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All intra-regular generalized hypersubstitutions of type (2)

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Abstract. A generalized hypersubstitution of type τ maps each operation symbol of the type to a term of the type, and can be extended to a mapping defined on the set of all terms of this type. The set of all such generalized hypersubstitutions forms a monoid. An element a of a semigroup S is intra-regular if there is $b \in S$ such that $a = baab$. In this paper, we determine the set of all intra-regular elements of this monoid for type $\tau = (2)$.

1 Introduction

A solid variety is a variety in which every identity holds as a hyperidentity, that is, we substitute not only elements for the variables but also term operations for the operation symbols. The notions of hyperidentities and hypervarieties of a given type τ without nullary operations were studied by J. Aczèl [1], V. D. Belousov [2], W.D. Neumann [8] and W. Taylor [13]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution, introduced by K. Denecke et al. [5]. The concept of a generalized hypersubstitution was introduced by S. Leeratanavalee and K. Denecke [7]. The authors

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defined a binary operation on the set of all generalized hypersubstitutions and proved that this set together with the binary operation forms a monoid. In 2010, W. Puninagool and S. Leeratanavalee determined all regular elements of this monoid for type $\tau = (n)$, see [10]. The set of all completely regular elements of this monoid of type $\tau = (n)$ was determined by A. Boonmee and S. Leeratanavalee [3]. Furthermore, we found that every completely regular element is intra-regular. In the present paper, we show that the set of all completely regular elements and the set of all intra-regular elements of type $\tau = (2)$ are the same.

Let $n \geq 1$ be a natural number and let $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-element set which is called an n-*element alphabet* and let its elements be called *variables*. Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of variables and $\{f_i \mid i \in I\}$ be a set of n_i-ary operation symbols, which is disjoint from X, indexed by the set I. To every n_i-ary operation symbol f_i we assign a natural number $n_i \geq 1$, called the *arity* of f_i. The sequence $\tau = (n_i)_{i \in I}$ is called the *type*. For $n \geq 1$, an n-*ary term* of type τ is defined in the following inductive way:

(i) Every variable $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

The smallest set which contains $x_1, \ldots, x_n$ and is closed under any finite number of applications of (ii) is denoted by $W_\tau(X_n)$, and is called the set of all n- ary terms of type τ. The set $W_\tau(X) := \cup_{n=1}^{\infty} W_\tau(X_n)$ is called the set of all terms of type τ.

A generalized hypersubstitution of type $\tau = (n_i)_{i \in I}$ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. Let $Hyp_G(\tau)$ be the set of all generalized hypersubstitutions of type τ. In general, the usual composition of mappings can be used as a binary operation on mappings. But in the case of $Hyp_G(\tau)$ this can not be done immediately. To define a binary operation on this set, we define inductively the concept of a generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, s_2, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

We extend any generalized hypersubstitution σ to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\widehat{\sigma}[x] := x \in X$,

(ii) $\widehat{\sigma}[f_i(t_1, t_2, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i assuming that $\widehat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Now, we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. Let σ_{id} be the hypersubstitution which maps each n_i−ary operation symbol f_i to the term $f_i(x_1, x_2, \ldots, x_{n_i})$. Then $\underline{Hyp_G(\tau)} = (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid [7].

From now on, we introduce some notations which will be used throughout this paper. For a type $\tau = (n)$ with an n−ary operation symbol f and $t \in W_{(n)}(X)$, we denote
σ_t - the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,
$\mathrm{var}(t)$ - the set of all variables occurring in the term t,
$\mathrm{vb}^t(x)$ - the total number of x-variable occurring in the term t.

For a term $t \in W_{(n)}(X)$, the set $\mathrm{sub}(t)$ of its subterms is defined as follows ([11], [12]):

(i) if $t \in X$, then $\mathrm{sub}(t) = \{t\}$,

(ii) if $t = f(t_1, \ldots, t_n)$, then $\mathrm{sub}(t) = \{t\} \cup \mathrm{sub}(t_1) \cup \ldots \cup \mathrm{sub}(t_n)$.

