phi + \mu_v(1-q)}{\alpha} - 1} dx. \end{aligned}$$

Taking limit $z \to 1$ in equation (21), we get

$$\begin{aligned} P_0(1) = \Bigg\{ \left(\frac{\mu_v}{\alpha} - 1\right) A(1) P_{00} - \frac{\mu_v(1-q)}{\alpha} B(1) P_{00} \\ - \left[\frac{(\phi + (1-q)\lambda)}{\alpha} P_{00} + \frac{q(1-p)\mu_b P_{12}}{\alpha} \right] C(1) \Bigg\} \lim_{z \to 1} (1-z)^{-\left(\frac{\phi + \mu_v(1-q)}{\alpha}\right)}. \end{aligned}$$

Since $0 \leq P_0(1) = \sum_{n=0}^{\infty} P_{n,0} \leq 1$ and $\lim_{z \to 1} (1-z)^{-\left(\frac{\phi + \mu_v(1-q)}{\alpha}\right)} \to \infty$,we must have

$$P_{12} = S_1 P_{00}, \tag{22}$$

where

$$S_1 = \left[\frac{(\mu_v - \alpha)\frac{A(1)}{C(1)} - \mu_v(1-q)\frac{B(1)}{C(1)} - (\phi + (1-q)\lambda)}{q(1-p)\mu_b} \right].$$

Substituting equation (22) into equation (21), we get

$$P_0(z) = P_{00}\Bigg\{\left(\tfrac{\mu_v}{\alpha}-1\right)\left[A(z)-\tfrac{A(1)}{C(1)}C(z)\right] - \tfrac{\mu_v(1-q)}{\alpha}\left[B(z)-\tfrac{B(1)}{C(1)}C(z)\right]\Bigg\} z^{-(\frac{\mu_v}{\alpha}q-1)}(1-z)^{-(\frac{\phi+\mu_v(1-q)}{\alpha})}.$$

Substituting equation (22) into equation (11), we obtain

$$P_0(1) = HP_{00}, \tag{23}$$

where

$$H = \left[\frac{(\mu_v-\alpha)\frac{A(1)}{C(1)} + \mu_v(1-q)\left(1-\frac{B(1)}{C(1)}\right)}{\phi+\mu_v(1-q)}\right].$$

From equations (15), (16), and (22), we find

$$P_1(1) = \frac{\xi S_1}{\theta K(1) T_0} P_{00}. \tag{24}$$

From equations (3) and (22), we get

$$P_{10} = V_1 P_{00}. \tag{25}$$

where $V_1 = \dfrac{\lambda-(1-p)\mu_b S_1}{\mu_v}$.

From equations (4)(for $n=1$) and (25), we obtain

$$P_{20} = V_2 P_{00}, \tag{26}$$

where $V_2 = f_0 V_1 - \frac{\lambda}{q\mu_v+\alpha}V_0$, $f_0 = \frac{\lambda+\mu_v+\phi}{q\mu_v+\alpha}$ and $V_0 = 1$.

From equations (4) (for $n=2$) and (25)-(26), we get

$$P_{30} = V_3 P_{00}, \tag{27}$$

where $V_3 = f_1 V_2 - \frac{\lambda}{q\mu_v+2\alpha}V_1$ and $f_1 = \frac{\lambda+\mu_v+\phi+\alpha}{q\mu_v+2\alpha}$.

From equations (4)(for $n=3$) and (26)-(27), we get

$$P_{40} = V_4 P_{00}, \tag{28}$$

where $V_4 = f_2V_3 - \frac{\lambda}{q\mu_v+3\alpha}V_2$ and $f_2 = \frac{\lambda+\mu_v+\phi+2\alpha}{q\mu_v+3\alpha}$.

Then, recursively, it yields

$$P_{n0} = V_nP_{00},$$

where

$$V_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{\lambda-(1-p)\mu_b S_1}{\mu_v}, & \text{if } n = 1, \\ f_{n-2}V_{n-1} - \frac{\lambda}{q\mu_v+(n-1)\alpha}v_{n-2}, & \text{if } n \geq 2, \end{cases}$$

with

$$f_{n-2} = \frac{\lambda + \mu_v + \phi + (n-2)\alpha}{q\mu_v + (n-1)\xi}.$$

Next, substituting equations (10) and (11) into equation (9), we get

$$P_2(z) = \frac{(z\phi+(1-q)\mu_v)P_0(z)+\theta zP_1(z)-z(\phi+\mu_v(1-q))P_0(1)-z\theta P_1(1)}{(1-z)(\lambda z-\mu_b)} - \frac{\mu_v(1-q)}{\lambda z-\mu_b}P_{00}. \quad (29)$$

Applying L'Hospital's rule to equation (29), we get

$$P_2(1) = \frac{(\phi + \mu_v(1-q))P_0'(1) + \theta P_1'(1) - \mu_v(1-q)P_0(1)}{\mu_b - \lambda} + \frac{\mu_v(1-q)}{\mu_b - \lambda}P_{00}. \quad (30)$$

This implies

$$P_0'(1) = \frac{(\mu_b - \lambda)P_2(1) + \mu_v(1-q)(P_0(1) - P_{00}) - \theta P_1'(1)}{\phi + \mu_v(1-q)}. \quad (31)$$

Equation (7) can be rewritten as

$$P_1'(z) = \frac{[\lambda(1-z) + \theta]P_1(z) - p\mu_bP_{12} - \theta P_{01}}{\xi(1-z)}$$

