Acta Univ. Sapientiae, Informatica **11,** 1 (2019) 24–40

DOI: 10.2478/ausi-2019-0002

A study on the pendant number of graph products

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Abstract. A path decomposition of a graph is a collection of its edge disjoint paths whose union is G. The pendant number Π_p is the minimum number of end vertices of paths in a path decomposition of G. In this paper, we determine the pendant number of corona products and rooted products of paths and cycles and obtain some bounds for the pendant number for some specific derived graphs. Further, for any natural number n, the existence of a connected graph with pendant number n has also been established.

1 Introduction

We refer to West [1] and Harary[2] for terms and definitions in graph theory. All graphs we consider in this paper are undirected, simple, finite and connected.

Computing Classification System 1998: G.2.2
Mathematics Subject Classification 2010: 05C70, 05C76, 05C38
Key words and phrases: graph operations, path decomposition, pendant number

Partition of a graph G into its subgraphs is also termed as *decomposition* of G. A *path-decomposition* of a graph G is the partitioning of its edges into subgraphs S_i, $1 \leq i \leq n$, where each of the subgraph S_i is a path in G.

Definition 1 [3] The *pendant number* of a graph G, denoted by $\Pi_p(G)$, is the least number of vertices in a graph such that they are the end vertices of a path in a given path decomposition of a graph G. If $V_p(G)$ denotes the set of all $u \in V(G)$ such that u is an end vertex of a path in P-decomposition in G, then $\Pi_p(G) = \min\{|V_p(G)|\}$.

An introductory study on pendant number of graphs is available in [3]. A similar study on the star number of graphs can be seen in [4]. For the discussions in this paper, we use the following theorems.

Theorem 2 [3] *Let G be a connected graph with n vertices. If G has l odd degree vertices, then $l \leq \Pi_p(G) \leq n$.*

Theorem 3 [3] *Let T be tree on n vertices of which k vertices are of even degree. Then, $\Pi_p(T) = n - k$.*

Theorem 4 [3] *For a unicyclic graph G of order n; $n \geq 3$ with l odd degree vertices, we have*

$$\Pi_p(G) = \begin{cases} 2 & \text{if } m = 0; \\ l+1 & \text{if } m = 1; \\ l & \text{otherwise,} \end{cases}$$

where m is the number of vertices on C with $\deg(v) \geq 2$.

Proposition 5 [3] *If G is the cycle C_n on $n \geq 3$ vertices, then $\Pi_p(G) = 2$.*

2 Properties of pendant number

Even though there is no direct relation between the pendant number and other known and popular graph parameters, it is observed that the pendant number is highly influenced by the number of odd degree vertices in a graph. Moreover, the pendant number of a graph G has plenty of interesting properties, some of which, we deal with in the following discussion.

Definition 6 A *Π_p-realisation* of a positive integer $k \geq 2$ is a minimal connected graph G, whose pendant number is k.

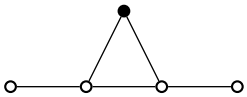
By the *one-point union* of a collection of graphs (possibly with different order), we mean a graph obtained by replacing some or all edges of a path P by some graphs in the collection. In view of this notion, the following theorem establishes the existence of Π_p-realisation for any given positive integer.

Theorem 7 *For every positive integer $k \geq 2$, there exists a Π_p-realisation for k.*

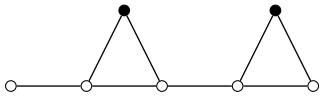
Proof. We can iteratively construct a (minimal) connected graph with pendant number k using 1-point union of K_2 and K_3 (in alternative manner) as shown in Figure 1. We note that for an even integer k, taking the one-point union of $\frac{k}{2}$ number of K_2's and $\frac{k}{2} - 1$ number of K_3's alternatively, we can construct a graph with $\Pi_p(G) = k$, whereas for an odd integer k, taking the one-point union of $\frac{k-1}{2}$ number of K_2's and K_3's alternatively, we can construct a graph with $\Pi_p(G) = k$. This completes the proof.

(a) A graph with $\Pi_p = 2$.

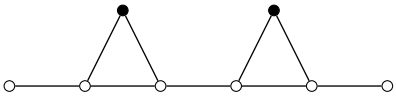
(b) A graph with $\Pi_p = 3$.



(c) A graph with $\Pi_p = 4$.



(d) A graph with $\Pi_p = 5$.



(e) A graph with $\Pi_p = 6$.