Example 1 Let $\tau = (2)$ and $t \in W_{(2)}(X)$ where $t = f(t_1, t_2)$ with $t_1 = f(x_3, f(x_1, x_4))$ and $t_2 = f(f(x_7, x_1), f(x_2, x_1))$. Then

$\mathrm{var}(t) = \{x_1, x_2, x_3, x_4, x_7\}$
$\mathrm{vb}^t(x_1) = 3$, $vb^t(x_2) = 1$, $vb^t(x_3) = 1$, $vb^t(x_4) = 1$, $vb^t(x_7) = 1$,
$\mathrm{sub}(t_1) = \{t_1, f(x_1, x_4), x_1, x_3, x_4\}$,
$\mathrm{sub}(t_2) = \{t_2, f(x_7, x_1), f(x_2, x_1), x_1, x_2, x_7\}$,
$\mathrm{sub}(t) = \{t, t_1, t_2, f(x_1, x_4), f(x_7, x_1), f(x_2, x_1), x_1, x_2, x_3, x_4, x_7\}$.

2 Sequence of terms

In this section, we construct some tools used to characterize all intra-regular elements in $Hyp_G(2)$. These tools are called the *sequence* of a term and the *depth* of a term, respectively.

Definition 1 *Let* $t \in W_{(n)}(X) \setminus X$ *where* $t = f(t_1 \ldots, t_n)$ *for some* $t_1, \ldots t_n \in W_{(n)}(X)$*. For each* $s \in \mathrm{sub}(t)$*,* $s \neq t$*, a set* $\mathrm{seq}^t(s)$ *of sequences of* s *in* t *is defined by where* $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ *by the formula* $\pi_{i_l}(f(t_1, \ldots, t_n)) = t_{i_l}$*. Maps* π_{i_l} *are defined for* $i_l = 1, 2, \ldots, n$*.*

Example 2 *Let* $t \in W_{(4)}(X)$ *where* $t = f(t_1, t_2, t_3, t_4)$ *such that* $t_1 = f(x_3, x_1, s, x_4)$*,* $t_2 = x_4$*,* $t_3 = (f(x_7, s, x_1, x_4), x_4, f(x_8, f(x_3, x_1, s, x_4), x_2, f(x_3, x_1, s, x_4)), s)$ *and* $t_4 = s$ *for some* $s \in W_{(4)}(X)$*. Then*

$$\mathrm{seq}^t(s) = \{(1,3), (3,1,2), (3,3,2,3), (3,3,4,3), (3,4), (4)\},$$
$$\mathrm{seq}^{t_3}(s) = \{(1,2), (3,2,3), (3,4,3), (4)\},$$
$$\mathrm{seq}^t(t_1) = \{(1), (3,3,2), (3,3,4)\}$$
$$\mathrm{seq}^t(x_4) = \{(1,4), (2), (3,1,3)\}.$$

Lemma 1 ([4]) *Let* $t, s \in W_{(n)}(X) \setminus X$*,* $x \in \mathrm{var}(t)$ *and* $\mathrm{var}(s) \cap X_n = \{x_{z_1}, \ldots, x_{z_k}\}$*. If* $(i_1, \ldots, i_m) \in \mathrm{seq}^t(x)$ *where* $i_1, \ldots, i_m \in \{z_1, \ldots, z_k\}$ *then* $x \in \mathrm{var}(\widehat{\sigma}_s[t]) = \mathrm{var}(\sigma_s \circ_G \sigma_t)$ *and there is* $(a_{i_1}, \ldots, a_{i_m}) \in \mathrm{seq}^{\widehat{\sigma}_s[t]}(x)$ *where* a_{i_j} *is a sequence of natural numbers* $j_1, \ldots, j_h$ *such that* $(j_1, \ldots, j_h) \in \mathrm{seq}^s(x_{i_j})$ *for all* $j \in \{1, \ldots, m\}$*.*

Let $t \in W_{(n)}(X) \setminus X$, and $t_i \in \mathrm{sub}(t)$. It can be possible that t_i occurs in the term t more than once, we denote

$t_i^{(j)}$ - subterm t_i occurring in the j^{th} order of t (from the left).