Applying L'Hospital's rule, we have

$$P_1'(1) = \frac{\lambda}{\theta + \xi}P_1(1). \quad (32)$$

Further, equation (8) can be rewritten as

$$P_0'(z) = \frac{1}{\alpha z(1-z)}\left([(1-z)(\lambda z - \mu_v + \alpha) + \mu_v(1-q) + z\phi]P_0(z)\right.$$
$$\left. -[z\phi - (1-z)(\mu_v - \alpha) + (1-q)(\lambda z + \mu_v)]P_{00} - q(1-p)z\mu_bP_{11}\right).$$

Applying L'Hospital's rule, we have

$$P_0'(1) = \frac{(\lambda + \alpha - \mu_v - \phi)P_0(1) + (\mu_v + \phi - \alpha + \lambda(1-q) + q\mu_b(1-p)S_1)P_{00}}{\alpha + \phi + \mu_v(1-q)}. \tag{33}$$

Next, substituting equations (32) and (33) into (30), we obtain

$$\begin{aligned} P_2(1) = &\left[\frac{(\phi + \mu_v(1-q))(\lambda + \alpha - \mu_v - \phi)}{(\alpha + \phi + \mu_v(1-q))(\mu_b - \lambda)} - \frac{\mu_v(1-q)}{\mu_b - \lambda}\right] P_0(1) \\ &+ \frac{\lambda\theta P_1(1)}{(\theta + \xi)(\mu_b - \lambda)} + (\phi + \mu_v(1-q)) \\ &\left[\frac{\mu_v + \phi - \alpha - \lambda(1-q) + q\mu_b(1-p)S_1}{(\alpha + \phi + \mu_v(1-q))(\mu_b - \lambda)} + \frac{\mu_v(1-q)}{\mu_b - \lambda}\right] P_{00}. \end{aligned} \tag{34}$$

Using equations (23)-(24) and (34), and normalization condition, we can get the value of P_{00}. Next, we need to write $P_{n,2}$ in terms of $P_{0,0}$.
Substituting equations (15), (18), (22), and (25)-(26) into equation (5), we get

$$P_{22} = S_2 P_{00}, \tag{35}$$

where $S_2 = (1+\rho)S_1 - \frac{\theta S_1}{\mu_b T_0} U_1 - \frac{\phi V_1 + V_2 \mu_v (1-q) V_1}{\mu_b}$, $\rho = \frac{\lambda}{\mu_b}$.

Substituting equations (15), (19), (22), and (26)-(27) into equation (6) (for $n = 2$), we obtain

$$P_{32} = S_3 P_{00}, \tag{36}$$

where $S_3 = (1+\rho)S_2 - \rho S_1 - \frac{\theta S_1}{\mu_b T_0} U_2 - \frac{\phi V_2 + \mu_v (1-q) V_3}{\mu_b}$.

Substituting equations (15), (20), (27)-(28), and (35)-(36) into equation (6) (for $n = 3$), we find

$$P_{42} = S_4 P_{00},$$

where $S_4 = (1+\rho)S_3 - \rho S_2 - \frac{\theta S_1}{\mu_b T_0} U_3 - \frac{\phi V_3 + \mu_v (1-q) V_4}{\mu_b}$.

Then, recursively, it yields

$$P_{n2} = S_n P_{00},$$

where

$$S_n = \begin{cases} 1, & \text{if } n = 1, \\ (1+\rho)S_{n-1} - \rho S_{n-2} - \frac{\theta S_1}{\mu_b T_0} U_{n-1} - \frac{\phi V_{n-1} + \mu_v (1-q) V_n}{\mu_b}, & \text{if } n \geq 2, \end{cases}$$

with $S_0 = 0$.

5 Performance measures

As the steady-state probabilities are obtained one can easily derive the various performance measures of the model.

– The probability that the system is in working vacation $(P_0(1))$.

$$P_0(1) = \left[\frac{(\mu_v - \alpha)\frac{A(1)}{C(1)} + \mu_v(1-q)\left(1 - \frac{B(1)}{C(1)}\right)}{\phi + \mu_v(1-q)} \right] P_{00}.$$

– The probability that the system is in vacation period $(P_1(1))$.

$$P_1(1) = \frac{\xi S_1}{\theta K(1) T_0} P_{00}.$$

– The probability that the system is in busy period $(P_2(1))$.

$$\begin{aligned} P_2(1) = &\left[\frac{(\phi+\mu_v(1-q))(\lambda+\alpha-\mu_v-\phi)}{(\alpha+\phi+\mu_v(1-q))(\mu_b-\lambda)} - \frac{\mu_v(1-q)}{\mu_b-\lambda} \right] P_0(1) + \frac{\lambda\theta P_1(1)}{(\theta+\xi)(\mu_b-\lambda)} \\ &+ (\phi+\mu_v(1-q)) \left[\frac{\mu_v+\phi-\alpha-\lambda(1-q)+q\mu_b(1-p)S_1}{(\alpha+\phi+\mu_v(1-q))(\mu_b-\lambda)} + \frac{\mu_v(1-q)}{\mu_b-\lambda} \right] P_{00} \end{aligned}$$

Substituting equation (23) into equation (33), we get the expected number of customers when the system is on working vacation period $(E(L_0))$.

$$E(L_0) = P_0'(1) = \left[\frac{(\lambda+\alpha-\mu_v-\phi)H + \mu_v+\phi-\alpha+\lambda(1-q)+q\mu_b(1-p)S_1}{\alpha+\phi+\mu_v(1-q)} \right] P_{00}.$$

Substituting equation (24) into equation (32), we get the expected number of customers when the system is on vacation period $(E(L_1))$.