Figure 1: A Π_p realisation of the given positive integer $k \geq 2$.

□

Remark 8 It has been determined that for every $k \in \mathbb{N}$, there exists a graph with $\Pi_p(G) = k$. More precisely, for every $k \in \mathbb{N}$, there exists a graph with cycles such that $\Pi_p(G) = k$ and there exists an acyclic graph corresponding to any even natural number.

Theorem 9 *If u, v are two non-adjacent vertices in a graph G of order n, then $|\Pi_p(G + uv) - \Pi_p(G)| \leq 2$.*

Proof. Let G be a graph with $\Pi_p(G) > 2$. Let u and v be two non-adjacent vertices in G. Here, one may consider the following two cases:

Case-1: If u and v are of even degree and neither of them is an end point of a path in the decomposition of G, then in $G + uv$, u and v are of odd degree and become end points of some path in G, while the degrees of all other vertices remain the same as those in G. Hence, in this case, $\Pi_p(G + uv) = \Pi_p(G) + 2$ (see Figure 2 for illustration).

(a) A graph G with no even degree vertex as end point of a path.

(b) A graph $G+uv$ with $\Pi_p(G+uv) = \Pi_p(G) + 2$.

Figure 2

Case-2: Let P and P' be two (vertex-) disjoint paths in G such that u is an end vertex of P in G and v is an end vertex of P' in G. Join the edge uv. Now the path $P+uv+P'$ becomes a longer path, in which neither u nor v is an end point. In this case, $\Pi_p(G+uv) = \Pi_p(G) - 2$ (see Figure 3 for illustration).

(a) A graph G with end vertices of paths.

(b) A graph $G+uv$ with $\Pi_p(G+uv) = \Pi_p(G) - 2$.

Figure 3

It can be verified that in all other possible cases, $\Pi_p(G + uv)$ lies between $\Pi_p(G) - 2$ and $\Pi_p(G) + 2$. This completes the proof. □

Invoking the results mentioned, we discuss some immediate observations in this section. By *one-point union* of cycles, we mean a graph obtained from a path by replacing its edges by cycles (possibly of different order). The following result provides the pendant number of one-point union of cycles.

Proposition 10 *If* G *is the one-point union of cycles, then* $\Pi_p(G) = 2$.

Proof. The proof is clear from Figure 4. The two edge disjoint (u, v)-paths of the one-point union of cycles are illustrated in the figure. □

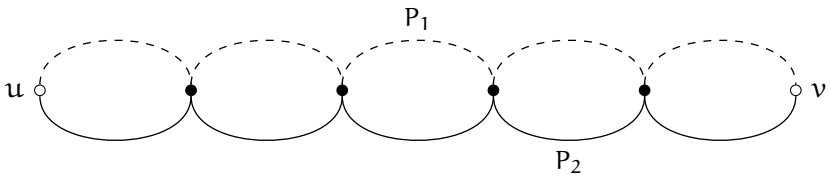
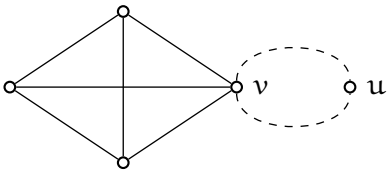


Figure 4: One-point union of cycles.

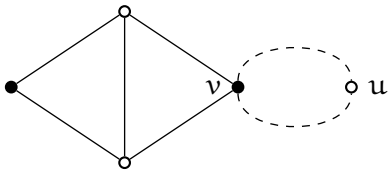
Proposition 11 *Let* G *be a graph, which is neither a cycle nor a one-point union of cycles. Then, the one-point union of* G *and a cycle* C_n *has the pendant number* $\Pi_p(G) + 1$.

Proof. Let G^* be the one-point union of a given graph G and a cycle C_n and let v be the vertex common to G and C_n in G^*. If v is a pendant vertex of any path decomposition of G, it will be a pendant vertex of some paths in G^* also. It can be taken as an end vertex of a path in C_n too. The other end vertex of this path can be arbitrarily chosen on C_n. If v is not a pendant vertex of any path decomposition of G, some paths passing through v in G can be extended to a vertex u of C_n in G^*. Hence, in this case also, one vertex of C_n will be a pendant vertex in G^*, other than the pendant vertices in G.

Therefore, in both cases, the pendant number is $\Pi_p(G) + 1$ (see Figure 5). □



(a) The one-point union of G and C_n with v as an end vertex of paths.