Definition 2 Let $t \in W_{(n)}(X) \setminus X$ where $t = f(t_1, \ldots, t_n)$ for some $t_1, \ldots, t_n \in W_{(n)}(X)$ and let $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ by the formula $\pi_{i_l}(t) = \pi_{i_l}(f(t_1, \ldots, t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, 2, \ldots, n$. For each $s^{(j)} \in \mathrm{sub}(t)$ for some $j \in \mathbb{N}$, we denote the sequence of $s^{(j)}$ in t by $\mathrm{seq}^t(s^{(j)})$ and denote the depth of $s^{(j)}$ in t by $\mathrm{depth}^t(s^{(j)})$. If $s^{(j)} = \pi_{i_m} \circ \ldots \circ \pi_{i_1}(t)$ for some $m \in \mathbb{N}$, then

$$\mathrm{seq}^t(s^{(j)}) = (i_1, \ldots, i_m) \quad \text{and} \quad \mathrm{depth}^t(s^{(j)}) = m.$$

Example 3 *Let* $\tau = (3)$ *and let* $t \in W_{(3)}(X) \setminus X$ *where* $t = f(t_1, t_2, t_3)$ *such that* $t_1 = x_5$*,* $t_2 = f(x_3, f(x_4, f(x_2, x_7, x_{10}), x_5), x_5)$ *and* $t_3 = f(f(x_5, x_4, f(x_2, x_7, x_{10})), x_1, x_6)$*. Then*

$$\mathrm{seq}^t(x_5^{(1)}) = (1) \quad \text{and} \quad \mathrm{depth}^t(x_5^{(1)}) = 1;$$
$$\mathrm{seq}^t(x_5^{(2)}) = (2,2,3) \quad \text{and} \quad \mathrm{depth}^t(x_5^{(2)}) = 3;$$
$$\mathrm{seq}^t(x_5^{(3)}) = (2,3) \quad \text{and} \quad \mathrm{depth}^t(x_5^{(3)}) = 2;$$

$$\begin{aligned}
seq^t(x_5^{(4)}) &= (3,1,1) \quad \text{and} \quad depth^t(x_5^{(4)}) = 3;\\
seq^t(f(x_2,x_7,x_{10})^{(1)}) &= (2,2,2) \quad \text{and} \quad depth^t(f(x_2,x_7,x_{10})^{(1)}) = 3;\\
seq^t(f(x_2,x_7,x_{10})^{(2)}) &= (3,1,3) \quad \text{and} \quad depth^t(f(x_2,x_7,x_{10})^{(2)}) = 3;\\
seq^{t_3}(f(x_2,x_7,x_{10})^{(1)}) &= (1,3) \quad \text{and} \quad depth^{t_3}(f(x_2,x_7,x_{10})^{(1)}) = 2;\\
seq^t(x_{10}^{(1)}) &= (2,2,2,3) \quad \text{and} \quad depth^t(x_{10}^{(1)}) = (4);\\
seq^t(x_{10}^{(2)}) &= (3,1,3,3) \quad \text{and} \quad depth^t(x_{10}^{(2)}) = 4;\\
seq^{t_3}(x_{10}^{(1)}) &= (1,3,3) \quad \text{and} \quad depth^{t_3}(x_{10}^{(1)}) = 3.
\end{aligned}$$

Let $t, s_1, s_2, \ldots, s_k \in W_{(n)}(X) \setminus X$ and $x_i \in var(t)$. We donote

$x_i^{(j)}$ - variable x_i occurring in the j^{th} order of t (from the left);

$x_i^{(j,j_1)}$ - variable $x_i^{(j)}$ occurring in the j_1^{th} order of $\widehat{\sigma}_{s_1}[t]$ (from the left);

$x_i^{(j,j_1,j_2)}$ - variable $x_i^{(j,j_1)}$ occurring in the j_2^{th} order of $\widehat{\sigma}_{s_2}[\widehat{\sigma}_{s_1}[t]]$ (from the left).