$$E(L_1) = P_1'(1) = \frac{\lambda\xi S_1}{\theta(\theta+\xi)K(1)T_0} P_{00}.$$

Equation (9) can be rewritten as

$$\begin{aligned} P_2(z) = &\frac{(z\phi + (1-q)\mu_v)P_0(z) + \theta z P_1(z) - z(\phi+\mu_v(1-q))P_0(1) - z\theta P_1(1)}{((1-z)(\lambda z - \mu_b)} \\ &- \frac{\mu_v(1-q)}{\lambda z - \mu_b} P_{00}. \end{aligned}$$

Differentiating the above equation and applying L'Hospital's rule, we get

$$\begin{aligned} E(L_2) = P_2'(1) = &\tfrac{\phi+\mu_v(1-q)}{2(\mu_b-\lambda)} P_0''(1) + \tfrac{(\lambda\mu_v(1-q)+\mu_b\phi)}{(\mu_b-\lambda)^2} P_0'(1) + \tfrac{\theta}{2(\mu_b-\lambda)} P_1''(1) \\ &+ \tfrac{(\lambda\mu_v(1-q)+\mu_b\phi)}{(\mu_b-\lambda)^2} P_0'(1) + \tfrac{\lambda\mu_v(1-q)}{(\mu_b-\lambda)^2}(P_{00} - P_0(1)) + \tfrac{\lambda\mu_v(1-q)}{(\mu_b-\lambda)^2} P_{00}. \end{aligned} \quad (37)$$

Differentiating equation (7) twice with respect to z and letting $z = 1$, we obtain

$$\frac{P_1''(1)}{2} = \frac{\lambda}{\theta + 2\xi} P_1'(1). \tag{38}$$

Differentiating equation (8) twice with respect to z and letting $z = 1$, we obtain

$$\frac{P_0''(1)}{2} = \frac{(\lambda - \mu_v - \phi)P_0'(1) + \lambda P_0(1)}{\phi + 2\alpha + \mu_v(1-q)}. \tag{39}$$

Substituting equations (38) and (39) into equation (37), we get the expected number of customer when the server is busy $(E(L_2))$.

$$E(L_2) = \frac{1}{\mu_b-\lambda}\left[\frac{(\phi+\mu_v(1-q))(\lambda-\mu_v-\phi)}{\phi+2\alpha+\mu_v(1-q)} + \frac{\lambda\mu_v(1-q)+\phi\mu_b}{\mu_b-\lambda}\right]P_0'(1) + \frac{1}{\mu_b-\lambda} \times \left[\frac{\lambda\theta}{\theta+2\xi} + \frac{\phi\mu_b}{\mu_b-\lambda}\right]P_1'(1) + \frac{\lambda\mu_v(1-q)}{(\mu_b-\lambda)^2}P_{00} + \frac{\lambda}{\mu_b-\lambda}\left[\frac{\phi+\mu_v(1-q)}{\phi+2\alpha+\mu_v(1-q)} - \frac{\mu_v(1-q)}{\mu_b-\lambda}\right]P_0(1).$$

The expected number of customers in the system can be computed as $E(L) = E(L_0) + E(L_1) + E(L_2)$.

– The average rate of abandonment of customers due to impatience (R_a).

$$R_a = \alpha\sum_{n=0}^{\infty}(n-1)P_{n,0} + \xi\sum_{n=0}^{\infty} nP_{n,1} = \alpha(E[L_0] - (P_0(1) - P_{00})) + \xi E[L_1].$$

6 Stochastic decomposition of the model

The stochastic decomposition structures for the mean queue length and mean waiting times at stationary state are expressed in the following Theorems.

Theorem 1 *If $\lambda < \mu_b$, the stationary queue length L can be decomposed into the sum of two independent random variables as $L = L_0 + L_d$, where L_0 is the stationary queue length of a classical $M/M/1$ queue without vacations and L_d is the additional queue length due to the effect of working vacation or vacation with its pgf as*

$$L_d(z) = \left(\frac{1}{1-\rho}\right)\left\{\left[1 - \rho z - \frac{(\phi z+\mu_v(1-q))}{\mu_b(1-z)}\right]P_0(z) + z\left[\frac{\phi+\mu_v(1-q)}{\mu_b(1-z)}\right]P_0(1) + \left[1-\rho z - \frac{\theta z}{\mu_b(1-z)}\right]P_1(z) + \frac{\theta z}{\mu_b(1-z)}P_1(1) + \frac{\mu_v(1-q)}{\mu_b}P_{00}\right\}. \tag{40}$$

Proof. Consider

$$\begin{aligned}
L(z) &= P_0(z) + P_1(z) + P_2(z)\\
&= \left[1 + \frac{\phi z + \mu_v(1-q)}{(1-z)(\lambda z - \mu_b)}\right] P_0(z) + \left[1 + \frac{\theta z}{(1-z)(\lambda z - \mu_b)}\right] P_1(z)\\
&\quad - z\left[\frac{\phi + \mu_v(1-q)}{(1-z)(\lambda z - \mu_b)}\right] P_0(1) - \left[\frac{\theta z}{(1-z)(\lambda z - \mu_b)}\right] P_1(1) - \frac{\mu_v(1-q)}{\lambda z - \mu_b} P_{00}\\
&= \left(\frac{\mu_b - \lambda}{\mu_b - \lambda z}\right)\left\{\left[\frac{\mu_b - \lambda z}{\mu_b - \lambda} - \frac{(\phi z + \mu_v(1-q))}{(\mu_b - \lambda)(1-z)}\right] P_0(z)\right.\\
&\quad + z\left[\frac{\phi + \mu_v(1-q)}{(1-z)(\mu_b - \lambda)}\right] P_0(1) + \left[\frac{\mu_b - \lambda z}{\mu_b - \lambda} - \frac{\theta z}{(\mu_b - \lambda)(1-z)}\right] P_1(z)\\
&\quad \left. + \left[\frac{\theta z}{(1-z)(\mu_b - \lambda)}\right] P_1(1) + \frac{\mu_v(1-q)}{\mu_b - \lambda} P_{00}\right\} = \frac{(1-\rho)}{1-\rho z} \times L_d(z),
\end{aligned}$$