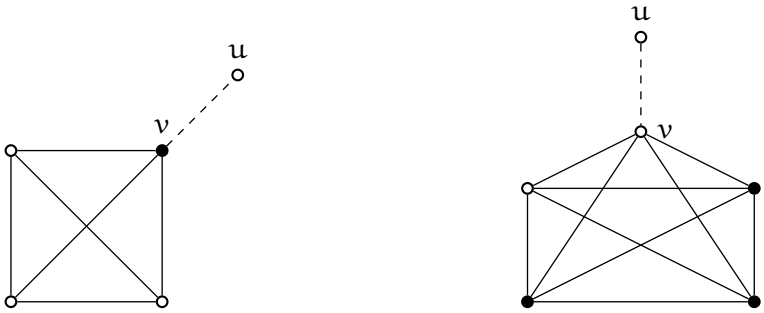
(b) The one-point union of G and C_n with v not as an end vertex of paths.

Figure 5

Proposition 12 *If* G *is a graph with cycles such that* v *is an end vertex of a path and* G^* *is the one-point union of* G *and a path* P_m *joined at* v*, then* $\Pi_p(G) \leq \Pi_p(G^*) \leq \Pi_p(G) + 1$.

Proof. If v is an end vertex of a single path, then the path can be extended to P_m so that $u \in P_m$ is the end vertex of the resultant path (see Figure 6a). If v is the end vertex of more than one path, then v together with $u \in P_m$ become the end vertices of the new path (see Figure 6b). □

Example 13 Let $G = K_n; n$ even and G^* be the (n, m)-*shovel graph*, which is the one-point union of K_n and P_m (that is, $(K_n \dot{\cup} P_m)$) [7]. Then, $\Pi_p(G^*) = \Pi_p(G)$. Let $G = K_n$; n odd and G^* be the (n, m)-shovel graph. Then, $\Pi_p(G^*) = \Pi_p(G) + 1$.



(a) A graph G with $\Pi_p(G^*) = \Pi_p(G)$ (b) A graph G with $\Pi_p(G^*) = \Pi_p(G)+1$.

Figure 6

A maximal, bi-connected, edge-disjoint subgraph of a graph G is called a *block* of G. Note that two blocks may have utmost one vertex in common.

Proposition 14 *If* $B_1, B_2, \ldots, B_k$ *are* k *distinct blocks of a graph* G*, then* $\Pi_p(G) \leq \Sigma_{i=1}^{k} \Pi_p(B_i)$.

The Figure 4 explains the case when the equality in the above result holds. The next proposition establishes an illustration of the case where there is the strict inequality in the above proposition.

Definition 15 A *non-uniform friendship graph* F_n^* is defined as the graph obtained by joining n cycles (need not be of same order) to a common vertex. Cycles in this graph are called as *petals* of F_n^*.

The following result discusses the pendant number of non-uniform friendship graphs.

Proposition 16 *For a non-uniform friendship graph* F_n^* *on* n *petals,* $\Pi_p(F_n^*) = n$.

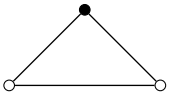
Proof. Let there be n petals in F_n^*. Let v_0 be the common vertex and choose n vertices $v_1, v_2, \ldots, v_n$ randomly from each petal. Join the path from v_1 to v_2 through v_0. Then, v_1 and v_2 becomes the end vertices of a path. Similarly, construct the paths from v_2 to v_3, from v_3 to v_4, ..., from v_{n-1} to v_n, and from v_n to v_1. There are n such paths with end vertices of every path is a starting point of another path. Hence, $\Pi_p(G) = n$. □

This class of non-uniform friendship graphs is another example for Π_p-realisation of positive integers. This fact is immediate from the above result. We consider $\delta(G)$ as the lowest degree among all the degrees of the vertices of the graph G.

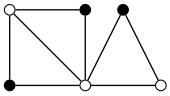
Theorem 17 *There exists a connected graph* G *with* $\delta(G) \geq 2$ *corresponding to any natural number* $k \geq 2$.

