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A result regarding monotonicity of the Gamma function

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Abstract. In this paper we analyze the monotony of the function $\frac{\ln\Gamma(x)}{\ln(x^2+\tau)-\ln(x+\tau)}$, for $\tau > 0$. Such functions have been used from different authors to obtain inequalities concerning the gamma function.

1 Introduction

In [8] the author proved the following double inequality:

$$\frac{x^2+1}{x+1} \le \Gamma(x+1) \le \frac{x^2+2}{x+2},\ x \in [0,1]. \tag{1}$$

In [12] the authors improved this inequality proving that

$$\left(\frac{x^2+1}{x+1}\right)^{2(1-\gamma)} \le \Gamma(x+1) \le \left(\frac{x^2+1}{x+1}\right)^{\gamma},\ x \in [0,1]. \tag{2}$$

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Other improvements of (1) can be found in [9], [10] and [11]. The inequality (2) is equivalent to

$$2(1-\gamma) > \frac{\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)} > \gamma, \quad x \in (0,1).$$

The authors of [12] proved inequality (2) using the monotony of the function

$$g:(0,\infty)\to\mathbb{R}, \quad g(x) = \frac{\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)}.$$

In connection with this function they formulated the following conjecture: if $\tau > 0$, then the mapping $u_\tau : (0,\infty) \to \mathbb{R}$ defined by

$$u_\tau(x) = \begin{cases} \frac{\ln\Gamma(x)}{\ln(x^2+\tau)-\ln(x+\tau)}, & x \neq 1 \\ -(1+\tau)\gamma, & x = 1 \end{cases} \tag{3}$$

is strictly increasing. This conjecture was confirmed for $\tau \in (0,1)$ in [6]. We found a counterexample regarding this conjecture: if $\tau = 1000$, then

$$u_\tau(11) = \frac{\ln\Gamma(11)}{\ln\frac{1121}{1011}} = \frac{\ln 3628800}{\ln\frac{1121}{1011}} < \frac{\ln 24^5}{\ln\frac{1121}{1011}} = \frac{\ln 24}{\ln\left(\frac{1121}{1011}\right)^{\frac{1}{5}}} = \frac{\ln\Gamma(5)}{\ln\left(\frac{1121}{1011}\right)^{\frac{1}{5}}}$$
$$< \frac{\ln\Gamma(5)}{\ln\left(\frac{1025}{1005}\right)} = u_\tau(5).$$

Numerical results suggest that there is a value $\tau_0 \in (212, 213)$ such that if $\tau \in (0,\tau_0)$ then u_τ is strictly increasing. We will prove a partial result regarding this question.

Theorem 1 *The function u_τ is strictly increasing on the interval $(0,\infty)$ for all τ, $0 < \tau \leq 25$.*

2 Preliminaries

In order to prove our main results we need the following lemmas.

Lemma 1 [3] *Let $h, k : [a,b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further let $k'(x) \neq 0$, $x \in (a,b)$. If h'/k' is strictly increasing (resp. decreasing) on (a,b), then the functions*

$$x \longmapsto \frac{h(x)-h(a)}{k(x)-k(a)} \qquad x \longmapsto \frac{h(x)-h(b)}{k(x)-k(b)}$$

are also strictly increasing (resp. decreasing) on (a,b).

Lemma 2 *If* $\tau > 1$, *then the function* $u_\tau : (0,\infty) \to \mathbb{R}$ *defined by*

$$u_\tau(x) = \begin{cases} \frac{\ln\Gamma(x)}{\ln(x^2+\tau)-\ln(x+\tau)}, & x \neq 1 \\ -(1+\tau)\gamma, & x = 1 \end{cases}$$

is strictly increasing on the interval $(0, x_1)$, *where* x_1 *is the positive root of the equation* $x^2 + 2\tau x - \tau = 0$.