where $L_d(z)$ can be expressed in series expansion as

$$\begin{aligned}
L_d(z) &= \left(\frac{1}{1-\rho}\right)\left\{\left[1 - \rho z - \frac{(\phi z + \mu_v(1-q))}{\mu_b(1-z)}\right] P_0(z) + z\left[\frac{\phi + \mu_v(1-q)}{\mu_b(1-z)}\right] P_0(1)\right.\\
&\quad \left. + \left[1 - \rho z - \frac{\theta z}{\mu_b(1-z)}\right] P_1(z) + \frac{\theta z}{\mu_b(1-z)} P_1(1) + \frac{\mu_v(1-q)}{\mu_b} P_{00}\right\}\\
&= \frac{1}{1-\rho}\left\{\sum_{n=0}^{\infty} P_{n,0} z^n - \rho \sum_{n=0}^{\infty} P_{n,0} z^{n+1} + \frac{\phi}{\mu_b} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} P_{n+k,0} z^n\right.\\
&\quad + \frac{\mu_v(1-q)}{\mu_b} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} P_{n+k+1,0} z^n + \sum_{n=0}^{\infty} P_{n,1} z^n - \rho \sum_{n=0}^{\infty} P_{n,1} z^{n+1}\\
&\quad \left. + \frac{\phi}{\mu_b} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} P_{n+k,1} z^n\right\} = \sum_{n=0}^{\infty} t_n z^n,
\end{aligned}$$

such that $t_0 = \frac{1}{1-\rho}(P_{00} + P_{01})$, and

$$\begin{aligned}
t_n = \frac{1}{1-\rho}&\left\{P_{n,0} - \rho P_{n-1,0} + \frac{\phi}{\mu_b}\sum_{k=0}^{\infty} P_{n+k,0} + \frac{\mu_v(1-q)}{\mu_b}\sum_{k=0}^{\infty} P_{n+k+1,0}\right.\\
&\left. + P_{n,1} - \rho P_{n-1,1} + \frac{\phi}{\mu_b}\sum_{k=0}^{\infty} P_{n+k,1}\right\}, \quad n \geq 1.
\end{aligned}$$

Now, we show that $\sum_{n=0}^{\infty} t_n = 1$ for $t_n \in [0,1]$.

$$\sum_{n=0}^{\infty} t_n = \frac{1}{1-\rho}\Bigg\{(1-\rho)\sum_{n=0}^{\infty} P_{n,0} + \frac{\phi}{\mu_b}\sum_{n=1}^{\infty} nP_{n,0} + \frac{(1-q)\mu_v}{\mu_b}\sum_{n=1}^{\infty}(n-1)P_{n,0}$$
$$+ (1-\rho)\sum_{n=0}^{\infty} P_{n,1} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty} nP_{n,1}\Bigg\}$$
$$= \frac{1}{1-\rho}\Bigg\{(1-\rho)\sum_{n=1}^{\infty} P_{n,0} + \left(\frac{\phi+\mu_v(1-q)}{\mu_b}\right)\sum_{n=1}^{\infty} nP_{n,0}$$
$$- \frac{\mu_v(1-q)}{\mu_b}\sum_{n=1}^{\infty}(n-1)P_{n,0} + (1-\rho)\sum_{n=0}^{\infty} P_{n,1} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty} nP_{n,1}\Bigg\}.$$

Applying equation (31), we get

$$\sum_{n=0}^{\infty} t_n = \frac{1}{1-\rho}\Bigg\{(1-\rho)\sum_{n=1}^{\infty} P_{n,0} - \frac{\mu_v(1-q)}{\mu_b}\sum_{n=1}^{\infty} P_{n,0} + (1-\rho)\sum_{n=0}^{\infty} P_{n,1} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty} nP_{n,1}$$
$$+ \left(\frac{\phi+\mu_v(1-q)}{\mu_b}\right)\left[\frac{(\mu_b-\lambda)P_2(1) + \mu_v(1-q)(P_0(1)-P_{00}) - \theta P_1'(1)}{\phi+\mu_v(1-q)}\right]\Bigg\} +$$
$$= \sum_{n=0}^{\infty} P_{n,0} + 1 - P_0(1) - P_1(1) - \frac{(1-q)\mu_v}{\mu_b(1-\rho)}P_{00} + \frac{(1-q)\mu_v}{\mu_b(1-\rho)}P_{00} + \sum_{n=0}^{\infty} P_{n,1} = 1.$$

Hence, $L_d(z)$ is a PGF of the additional queue length due to the Bernoulli schedule vacation interruption. □