Similarly,

$x_i^{(j,j_1,j_2,\ldots,j_k)}$ - variable $x_i^{(j,j_1,\ldots,j_{k-1})}$ occurring in the j_k^{th} order of $\widehat{\sigma}_{s_k}[\widehat{\sigma}_{s_{k-1}}[\ldots[\widehat{\sigma}_{s_2}[\widehat{\sigma}_{s_1}[t]]\ldots]$ (from the left).

Theorem 1 *Let* $t, s \in W_{(n)}(X) \setminus X$ *and* $x_i^{(j)} \in var(t)$ *for some* $i, j \in \mathbb{N}$ *and let* $seq^t(x_i^{(j)}) = i_1, \ldots, i_m$. *Then* $x_{i_1}, \ldots, x_{i_m} \in var(s) \cap X_n$ *if and only if* $x_i^{(j,j_1)} \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ *for some* $j_1 \in \mathbb{N}$ *and* $seq^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = (a_{i_1}, \ldots, a_{i_m})$ *where* a_{i_l} *is a sequence of natural number* $p_1, \ldots, p_q$ *such that* $(p_1, \ldots, p_q) = seq^s(x_{i_l}^{h_l})$ *for some* $h_l \in \mathbb{N}$ *and for all* $l \in \{1, \ldots, m\}$.

Proof.($\Rightarrow$) By Lemma 1.
($\Leftarrow$) Assume that $x_i^{(j,j_1)} \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ for some $j_1 \in \mathbb{N}$ and $seq^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = (a_{i_1}, \ldots, a_{i_m})$ where a_{i_l} is a sequence of natural number $p_1, \ldots, p_q$ such that $(p_1, \ldots, p_q) = seq^s(x_{i_l}^{h_l})$ for some $h_l \in \mathbb{N}$ and for all $l \in \{1, \ldots, m\}$. Then

$$\nu b^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = \nu b^s(x_{i_1}) \times \nu b^s(x_{i_2}) \times \ldots \times \nu b^s(x_{i_m}).$$

Suppose that $x_{i_k} \notin var(s) \cap X_n$ for some $1 \leq k \leq m$, so $\nu b^s(x_{i_z}) = 0$, i.e. $\nu b^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = 0$, which contradicts to our assumption. Hence $x_{i_1}, \ldots, x_{i_m} \in var(s) \cap X_n$. □

Example 4 Let $\tau = (3)$ and let $t = f(x_2, f(x_4, x_5, x_2), f(x_2, x_6, x_7))$ and $s = f(x_3, x_1, x_3)$. Then $seq^t(x_2^{(1)}) = (1)$, $seq^t(x_2^{(2)}) = (2,3)$, $seq^t(x_2^{(3)}) = (3,1)$

and $\mathrm{seq}^t(x_7^{(1)}) = (3,3)$. By Theorem 1 , there is $x_2^{(1,h)}, x_2^{(3,k_1)}, x_2^{(3,k_2)}, x_7^{(1,l_1)}, x_7^{(1,l_2)}, x_7^{(1,l_3)}, x_7^{(1,l_4)} \in \mathrm{var}(\widehat{\sigma}_s[t])$ for some $h, k_1, k_2, l_1, l_{2,3}, l_4 \in \mathbb{N}$ and

$$\begin{aligned}
\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(1,h)}) &= (2) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(1,2)}) \text{ where } \mathrm{seq}^s(x_1^{(1)}) = (2)\\
\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,k_1)}) &= (1,2) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,1)}) \text{ where } \mathrm{seq}^s(x_3^{(1)}) = (1) \text{ and } \mathrm{seq}^s(x_1^{(1)}) = (2)\\
\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,k_2)}) &= (3,2) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,3)}) \text{ where } \mathrm{seq}^s(x_3^{(2)}) = (3) \text{ and } \mathrm{seq}^s(x_1^{(1)}) = (2)\\
\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_1)}) &= (1,1) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,1)}) \text{ where } \mathrm{seq}^s(x_3^{(1)}) = (1) \text{ and } \mathrm{seq}^s(x_3^{(1)}) = (1)\\
\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_2)}) &= (1,3) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,2)}) \text{ where } \mathrm{seq}^s(x_3^{(1)}) = (1) \text{ and } \mathrm{seq}^s(x_3^{(2)}) = (3)\\
\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_3)}) &= (3,1) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,3)}) \text{ where } \mathrm{seq}^s(x_3^{(2)}) = (3) \text{ and } \mathrm{seq}^s(x_3^{(1)}) = (1)\\
\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_4)}) &= (3,3) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,4)}) \text{ where } \mathrm{seq}^s(x_3^{(2)}) = (3) \text{ and } \mathrm{seq}^s(x_3^{(2)}) = (3).
\end{aligned}$$