Proof. According to [4] we have $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}$. It is easily seen that $\frac{1}{2} > x_1 = \frac{\tau}{\tau+\sqrt{\tau^2+\tau}} > \frac{1}{4}$. If $x \in (0, x_1)$, then $\frac{\tau-2\tau x-x^2}{(x^2+\tau)(x+\tau)} > 0$, $\frac{1}{x} + \gamma - \sum_{n=1}^{\infty} \frac{x}{n(n+x)} > 0$, $\Gamma(x) > 1$, and this implies

$$u_\tau'(x) = \frac{\left(\frac{1}{x} + \gamma - \sum\limits_{n=1}^{\infty} \frac{x}{n(n+x)}\right) \ln\frac{x+\tau}{x^2+\tau} + \frac{\tau-2\tau x-x^2}{(x^2+\tau)(x+\tau)} \ln\Gamma(x)}{\ln^2\left(\frac{x^2+\tau}{x+\tau}\right)} > 0.$$

Thus u_τ is strictly increasing on the interval $(0, x_1)$. □

Lemma 3 *The unique positive root of the equation* $\psi(x) = -\frac{1}{x} - \gamma + \sum\limits_{n=1}^{\infty} \frac{x}{(n+x)n} = 0$ *is* $x_2 = 1.4616\ldots$. *If* $\tau > 1$, *then the function*

$$v : (x_1, \infty) \to \mathbb{R}, \quad v(x) = \frac{-\frac{1}{x} - \gamma + \sum\limits_{n=1}^{\infty} \frac{x}{(n+x)n}}{\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}}, \tag{4}$$

is strictly increasing on the interval (x_1, x_2), *where* x_1 *is defined in Lemma 2.*

Proof. We have $v'(x) = \frac{A(x)}{\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right)^2}$, where

$$\begin{aligned} A(x) = &\left(\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) \\ &+ \left(\frac{1}{x} + \gamma - \sum_{n=1}^{\infty} \frac{x}{n(n+x)}\right)\left(\frac{-2x^2+2\tau}{(x^2+\tau)^2} + \frac{1}{(x+\tau)^2}\right). \end{aligned} \tag{5}$$

Since $\frac{1}{3} < x_1$, and the following inequalities hold

$$\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} > \frac{1}{x^2} + \frac{\gamma}{x} - \sum_{n=1}^{\infty} \frac{1}{n(n+x)} > 0, \; x \in (\frac{1}{3}, x_2),$$

$$\text{and} \quad \frac{2x}{x^2+\tau} - \frac{1}{x+\tau} = \frac{x^2+2\tau x-\tau}{(x+\tau)(x^2+\tau)} > 0, \quad x \in (x_1, x_2),$$

it follows that

$$\begin{aligned}
A(x) &> \left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{2x}{x^2+\tau}-\frac{1}{x+\tau}\right)\\
&+\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{-2x^3+2\tau x}{(x^2+\tau)^2}+\frac{x}{(x+\tau)^2}\right)\\
&=\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{2x^3+2\tau x}{(x^2+\tau)^2}-\frac{x+\tau}{(x+\tau)^2}+\frac{-2x^3+2\tau x}{(x^2+\tau)^2}+\frac{x}{(x+\tau)^2}\right)\\
&=\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{4\tau x}{(x^2+\tau)^2}-\frac{\tau}{(x+\tau)^2}\right)\\
&=\tau\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{x^3(4-x)+6\tau x^2+\tau^2(4x-1)}{(x^2+\tau)^2(x+\tau)^2}\right)\\
&>0,\ x\in(x_1,x_2),
\end{aligned}$$

and we get $\nu'(x)>0,\quad x\in(x_1,x_2)$. Thus ν is a strictly increasing function on the interval (x_1,x_2). □

Lemma 4 *Suppose* $\tau>1$. *The equation* $\psi(x)=\psi'(x)$ *has a unique positive root* $x_3=2.2324\ldots$. *The function* $\nu:(x_1,\infty)\to\mathbb{R}$ *defined by (4) is strictly increasing on the interval* (x_2,x_3).