Theorem 2 *If $\lambda < \mu_b$, the stationary waiting time can be decomposed into the sum of two independent random variables as $W = W_0 + W_d$, where W_0 is the waiting time of a customer corresponding to classical M/M/1 queue which has an exponential distribution with the parameter $\mu_b(1-\rho)$ and W_d is the additional delay due to due to the effect of working vacation or vacation with its Laplace-Stieltjes transform (LST).*

$$W_d^*(s) = \frac{1}{(\mu_b-\lambda)s}\Bigg\{[(\mu_b-\lambda+s)s - \phi(\lambda-s) - \lambda(1-q)\mu_v]\,P_0\left(1-\frac{s}{\lambda}\right)$$
$$+ [(\mu_b-\lambda+s)s - \theta(\lambda-s)]\,P_1\left(1-\frac{s}{\lambda}\right)$$
$$+(\lambda-s)(\phi+\mu_v(1-q))P_0(1) + (\lambda-s)\theta P_1(1) + (1-q)\mu_v sP_{00}\Bigg\}.$$

Proof. The relationship between the probability generating function L and LST of waiting time [12] is given by

$$L(z) = W^*(\lambda(1-z)).$$

Assume that $s = \lambda(1-z)$, so $z = 1 - \frac{s}{\lambda}$ and $1 - z = \frac{s}{\lambda}$. Applying the relations in equation (40), we obtain the desired result. □

7 Cost model

Practically, queueing managers are interested in minimizing operating cost of unit time. In this part of paper, we first formulate a steady-state expected cost function per unit time, where the service rate (μ_b) is the decision variable. Our main goal is to determine the optimum value of μ_b in order to minimize the expected cost function. To this end, we have to define the following cost elements:

- C_1 : Cost per unit time when the server is on working during regular busy period.
- C_2 : Cost per unit time when the server is on vacation period.
- C_3 : Cost per unit time when the server is on busy period.
- C_4 : Cost per service per unit time during regular busy period.
- C_5 : Cost per service per unit time during working vacation period.
- C_6 : Cost per unit time when a customer reneges.
- C_7 : Holding cost per customer per unit time.

Let $\mathcal{T}_c$ be the total expected cost per unit time of the system:

$$\mathcal{T}_c = C_1 P_0(1) + C_2 P_1(1) + C_3 P_2(1) + \mu_b C_4 + \mu_v C_5 + C_6 R_{ren} + C_7 E[L].$$

7.1 The optimization study

In this subsection we focus on the optimization of the service rate (μ_b) in different cases in order to minimize the cost function $\mathcal{T}_c$. We solve the stated optimization problem using QFSM method.

Given a 3-point pattern, we can fit a quadratic function through corresponding functional values that has a unique minimum, x^q, for the given objective function $\mathcal{T}_c(x)$. Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with optimum x^q. The unique optimum x^q of the quadratic function agreeing with $\mathcal{T}_c(x)$ at 3-point operation (x^l, x^m, x^u) is given by

$$x^q \cong \frac{1}{2}\left[\frac{\mathcal{T}_c(x^l)((x^m)^2-(x^u)^2)+\mathcal{T}_c(x^m)((x^u)^2-(x^l)^2)+\mathcal{T}_c(x^u)((x^l)^2-(x^m)^2)}{\mathcal{T}_c(x^l)(x^m-x^u)+\mathcal{T}_c(x^m)(x^u-x^l)+\mathcal{T}_c(x^u)(x^l-x^m)}\right].$$

The optimization problem can be illustrated mathematically as:

$$\text{Minimize: } \mathcal{T}_c(\mu_b) = C_1P_0(1)+C_2P_1(1)+C_3P_2(1)+\mu_bC_4+\mu_vC_5+C_6R_{ren}+C_7E[L].$$

Suppose that all system parameters have fixed values, and the only controlled parameter is the service rate (μ_b).

8 Numerical results

In this section, we provide numerical experiments to illustrate how different system parameters affect some system characteristics.

The system parameters chosen are presented in Tables and Figures given in the following items:

- Table 1 and Figure 2 : $\lambda = 2.4, \mu_v = 3.0, p = 0.3, q = 0.8, \theta = 1.8, \phi = 0.8, \alpha = 0.1$, and $\xi = 1.9$.
- Table 2 : $\mu_v = 2.6, p = 0.4, \theta = 1.4, \phi = 0.8, \alpha = 0.1$, and $\xi = 1.2$.
- Table 3 : $\lambda = 3.2, q = 0.6, \theta = 1.1, \phi = 0.7, \alpha = 0.3$, and $\xi = 1.7$.
- Table 4 : $\lambda = 3.0, q = 0.7, \theta = 0.8, \phi = 0.2, \mu_v = 2.4$, and $p = 0.4$.
- Table 5 : $\lambda = 2.8, q = 0.8, \alpha = 0.2, \xi = 1.5, \mu_v = 2.2$, and $p = 0.4$.
- Figure 3 : $\mu_b = 4.5, \mu_v = 2.6, \alpha = 0.1, \xi = 1.2, \phi = 0.8, p = 0.4$, and $\theta = 1.4$.
- Figure 4 : $\lambda = 3.4, \mu_v = 2.6, \alpha = 0.1, \xi = 1.2, \phi = 0.8$, and $\theta = 1.4$.

- Figure 5 : $\mu_b = 4.7, q = 0.9, \alpha = 0.2, \xi = 1.2, \phi = 0.3, \theta = 0.7$, and $p = 0.4$.
- Figures 6-8 : $\lambda = 3.0, \mu_b = 4.5, \mu_v = 2.6, q = 0.7, \xi = 1.2, \theta = 1.4$, and $p = 0.4$.
- Figures 7-9 : $\lambda = 3.0, \mu_b = 4.5, \mu_v = 2.6, q = 0.7, \alpha = 0.4, \phi = 0.6$, and $p = 0.5$.

Table 1: Search for the optimum service rate μ_b^* during regular busy period.