Proof. A (minimal) connected graph G with $\delta(G) \geq 2$ can be iteratively constructed corresponding to any natural number $k \geq 2$. Consider a cycle C_k with chords and a cycle C_3 in alternative manner. For an even integer $2k$, take the cycle C_{2k} with every vertex has exactly one chord results in an even number as pendant number. For an odd integer $2k+1$, take the one point union of the cycle C_{2k} with every vertex have exactly one chord and a C_3 attached to any one of its vertices, result in an odd number as pendant number (by Theorem 11). That is, consider the cycle C_3 to get the pendant number 2 (see Figure 7a). The one-point union of a diamond with C_3 has the pendant number 3 (see Figure 7b). The cycle C_4 with every vertex having exactly one chord has the pendant number 4 (see Figure 7c). The one point union of the above C_4 and a C_3 has the pendant number 5 (see Figure 7d). The cycle C_6 with every vertex having exactly one chord has the pendant number 6 (see Figure 7e). The one point union of the above C_6 and a C_3 has the pendant number 7 (see Figure 7f). This process gives the pendant number of a (minimal) connected graph with $\delta(G) \geq 2$ for any natural number. This completes the proof. □

In view of the Theorem 17, instead of taking C_3, if one takes a cycle of desired length, then it leads to the existence of a graph with desired pendant number and desired number of vertices. Note that to get the required pendant

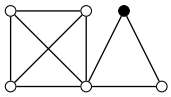


(a) A graph with $\kappa \geq 2$ and $\Pi_p = 2$.

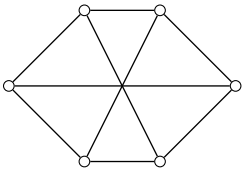


(b) A graph with $\kappa \geq 2$ and $\Pi_p = 3$.

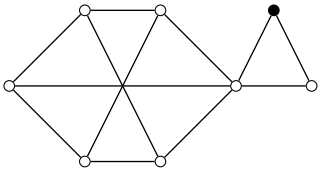
(c) A graph with $\kappa \geq 2$ and $\Pi_p = 4$.



(d) A graph with $\kappa \geq 2$ and $\Pi_p = 5$.



(e) A graph with $\kappa \geq 2$ and $\Pi_p = 6$.



(f) A graph with $\kappa \geq 2$ and $\Pi_p = 7$.

Figure 7

number, one must have at least the same number of odd vertices. It leads to the following result.

Corollary 18 *There exists a connected graph* G *of order* n *with* $\delta(G) \geq 2$ *corresponding to any natural number* $k \geq 2$ *and corresponding to any order* $n \geq 3$ *with desired pendant number.*

Theorem 19 *A graph* G *of order* $n \geq 3$*, with at least one even degree vertex has the pendant number at most* $n - 1$*.*

Proof. Since the result has already been proved for acyclic graphs (see Theorem 3), it is sufficient to prove the result for cyclic graphs on n vertices using the method of induction on the number of vertices.

The smallest graph with cycles with at least one even degree vertex is C_3 and $\Pi_p(C_3) = 2$. Hence the result is true for $n = 3$. Assume the result is true for $n = k$. Let G be a graph with k vertices and $\Pi_p(G) \leq k - 1$. Let v_1 be an even degree vertex, which is not an end vertex of any path in G.

Now let $n = k+1$. Add one more vertex v_{k+1} to the above graph G. Connect any number of vertices of G except v_1 to v_{k+1} (if v_1 is joined to v_{k+1}, then degree of v_1 becomes odd). For v_1 is not an end vertex of any path in G, $\Pi_p(G) \leq k$. Hence, the proof. □

Theorem 20 *Let* G *be a graph of order* n*, consisting of cycles and* $k > 0$ *even degree vertices. Then,* $n - k \leq \Pi_p(G)$*.*

Proof. Let us use the induction on k, viz., $k = 1, n-1$ and n. Let $k = 1$. Then, by Theorem 19, $n-1 \leq \Pi_p(G) \leq n-1$. Let $k = n-1$ then, $n-(n-1) = 1 \leq \Pi_p(G)$. Let $k = n$ then, $n - n = 0 \leq \Pi_p(G)$. Hence the result is true. □

Combining Theorem 19 and Theorem 20, it follows:

Theorem 21 *If a graph G, with $n \geq 3$ vertices, has k even degree vertices, then $n - k \leq \Pi_p(G) \leq n - 1$.*

Proposition 22 *If both G and its complement $\overline{G}$ are connected graphs with odd degree, then $|V(G)| \geq 5$.*

Proof. Let a graph G and its complement $\overline{G}$ be connected graphs such that all their vertices are of odd degree. Then, at least two vertices each of G and $\overline{G}$ will be with degree ≥ 3. If all vertices except one is of degree one in G, then this vertex must be isolated in $\overline{G}$. It implies that the number of vertices of G must be at least 5. □

A pineapple graph [9], denoted by K_n^m, is a graph obtained by appending m pendant edges to a vertex of a complete graph $K_n; m \geq 1, n \geq 3$. Let $\mathcal{A}$ be the collection of graphs given in Figure 8 and Figure 9. Even though $K_3^m \subseteq$ one-point union of a triangle and an odd degree tree, $\overline{K}_3^m$ is disconnected. Moreover, it is clear that the complement of one-point union of triangle and an odd degree tree always has $\Pi_p \leq n - 2$ (see Theorem 3 and Theorem 4).