Since $x_2 \notin \mathrm{var}(s)$, so $x_2^{(2,i)} \notin \mathrm{var}(\widehat{\sigma}_s[t])$ for all $i \in \mathbb{N}$. Consider,

$$\begin{aligned}
\widehat{\sigma}_s[t] &= \widehat{\sigma}_s[f(x_2^{(1)}, f(x_4, x_5, x_2^{(2)}), f(x_2^{(3)}, x_6, x_7^{(1)}))]\\
&= S^3(f(x_3, x_1, x_3), \widehat{\sigma}_s[x_2^{(1)}], \widehat{\sigma}_s[f(x_4, x_5, x_2^{(2)})], \widehat{\sigma}_s[f(x_2^{(3)}, x_6, x_7^{(1)})])\\
&= f(f(x_7^{(1,1)}, x_2^{(3,1)}, x_7^{(1,2)}), x_2^{(1,2)}, f(x_7^{(1,3)}, x_2^{(3,3)}, x_7^{(1,4)}))\\
&= f(f(x_7, x_2, x_7), x_2, f(x_7, x_2, x_7)).
\end{aligned}$$

Corollary 1 *Let* $t, s \in W_{(n)}(X)\backslash X$ *and* $x_i^{(j)} \in \mathrm{var}(t)$ *for some* $i, j \in \mathbb{N}$ *such that* $\mathrm{seq}^t(x_i^{(j)}) = (i_1, i_2, \ldots, i_m)$ *for some* $i_1, i_2, \ldots, i_m \in \{1, \ldots, n\}$ *and* $x_{i_k} \in \mathrm{var}(s)$ *for all* $1 \le k \le m$*. Then there is* $j_1 \in \mathbb{N}$ *such that*

$$\mathrm{depth}^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = \mathrm{depth}^s(x_{i_1}^{(l_1)}) + \mathrm{depth}^s(x_{i_2}^{(l_2)}) + \ldots + \mathrm{depth}^s(x_{i_m}^{(l_m)})$$

for some $l_1, l_2, \ldots, l_m \in \mathbb{N}$*, and*

$$\nu b^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = \nu b^s(x_{i_1}) \times \nu b^s(x_{i_2}) \times \ldots \times \nu b^s(x_{i_m}).$$

Let $\nu b^t(x_i) = d$.

If $x_i \in X_n$*, then* $\nu b^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} \nu b^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

If $x_i \in X \setminus X_n$ *where* $x_i \notin \text{var}(s)$, *then* $vb^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} vb^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

3 Main results

In this section, we will show that the set of all completely regular elements and the set of all intra-regular elements in $\text{Hyp}_G(2)$ are the same. First, we recall definitions of regular and completely regular elements and then we characterize all completely regular elements in $\text{Hyp}_G(2)$.

Definition 3 [6] An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$.

Definition 4 [9] An element a of a semigroup S is called *completely regular* if there exists $b \in S$ such that $a = aba$ and $ab = ba$.

Let $\sigma_t \in \text{Hyp}_G(2)$. We denote

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | \text{var}(t) \cap X_2 = \emptyset\}$;

$R_3 := \{\sigma_t | t = f(t_1, t_2)$ where $t_i = x_j$ for some $i, j \in \{1, 2\}$ and $\text{var}(t) \cap X_2 = \{x_j\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$

$CR(R_3) := \{\sigma_t | t = f(t_1, t_2)$ where $t_i = x_i$ for some $i \in \{1, 2\}$ and $\text{var}(t) \cap X_2 = \{x_i\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_f(x_2, x_1)\}$.