μ^l	μ^m	μ^u	$\mathcal{T}_c(\mu^l)$	$\mathcal{T}_c(\mu^m)$	$\mathcal{T}_c(\mu^u)$	μ^q	$\mathcal{T}_c(\mu^q)$
5.100000	5.400000	5.700000	410.484439	394.420852	391.963589	5.604179	391.910733
5.400000	5.604179	5.700000	394.420852	391.910733	391.963589	5.645648	391.857148
5.604179	5.645648	5.700000	391.910733	391.857148	391.963589	5.643959	391.856942
5.604179	5.643959	5.645648	391.910733	391.856942	391.857148	5.643089	391.856912
5.604179	5.643089	5.643959	391.910733	391.856912	391.856942	5.643048	391.856912
5.604179	5.643048	5.643089	391.910733	391.856912	391.856912	5.643033	391.856912
5.604179	5.643033	5.643048	391.910733	391.856912	391.856912	5.643032	391.856912
5.604179	5.643032	5.643033	391.910733	391.856912	391.856912	5.643031	391.856912

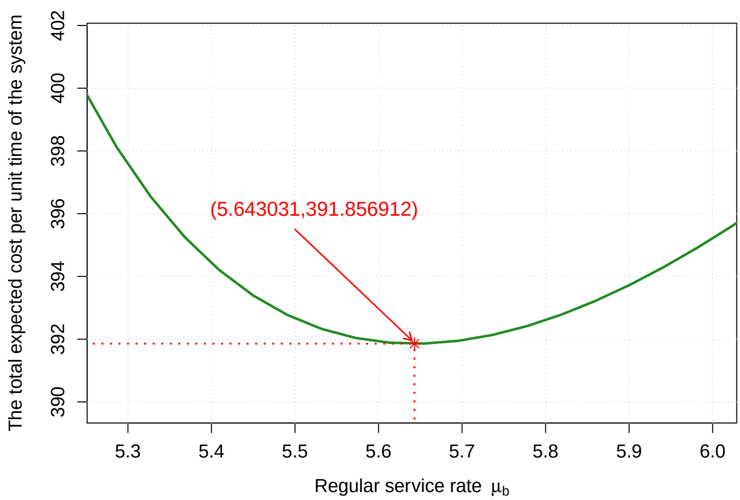

Figure 2: Effect of μ_b on $\mathcal{T}_c$.

Table 2: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of λ and $\bar{q}$.

	$\lambda = 3.5$		$\lambda = 4.5$		$\lambda = 5.5$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$\bar{q} = 0.3$	4.862986	352.655384	6.012295	410.465278	7.138987	466.276595
$\bar{q} = 0.6$	4.589849	333.020545	5.717871	387.784157	6.829078	440.910714
$\bar{q} = 0.9$	4.449532	323.932229	5.563229	377.161323	6.663980	429.046786

Table 3: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of μ_v and p.

	$\mu_v = 2.2$		$\mu_v = 2.5$		$\mu_v = 2.8$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$p = 0.3$	3.607634	279.308457	3.600697	286.217326	3.593235	293.371886
$p = 0.6$	3.313594	257.348468	3.310018	265.481915	3.306657	273.738369
$p = 0.9$	3.134604	243.821124	3.133657	252.623948	3.132829	261.457645

Table 4: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of α and ξ.

	$\alpha = 0.1$		$\alpha = 0.4$		$\alpha = 0.7$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$\xi = 0.5$	4.101791	324.408533	4.026891	319.266445	3.961865	315.540988
$\xi = 1.0$	4.126021	322.687741	4.045522	317.026212	3.975738	312.850216
$\xi = 1.5$	4.139667	322.657325	4.056594	316.842638	3.984583	312.525274

Table 5: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of θ and ϕ.

	$\theta = 0.8$		$\theta = 1.4$		$\theta = 2.0$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$\phi = 0.4$	4.026534	307.781311	4.023896	302.709691	4.020944	300.922353
$\phi = 0.8$	4.114485	312.824843	4.078663	302.785848	4.071751	298.539176
$\phi = 1.2$	4.228526	319.734187	4.143652	305.561523	4.112019	299.095820

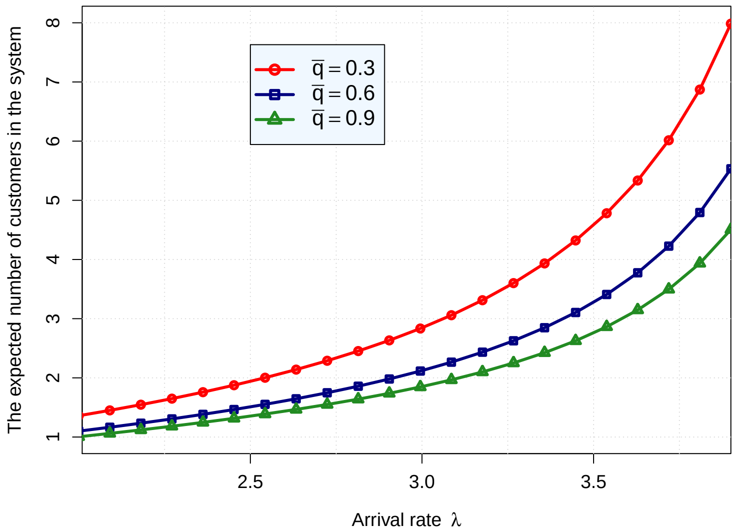

Figure 3: Effect of λ and $\bar{q}$ on $E[L]$.