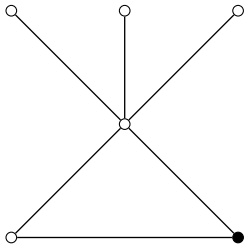


Figure 8: The graph $K_3^m; m$ odd.

The next proposition can be proved in a similar manner as that of Theorem 19.

Proposition 23 *Let G be a graph G of order $n \geq 4$ and $G \notin \mathcal{A}$. If G has at least two even degree vertices, then $\Pi_p(G) \leq n - 2$.*

Using Proposition 22 and Proposition 23, we have the next result associated a graph G and its complement $\overline{G}$.

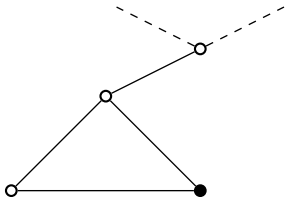


Figure 9: One-point union of K_3 and an odd degree tree.

Theorem 24 *If both the graphs* G *and its complement* $\overline{G}$ *are connected, then* $4 \leq \Pi_p(G) + \Pi_p(\overline{G}) \leq 2(n-1)$.

Proof. The lower bound of $\Pi_p(G)$ for any graph G is 2. Therefore, $2 \leq \Pi_p(G)$ and $2 \leq \Pi_p(\overline{G})$. Hence, $4 \leq \Pi_p(G) + \Pi_p(\overline{G})$.
To prove the other part, one may consider the following two cases:

Case-1: The number of vertices n of the graph G is odd. Since n is odd, at least one vertex each of G and $\overline{G}$ must be even. Thus, the result can be determined by using the Theorem 19, as $\Pi_p(G) \leq n-1$ and $\Pi_p(\overline{G}) \leq n-1$.

Case-2: The number of vertices n of the graph G is even. Let all the vertices of G be odd degree vertices. Hence, $\Pi_p(G) = n$. Thus, the degree of $\overline{G}$ must be even. By Proposition 22, $|V(G)| \geq 5$ and by Proposition 23, $\Pi_p(\overline{G}) = n-2$.

In both cases, it is determined that $\Pi_p(G) + \Pi_p(\overline{G}) \leq 2(n-1)$, completing the proof. □

3 Pendant number of graph products

The *rooted product* of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph obtained by taking $|V(G_1)|$ copies of G_2 and identifying one vertex (root) of each copy of G_2 to the corresponding vertex of G_1. The following result discusses the pendant number of the rooted products of cycles and paths.

Theorem 25 *Let* P_n *and* P_m *be any two paths and* C_n *and* C_m *be any two cycles. Then the pendant number of their rooted products are given by;*

(i) *For* $P_n \circ P_m$,

$$\Pi_p(P_n \circ P_m) = \begin{cases} 2(n-1) & \text{if the root vertex is a pendant vertex ;} \\ 2(n+1) & \text{if the root vertex is an internal vertex.} \end{cases}$$

(ii) *For* $C_n \circ P_m$, $\Pi_p(C_n \circ P_m) = 2n$.

(iii) *For* $C_n \circ C_m$, $\Pi_p(C_n \circ C_m) = n$.

Proof.

(i) Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n and $v_1, v_2, \ldots, v_m$ be the vertices of P_m.

Case-1: Let the root vertex of P_m be a pendant vertex. Then, in the rooted product $P_n \circ P_m$, one copy of P_m is joined to each of the vertices $u_1, u_2, \ldots, u_n$. The longest path includes the one-point union of P_n between two copies of P_m (situated at the end points of P_n), yielding 2 pendant vertices. The remaining $n-2$ vertices of P_n together with the pendant vertices in the remaining $n-2$ copies of P_m result in $2(n-2)$ pendant vertices (see Figure 10 for illustration). Therefore, $\Pi_p(G) = 2 + 2(n-2) = 2(n-1)$.