It was shown in [10] and [3] that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $\text{Hyp}_G(2)$ and $CR(\text{Hyp}_G(2)) := CR(R_3) \cup R_1 \cup R_2$ is the set of all completely regular elements in $\text{Hyp}_G(2)$, respectively.

Definition 5 [9] An element a of a semigroup S is called *intra-regular* if there is $b \in S$ such that $a = baab$.

Theorem 2 [3] *Let S be a semigroup and $a \in S$. If a is completely regular, then a is intra-regular.*

Corollary 2 [3] *Let $\sigma_t \in CR(\text{Hyp}_G(2))$. Then σ_t is intra-regular in $\text{Hyp}_G(2)$.*

Lemma 2 *Let $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X_2$. Then σ_t is not intra-regular in $\text{Hyp}_G(2)$.*

Proof. Let $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X_2$. For each $u \in X$, we get $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ and $\sigma_v \circ_G \sigma_t^2 \circ_G \sigma_u \neq \sigma_t$ for all $v \in W_{(2)}(X)$. Let $u, v \in W_{(2)}(X) \setminus X$ where $u = f(u_1, u_2)$ and $v = f(v_1, v_2)$ for some $u_1, u_2, v_1, v_2 \in W_{(2)}(X)$, we will show that $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. If $t_1 \in X \setminus X_2$ then $x_2 \notin \text{var}(t)$. By Theorem 1, $x_1 \notin \text{var}(\widehat{\sigma}_t[t]) = \text{var}(\sigma_t^2)$, i.e. $\text{var}(\sigma_t^2) \cap X_2 = \emptyset$. Hence $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. If $t_1 \in W_{(2)}(X) \setminus X$,

$$\sigma_t^2(f) = \widehat{\sigma}_t[t] = S^2(f(t_1, x_1), \widehat{\sigma}_t[t_1], x_1) = f(w_1, w_2)$$

where $w_1 = S^2(t_1, \widehat{\sigma}_t[t_1], x_1)$ and $w_2 = S^2(x_1, \widehat{\sigma}_t[t_1], x_1) = \widehat{\sigma}_t[t_1]$. Let $w = f(w_1, w_2)$. Since $t_1 \notin X$, so $w_1 \notin X$ and $w_2 = \widehat{\sigma}_t[t_1] \notin X$. Consider

$$\sigma_t^2 \circ_G \sigma_v(f) = \widehat{\sigma}_w[v] = S^2(f(w_1, w_2), \widehat{\sigma}_w[v_1], \widehat{\sigma}_w[v_2]) = f(s_1, s_2)$$

where $s_i = S^2(w_i, \widehat{\sigma}_w[v_1], \widehat{\sigma}_w[v_2])$ for all $i \in \{1, 2\}$. Since $w_i \notin X$ for all $i \in \{1, 2\}$, $s_i \notin X$ for all $i \in \{1, 2\}$. Then $\widehat{\sigma}_u[s_i] \notin X$ for all $i \in \{1, 2\}$. Consider

$$\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v(f) = S^2(f(u_1, u_2), \widehat{\sigma}_u[s_1], \widehat{\sigma}_u[s_2]) = f(r_1, r_2)$$

where $r_i = S^2(u_i, \widehat{\sigma}_u[s_1], \widehat{\sigma}_u[s_2])$ for all $i \in \{1, 2\}$. If $u_2 \in W_{(2)}(X) \setminus X$ or $u_2 \in X_2$ then $r_2 \notin X$. If $u_2 \in X \setminus X_2$ then $u_2 = r_2$. So $r_2 \neq x_1$. Therefore $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. Hence σ_t is not intra-regular in $Hyp_G(2)$. □

Lemma 3 *Let* $t = f(x_2, t_2)$ *where* $t_2 \in W_{(2)}(X) \setminus X_2$. *Then* σ_t *is not intra-regular in* $Hyp_G(2)$.

Proof. The proof is similar to the proof of Lemma 2. □

Lemma 4 *Let* $t = f(x_1, t_2)$ *where* $t_2 \in W_{(2)}(X) \setminus X_2$ *and* $x_2 \in \text{var}(t)$. *Then* σ_t *is not intra-regular in* $Hyp_G(2)$.