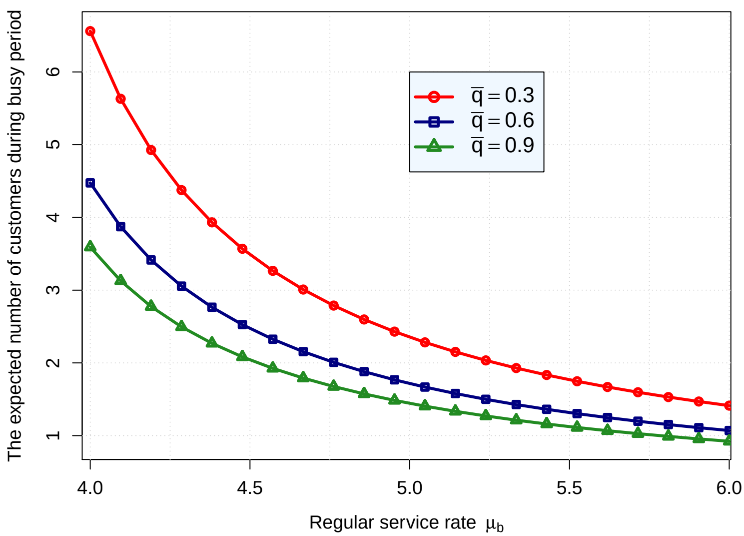

Figure 4: Effect of μ_b and $\bar{q}$ on $E[L_2]$.

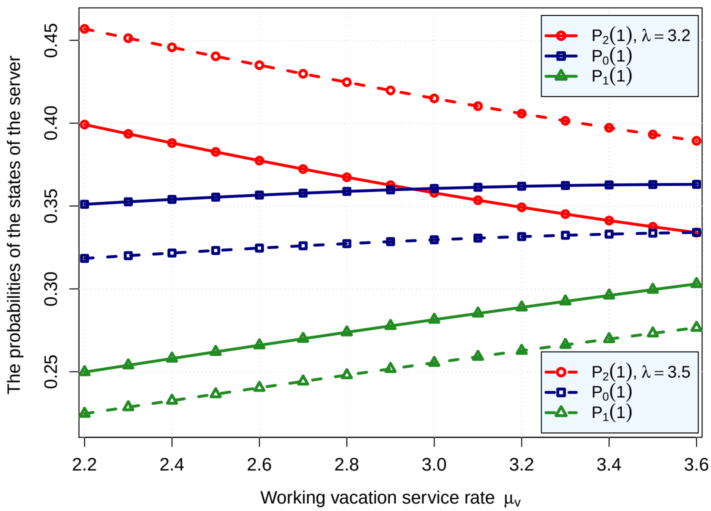

Figure 5: Effect of μ_v and λ on $P_0(1), P_1(1)$ and $P_2(1)$.

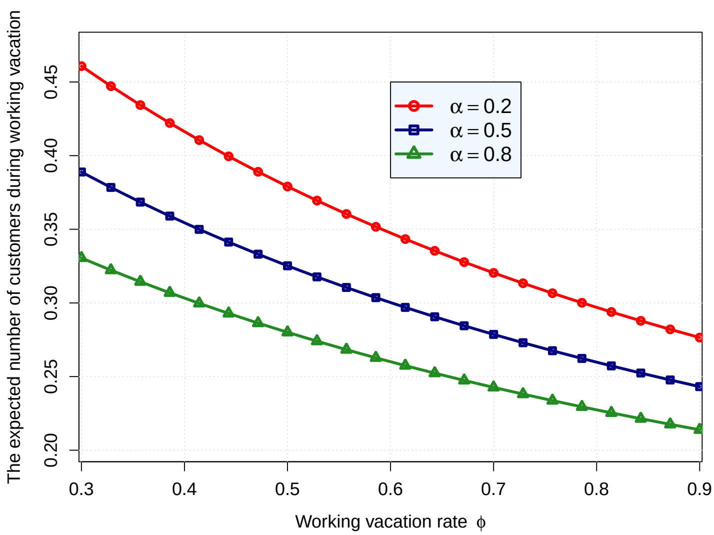

Figure 6: Effect of ϕ and α on $E[L_0]$.

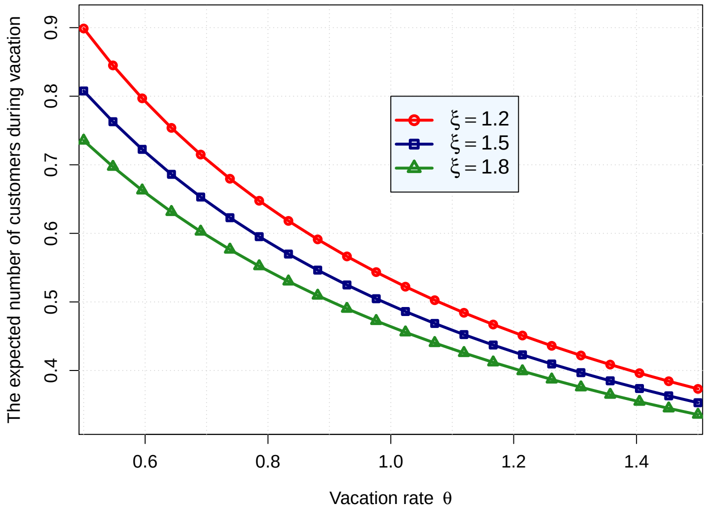

Figure 7: Effect of θ and ξ on $E[L_1]$.