Proof. We will prove this lemma in two steps. We have $x_2<\frac{3}{2}$.
In the first step we discuss the case $(x_2,\frac{3}{2})$.
According to the mean value theorem for every $x\in(x_2,\frac{3}{2})$ there are the values $c_x,d_x\in(x_2,x)$ such that $\psi(x)=\psi(x)-\psi(x_2)=\psi'(c_x)(x-x_2)$ and $\psi'(x_2)-\psi'(x)=-\psi''(d_x)(x-x_2)$. These two equalities imply

$$\psi(x)=\psi(x)-\psi(x_2)=\psi'(c_x)(x-x_2)<\psi'(x_2)\left(\frac{3}{2}-x_2\right)<\frac{4}{100}\psi'(x_2)$$

and

$$\begin{aligned}
\psi'(x_2)-\psi'(x)&=-\psi''(d_x)(x-x_2)=2(x-x_2)\left(\sum_{n=0}^{\infty}\frac{1}{(n+d_x)^3}\right)\\
&<\frac{8}{100}\psi'\left(\frac{3}{2}\right)\le\frac{8}{100}\psi'(x).
\end{aligned}$$

□

Thus we get $0 < \psi(x) < \frac{4}{100}(1 + \frac{8}{100})\psi'(x) < \frac{1}{12}\psi'(x),\ x \in (x_2, \frac{3}{2})$ and consequently

$$A(x) > \psi(x)\left(\frac{24x}{x^2+\tau} - \frac{12}{x+\tau} + \frac{2x^2-2\tau}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right)$$
$$= \psi(x)\frac{B(x)}{(x^2+\tau)^2(x+\tau)^2},$$

where $B(x) = 12x^5 + (1+36\tau)x^4 + 24\tau^2x^3 + 4\tau x^2(x-1) + \tau^2 x(26x-16) + 24x\tau^3 - 14\tau^3 - \tau^2$, and A is defined by (5). It is easily seen that if $x \in (x_2, \frac{3}{2})$, then $B(x) > 0$, and consequently $\nu'(x) > 0$, for $x \in (x_2, \frac{3}{2})$.
In the second step suppose $x \in (\frac{3}{2}, x_3)$. We have in this case $0 < \psi(x) \le \psi'(x)$, where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. A short calculation leads to

$$A(x) > \psi(x)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau} + \frac{2x^2-2\tau}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) = \psi(x)\frac{C(x)}{(x^2+\tau)^2(x+\tau)^2},$$

where $C(x) = x^5 + (1+3\tau)x^4 + 4\tau x^2(x-1) + \tau^2 x(4x-5) + \tau^2(2x^3-1) + \tau^3(2x-3) > 0,\ \ x \in (\frac{3}{2}, x_3)$. Consequently we obtain $\nu'(x) > 0,\ \ x \in (\frac{3}{2}, x_3)$, and the proof is completed.

Lemma 5 *If* $x \in [2,3)$, *then*

$$\frac{6}{7}(\ln x - \frac{7}{25}) > \ln\Gamma(x), \tag{6}$$

and

$$-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty}\frac{x}{n(n+x)} > \ln x - \frac{7}{25}. \tag{7}$$

If $\tau = 25$, *then*

$$\ln\frac{x^2+\tau}{x+\tau} \ge \frac{6}{7}\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right),\quad x \in [2.23, 3]. \tag{8}$$

Proof. Let $\nu_5 : [2,3] \to \mathbb{R}$ be the function defined by $\nu_5(x) = \frac{6}{7}(\ln x - \frac{7}{25}) - \ln\Gamma(x)$. We have

$$\nu_5'(x) = \frac{6}{7x} - \psi(x) = \frac{13}{7x} + \gamma - \sum_{n=1}^{\infty}\frac{x}{n(n+x)},$$

and

$$v_5''(x) = -\frac{6}{7x^2} - \psi'(x) = -\frac{13}{7x^2} - \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} < 0, \quad x \in [2,3].$$

The monotony of v_5' and the inequalities $v_5'(2) > 0, \ \ v_5'(3) < 0$ implies that the equation $v_5'(x) = 0$ has exactly one root $x_1 \in (2,3)$ and $v_5'(x) > 0, \ \ x \in (2, x_1)$, and $v_5'(x) < 0, \ \ x \in (x_1, 3)$.
The monotony of v_5 implies

$$v_5(x) \geq \min\{v_5(2), v_5(3)\} > 0, \quad x \in (2,3),$$

and thus the inequality (6) holds.
In order to prove (7), we define the function $v_6 : [2,3] \to \mathbb{R}$,

$$v_6(x) = \psi(x) - \ln x + \frac{7}{25} = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} - \ln x + \frac{7}{25}.$$