Case-2: Let the root vertex of P_m be an internal vertex. Then, both end vertices of all the n copies of P_m and the end vertices of P_n are the end vertices of some paths in the path decomposition of $P_n \circ P_m$ (see Figure 11 for illustration). Hence, $\Pi_p(G) = 2(n+1)$.

(ii) For $C_n \circ P_m$, the collection $V_p(G)$ (see 1) is constituted by all the vertices of C_n together with one end vertex of each of the n copies of the paths P_m on the other end as seen in Figure 12. Hence, $\Pi_p(G) = 2n$.

(iii) In the rooted product $C_n \circ C_m$, one copy of C_m is joined at each vertex of C_n. Now take any vertex of degree 2 of $C_n \circ C_m$ as the first vertex of a path. This path passes through some vertices of the same copy of C_m, say $C_{m(i)}$, passes through two adjacent vertices of C_n and passes through some vertices of the next copy $C_{m(i+1)}$ before it terminates at some vertex of $C_{m(i+1)}$. The next path starts from this end vertex in $C_{m(i+1)}$ and will end at some vertex of the next copy $C_{m(i+2)}$. Continuing like this, one can find out paths which cover all edges of $C_n \circ C_m$, as shown in Figure 13. Hence, $\Pi_p(C_n \circ C_m) = n$.

□

The *corona product* $G \odot H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G; $1 \leq i \leq |V(G)|$ (see [8]). When we consider the corona of two paths, for certain initial values of n and m, the pendant numbers of $P_n \odot P_m$ have already determined (see [3],[6]). They are:

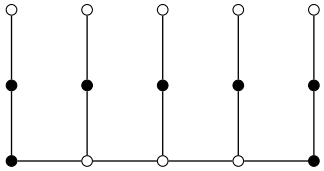


Figure 10: A rooted product $P_n \circ P_m$ with the root vertex as a pendant vertex.

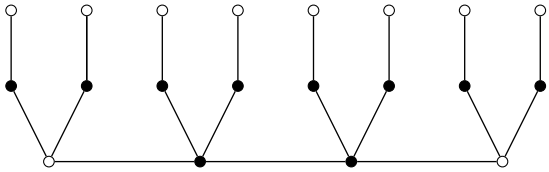


Figure 11: A rooted product $P_n \circ P_m$ with the root vertex as an internal vertex.

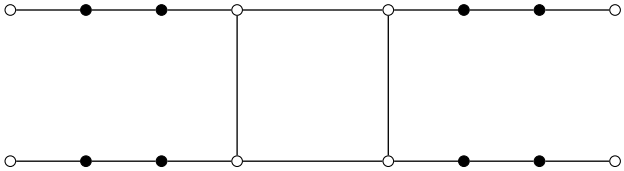


Figure 12: A rooted product $C_n \circ P_m$.

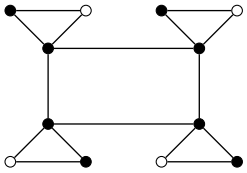


Figure 13: A rooted product $C_n \circ C_m$.

(i) We can exclude the cases $P_1 \odot P_1 = K_2$, $P_1 \odot P_2 = C_3$, $P_2 \odot P_1 = P_4$, whose pendant number is already determined in [3] as 2.

(ii) Since, $P_1 \odot P_m$; $m \geq 3$ is the $n-$ fan graph on $n+2$ vertices, its pendant number is determined in [6] as;

$$\Pi_p(P_1 \odot P_m) = \begin{cases} m-1 & \text{if } m \text{ is odd;} \\ m-2 & \text{if } m \text{ is even.} \end{cases}$$

(iii) $P_n \odot P_1$; $n \geq 3$ is the comb tree T, whose pendant number is $\Pi_p(P_n \odot P_1) = 2(n-1)$ (see[6]).

(iv) $P_n \odot P_2$; $n \geq 2$ is the lever graph L_n with pendant number $\Pi_p(P_n \odot P_2) = n+2$ (see [6]).

(v) Since $P_n \odot P_3$; $n \geq 2$ is the diamond neckalce graph D_n, the pendant number is $\Pi_p(P_n \odot P_3) = 2n$ (see[6]).