Proof. Assume that $t = f(x_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X_2$ and $x_2 \in \text{var}(t)$. Let $m = \max\{\text{depth}^t(x_2^{(i)}) | x_2^{(i)} \in \text{var}(t) \text{ for some } i \in \mathbb{N}\}$ $(*)$, then there exists $h \in \mathbb{N}$ such that $\text{seq}^t(x_2^{(h)}) = (i_1, i_2, \ldots, i_m)$ where $i_1, i_2, \ldots, i_m \in \{1, 2\}$. It means $x_2^{(h)} = \pi_{i_m} \circ \pi_{i_{m-1}} \circ \ldots \circ \pi_{i_1}(t)$ where maps $\pi_{i_1}, \ldots, \pi_{i_{m-1}}, \pi_{i_m}$ are defined on $W_{(2)}(X) \setminus X_2$ to $W_{(2)}(X)$. Since $x_2^{(h)} \in \text{var}(t_2)$, $\pi_{i_1}(t) = t_2$, i.e. $i_1 = 2$. So $\text{seq}^t(x_2^{(h)}) = (2, i_2, \ldots, i_m)$. By Theorem 1, there is $x_2^{(h,h_1)} \in \text{var}(\widehat{\sigma}_t[t]) = \text{var}(\sigma_t^2)$ for some $h_1 \in \mathbb{N}$ such that

$$\text{seq}^{\sigma_t^2}(x_2^{(h,h_1)}) = (2, i_2, \ldots, i_m, a_{i_2}, \ldots, a_{i_m})$$

where $(2, i_2, \ldots, i_m) = \text{seq}^t(x_2^{(h)})$ and a_{i_z} is a sequence of natural numbers such that $(a_{i_z}) = \text{seq}^s(x_{i_z}^{(h_{i_z})})$ for some $h_{i_z} \in \mathbb{N}$ and for all $2 \leq z \leq m$. [Note: $x_2^{(h)}$ is a variable x_2 occurring in the h^{th} order of t (from the left) and $x_2^{(h,h_1)}$ is a variable $x_2^{(h)}$ occurring in the h_1^{th} order of σ_t^2 (from the left)]. Instead of a sequence $a_{i_2}, \ldots, a_{i_m}$, we write a sequence of natural numbers $w_1, \ldots, w_d$ for some $d \in \mathbb{N}$ and $w_1, \ldots, w_d \in \{1, 2\}$. Then

$$\text{seq}^{\sigma_t^2}(x_2^{(h,h_1)}) = (2, i_2, \ldots, i_m, w_1, \ldots, w_d).$$

Suppose that there exist $u, v \in W_{(2)}(X)$ such that $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v = \sigma_t$ $(**)$, i.e. $u = f(x_1, u_2)$ and $v = f(x_1, v_2)$ for some $u_2, v_2 \in W_2(X)$ where $x_2 \in \text{var}(u_2) \cap \text{var}(v_2)$. Choose $x_2^{(j)} \in \text{var}(v)$ for some $j \in \mathbb{N}$. Then $\text{seq}^v(x_2^{(j)}) = (2, p_1, \ldots, p_q)$ for some $p_1, \ldots, p_q \in \{1, 2\}$ and for some $q \in \mathbb{N}$. By Theorem 1, there is $x_2^{(j,j_1)} \in \text{var}(\sigma_t^2 \circ_G \sigma_v)$ for some $j_1 \in \mathbb{N}$ such that

$$\text{seq}^{\sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1)}) = (2, i_2, \ldots, i_m, w_1, \ldots, w_d, a_{p_1}, \ldots, a_{p_q})$$