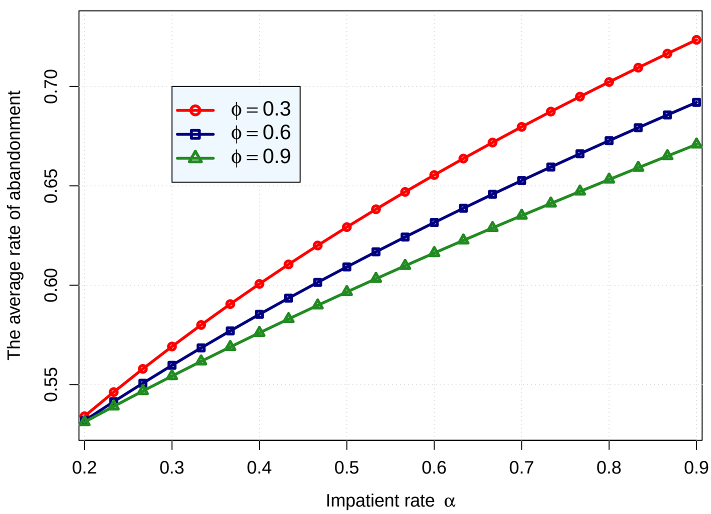

Figure 8: Effect of α and ϕ on R_a.

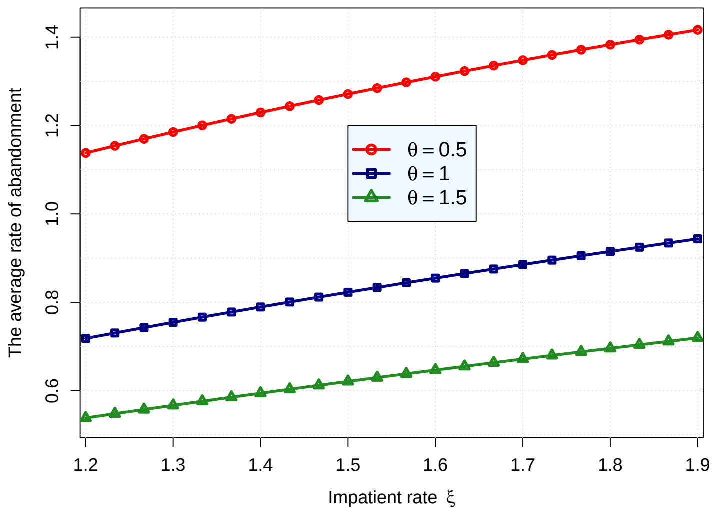

Figure 9: Effect of ξ and θ on R_a.

8.1 Discussion

– From Table 1 and Figure 2, we easily observe that the curve is convex. This proves that there exists some value of the service rate μ_b that minimizes the total expected cost function for the chosen set of model parameters. By adopting QFSM and choosing the initial 3-point pattern as $(\mu^l, \mu^m, \mu^u) = (5.10, 5.40, 5.70)$, and after finite iterations, we see that the minimum expected operating cost per unit time converges to the solution $\mathcal{T}_c = 391.856912$ at $\mu_b^* = 5.643031$.

– From Tables 2-5, we have:

– As intuitively expected, the optimum cost function $\mathcal{T}_c(\mu_b^*)$ increases with (λ), (μ_v), and (ϕ) and decreases with $(\overline{q})$, (p), (ξ), (α), and (θ). With the increasing of the arrival rate, the mean system size increases significantly. This increases significantly the optimum cost function $\mathcal{T}_c(\mu_b^*)$. Obviously, the increasing of the vacation rate increases the probability of the regular busy period which in turns decreases the mean system size. This results in the decreasing of the minimum expected cost. Further, the impatience rates either during vacation or working vacation periods lead to the decreasing of the mean number of customers in the systems which implies a decreasing in the optimal expected cost. Then, when the probability with which the server resumes its service during working vacation period to the regular service increases the customers are served faster. Consequently, $\mathcal{T}_c(\mu_b^*)$ decreases. The same when the probability that the server switches to the vacation period at which the customers may get impatient and leave the system. This yields to the decreasing of the mean number of customers in the system and consequently the total expected cost decreases accordingly. In addition, the decreasing of the optimum cost function $\mathcal{T}_c(\mu_b^*)$ with (ϕ) can be due to the choice of the system parameters.

– The average rate of abandonment (R_a) increases with (ξ) and (α) and decreases with (θ) and (ϕ). This is quite reasonable; the higher the impatience rate (resp. vacation and working vacation rate), the greater (resp. the lower) the average rate of reneging (R_a) and the smaller the mean number of customers in the system $(E(L_0))$ and $(E(L_1))$.

– With the increasing of (μ_b) and $(\bar{q})$, the mean number of customers in the system decreases. Obviously, the smaller (resp. greater) the mean service rate during regular busy period (resp. the probability that the server switches to the regular busy period), the higher the mean number of customers served and the smaller the mean system size during this period $(E(L_2))$.

– As it should be, the service rate (μ_v) decreases the probability that the server is in regular period $(P_2(1))$ and increases the probabilities that the server is on vacation and working vacation periods $(P_1(1))$ and $(P_0(1))$ respectively. Further, obviously, the increasing of the arrival rate (λ) increases $(P_0(1))$, $(P_1(1))$, and $(P_2(1))$.

9 Conclusion

The steady-state solution of an infinite-space single-server Markovian queueing system with working vacation (WV), Bernoulli schedule vacation interruption, and impatient customers has been presented. The proposed queueing system can be applied in diverse real life situations of day-to-day as well as industrial congestion problems including call centers, telecommunication networks, manufacturing system, and so on. The analytical results using probability generating function (PGF) technique are obtained. The performance indices derived may be helpful to the decision makers for improving the availability of the server.

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