Theorem 26 *The pendant number of the corona product* $P_n \odot P_m$ *with* $n \geq 2$ *and* $m \geq 4$ *of two paths* P_n *and* P_m *is given by*

$$\Pi_p(P_n \odot P_m) = \begin{cases} n(m-1)-2 & \text{if } m \text{ is odd;} \\ n(m-2)+2 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n and $v_1, v_2, \ldots, v_m$ be the vertices of P_m. Let $u_1, u_2, \ldots, u_n$ be the vertices on the root and $v_{11}, v_{12}, \ldots, v_{1m}$, $v_{21}, v_{22}, \ldots, v_{2m}$, ..., $v_{n1}, v_{n2}, \ldots, v_{nm}$ be the vertices on the crown. Then, the total number of vertices is $n(m+1)$. Let the j^{th} vertex of the i^{th} copy of P_m be denoted by $v_{i,j}$. The corona product of $P_n \odot P_m$ where $n \geq 2$ and $m \geq 4$ has two cases:

Case-1: Let m be odd. Since $v_{i1}, v_{im}; 1 \leq i \leq n$ and u_1, u_n are even degree ($= 2n+2$) and all other vertices are odd, one can make a path decomposition with the odd degree vertices as the end vertices of every path (see Figure 14) and it will be the least (see Theorem 2). Hence, $\Pi_p(G) = n(m+1) - (2n+2) = n(m-1) - 2$.

Case-2: Let m be even. Since $v_{i1}, v_{im}; 1 \leq i \leq n$ and $u_2, u_3, \ldots, u_{n-1}$ are even degree ($= 2n + (n-2)$) and all other vertices are odd, the path decomposition with minimum number of end vertices be the path decomposition with odd degree vertices as the end vertices of every path (see Figure 15). Hence,$\Pi_p(G) = n(m+1) - (2n+n-2) = n(m-2)+2$.

□

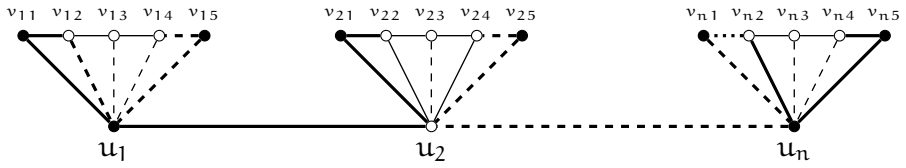


Figure 14: A corona product $P_n \odot P_m; m$ odd.

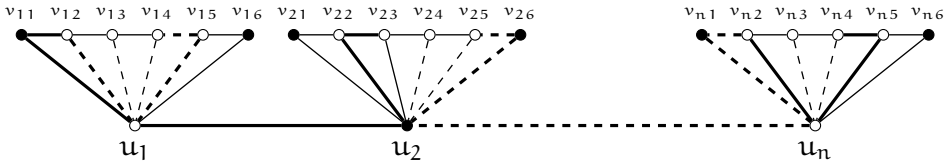


Figure 15: A corona product $P_n \odot P_m; m$ even.

Theorem 27 *The pendant number of the corona product of cycles and paths is given as:*

(i) For $C_n \odot P_m$,

$$\Pi_p(C_n \odot P_m) = \begin{cases} 2n & \text{if } m = 1; \\ n & \text{if } m = 2; \\ 2n\lfloor \frac{m-1}{2} \rfloor & \text{if } m \neq 1, 2. \end{cases}$$

(ii) The pendant number of the corona product $C_n \odot C_m$ *is given by,*

$$\Pi_p(C_n \odot C_m) = \begin{cases} nm & \text{if } m \text{ is even;} \\ n(m+1) & \text{if } m \text{ is odd.} \end{cases}$$

Proof.

(i) The first two parts of the result follow respectively from Part-(ii) and Part-(iii) of Theorem 25.

Let us now consider the corona product $C_n \odot P_m; m \neq 1, 2$. Note that for every vertex v of C_n, the degree of v in $C_n \odot P_m$ is $m + 2$ and for every vertex of each copy of P_m has degree one more than the degree of the corresponding vertex in P_m. Hence, the following two cases must be considered:

Case-1: Let m be odd. Then, in $C_n \odot P_m$, every vertex of C_n becomes odd degree vertex and all vertices of each copy of P_m, except two (corresponding to the end vertices of P_m) become odd. It is possible to find edge-disjoint paths in $C_n \odot P_m$ in such a way that the vertices of degree two are not pendant vertices of paths in the path decomposition of $C_n \odot P_m$ (see Figure 16 for example). Hence, $\Pi_p(C_n \odot P_m) = n + n(m-2) = n(m-1)$.