where $(2, i_2, \ldots, i_m, w_1, \ldots, w_d) = \text{seq}^{\sigma_t^2}(x_2^{(h,h_1)})$ and a_{p_z} is a sequence of natural numbers such that $(a_{p_z}) = \text{seq}^s(x_{p_z}^{(l_z)})$ for some $l_z \in \mathbb{N}$ and for all $1 \leq z \leq q$. [Note: $x_2^{(j)}$ is a variable x_2 occurring in the j^{th} order of v (from the left) and $x_2^{(j,j_1)}$ is a variable $x_2^{(j)}$ occurring in the j_1^{th} order of $\sigma_t^2 \circ_G \sigma_v$ (from the left)]. Instead of a sequence $a_{p_1}, \ldots, a_{p_q}$ we write a sequence of natural numbers $w_{d+1}, \ldots, w_k$ for some $k \in \mathbb{N}$ and $w_{d+1}, \ldots, w_k \in \{1, 2\}$. Then

$$\text{seq}^{\sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1)}) = (2, i_2, \ldots, i_m, w_1, \ldots, w_d, w_{d+1}, \ldots, w_k).$$

By Theorem 1, we have $x_2^{(j,j_1,j_2)} \in \text{var}(\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v)$ for some $j_2 \in \mathbb{N}$. By Corollary 1, we have

$$\begin{aligned} \text{depth}^{\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1,j_2)}) &= \text{depth}^u(x_2^{(b_1)}) + \text{depth}^u(x_{i_2}^{(b_2)}) + \ldots + \text{depth}^u(x_{i_m}^{(b_m)}) \\ &\quad + \text{depth}^u(x_{w_1}^{(b_{m+1})}) + \ldots + \text{depth}^u(x_{w_d}^{(b_{m+d})}) \\ &\quad + \text{depth}^u(x_{w_{d+1}}^{(b_{m+d+1})}) + \ldots + \text{depth}^u(x_{w_k}^{(b_{m+k})}) \\ &> m \end{aligned}$$

for some $b_1, \ldots, b_m, b_{m+1}, \ldots, b_{m+d}, b_{m+d+1}, \ldots, b_{m+k} \in \mathbb{N}$, which contradicts to $(*)$ and $(**)$. Therefore σ_t is not intra-regular in $\text{Hyp}_G(2)$. □

Lemma 5 *Let* $t = f(t_1, x_2)$ *where* $t_1 \in W_{(2)}(X) \setminus X_2$ *and* $x_1 \in \text{var}(t)$. *Then* σ_t *is not intra-regular in* $\text{Hyp}_G(2)$.

Proof. The proof is similar to the proof of Lemma 4. □

Lemma 6 *If* $t = f(t_1, t_2)$ *where* $t_1, t_2 \in W_{(2)}(X) \setminus X_2$ *and* $\mathrm{var}(t) \cap X_2 \neq \emptyset$ *then* σ_t *is not intra-regular in* $\mathrm{Hyp}_G(2)$.

Proof. Let $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \setminus X_2$ and $\mathrm{var}(t) \cap X_2 \neq \emptyset$.
Case1: $\mathrm{var}(t) \cap X_2 = \{x_i\}$ for some $i \in \{1, 2\}$. Let $j \in \{1, 2\}$ where $i \neq j$.

If j is occurring in $\mathrm{seq}^t(x_i^{(h)})$ for all $x_i^{(h)} \in \mathrm{var}(t)$ then $\mathrm{var}(\sigma_t^2) \cap X_2 = \emptyset$, i.e. $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.

If j is not occurring in $\mathrm{seq}^t(x_i^{(h)})$ for some $x_i^{(h)} \in \mathrm{var}(t)$ then $\mathrm{seq}^t(x_i^{(h)}) = (i_1, i_2, \ldots, i_m)$ where $i_1, i_2, \ldots, i_m \in \{i\}$ for some $m \in \mathbb{N}$. We can prove similar to the proof of Lemma 4, then $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.
Case2: $\mathrm{var}(t) \cap X_2 = X_2$. We can prove similar to the proof of Lemma 4, then $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.

Therefore σ_t is not intra-regular in $\mathrm{Hyp}_G(2)$. □

Theorem 3 $\mathrm{CR}(\mathrm{Hyp}_G(2))$ *is the set of all intra-regular elements in* $\mathrm{Hyp}_G(2)$.

Proof. By Corollary 2 and by Lemma 2 to 6. □

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