Case-2: Let m be even. Then, in $C_n \odot P_m$, every vertex of C_n remains as an even degree vertex and all vertices of each copy of P_m, except two (corresponding to the end vertices of P_m) become odd. It is possible to find edge-disjoint paths in $C_n \odot P_m$ in such a way that the even degree vertices are not pendant vertices of paths in the path decomposition of $C_n \odot P_m$ (see Figure 17 for example). Hence, $\Pi_p(C_n \odot P_m) = n(m-2)$.

Combining the above two cases, it is clear that $\Pi_p(C_n \odot P_m) = 2n\lfloor \frac{m-1}{2} \rfloor$.

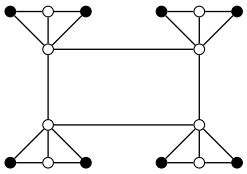


Figure 16: A corona product $C_n \odot P_m; m$ odd.

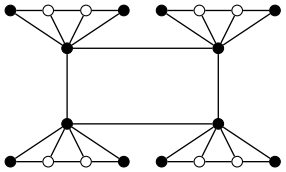


Figure 17: A corona product $C_n \odot P_m; m$ even.

(ii) Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n and $v_1, v_2, \ldots, v_m$ be the vertices of C_m. Let $u_1, u_2, \ldots, u_n$ be the vertices on the root and $v_{11}, v_{12}, \ldots, v_{1m}$, $v_{21}, v_{22}, \ldots, v_{2m}$, $\ldots$, $v_{n1}, v_{n2}, \ldots, v_{nm}$ be the vertices on the crown. In $C_n \odot C_m$, $v_{i1}, v_{i2}, v_{im}; 1 \leq i \leq n$ always have odd degree with $\deg(v_{ij}) = 3; 1 \leq j \leq m$. The total number of vertices in the corona product is $n(m+1)$. There are two possibilities for $C_n \odot C_m$.

Case-1: Let m be even. Since every vertex of each copy of C_m is joined to each vertex of C_n, the degree of C_n remains even, one can make a path decomposition in such a way that none of the vertices of C_n is the end vertex of any path (see Figure 18) and it will be the least too (see Theorem 2). Thus the pendant number becomes $n(m+1) - n = nm$.

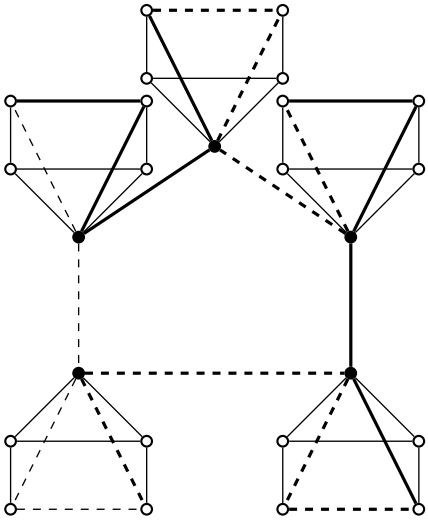


Figure 18: A corona product $C_n \odot C_m$; m *even*.

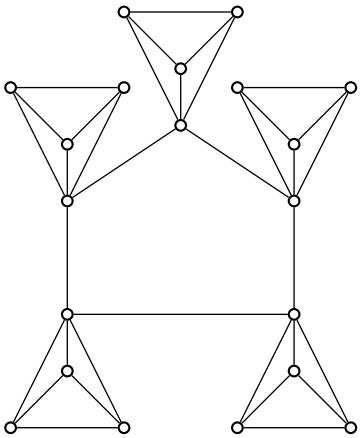


Figure 19: A corona product $C_n \odot C_m$; m odd.

Case-2: Let m be odd. Then, all the vertices of $C_n \odot C_m$ become odd. Thus, by Theorem 2, pendant number is $n(m+1)$ (see Figure 19), the entire vertex set of the resultant graph.

□

4 Conclusion

In this paper, we discussed certain properties of the pendant number of a given graph G. We have determined the pendant number of corona products and rooted products of paths and cycles. We have also obtained some bounds for the pendant number for some specific derived graphs. Corresponding to any natural number $\mathfrak{n}$, the existence of a connected graph with pendant number $\mathfrak{n}$ has also been established in this study. These studies can further be extended to other general graph classes, graph products and other graph operations. The relation of pendant numbers with other popular parameters like domination number, covering number, chromatic number etc. can also be studied.

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Received: February 19, 2019 • Revised: May 29, 2019