We have $v_6'(x) = -\frac{1}{x} + \psi'(x) = -\frac{1}{x} + \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} > 0, \ x \in [2,3]$, and consequently

$$v_6(x) \geq v_6(2) > 0, \ x \in [2,3].$$

Thus the inequality (7) holds.
The third inequality can be proved as follows.
Let $v_7 : [2.23, \infty) \to \mathbb{R}$ be the function defined by $v_7(x) = \ln \frac{x^2+\tau}{x+\tau} - \frac{6}{7}\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right)$. We have $v_7'(x) = \frac{D(x)}{(x^2+\tau)^2(x+\tau)^2}$, where $\alpha = \frac{6}{7}$ and $D(x) = x^5 + (3\tau + 3\alpha)x^4 + (2\tau^2 + 4\alpha\tau)x^3 + 2(1+\alpha)\tau^2 x^2 + (2\tau^3 - (4\alpha+1)\tau^2)x - (2\alpha+1)\tau^3 + \alpha\tau^2$. A suitable alignment in the numerator of v_7' shows that $v_7'(x) > 0, \ \ x \in [2.23, 3]$.
Thus we get

$$v_7(x) \geq v_7(2.23) > 0, \quad x \in [2.23, 3],$$

and the inequality (8) follows. □

Lemma 6 *If* $x \in [3, \infty)$, *then*

$$(x-2)(\ln x - \frac{1}{4}) > \ln \Gamma(x). \tag{9}$$

If $x \in [3, \infty)$, *then*

$$-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} > \ln x - \frac{1}{4}, \quad x \in (3, \infty). \tag{10}$$

If $x \in [3, \infty)$, *and* $\tau = 25$, *then*

$$\ln \frac{x^2+\tau}{x+\tau} \geq (x-2)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right), \quad x \in (3, \infty). \tag{11}$$

Proof. In order to prove inequality (9) we define the function $v_8 : [3, \infty) \to \mathbb{R}$ by $v_8(x) = (x-2)(\ln x - \frac{1}{4}) - \ln \Gamma(x)$. We have

$$v_8'(x) = \ln x - \frac{1}{4} + \frac{x-1}{x} + \gamma - \sum_{n=1}^{\infty} \frac{x}{n(n+x)},$$

and

$$v_8''(x) = \frac{1}{x} + \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}.$$

It is easily seen that

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)^2} < \sum_{n=1}^{\infty} \frac{1}{(n+x)(n-1+x)} = \frac{1}{x}, \ x \in [3, \infty).$$

Thus we have $v_8''(x) > 0, \ x \in [3, \infty)$, consequently v_8' is strictly increasing and

$$v_8'(x) > v_8'(3) = \ln 3 + \gamma - 1 - \frac{5}{12} > 0, \ x \in (3, \infty).$$

This means that v_8 is strictly increasing too and

$$v_8(x) > v_8(3) = \ln 3 - \frac{1}{4} - \ln 2 > 0, \quad x \in (3, \infty).$$

The inequality (10) can be proved as follows. Let the function $v_9 : [3, \infty) \to \mathbb{R}$ be defined by $v_9(x) = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} - \ln x + \frac{1}{4}$. We have

$$v_9'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} - \frac{1}{x}.$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{(n+x)^2} > \sum_{n=0}^{\infty} \frac{1}{(n+x)(n+1+x)} = \frac{1}{x}, \ x \in [3, \infty),$$

it follows that $v_9'(x) > 0, \ x \in [3, \infty)$, consequently v_9 is strictly increasing and

$$v_9(x) > v_9(3) = 1 + \frac{3}{4} - \gamma - \ln 3 > 0, \ x \in (3, \infty).$$

□

Finally, in order to prove (11), we define the function $\nu_{10} : [3, \infty) \to \mathbb{R}$ by $\nu_{10}(x) = \ln \frac{x^2+\tau}{x+\tau} - (x-2)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right)$, where $\tau = 25$. We have

$$\begin{aligned}\nu_{10}'(x) &= (x-2)\left(\frac{2x^2-2\tau}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right)\\ &= (x-2)\frac{x^4+4\tau x^3+2(\tau^2-2\tau)x^2-4\tau^2x-2\tau^3-\tau^2}{(x^2+\tau)^2(x+\tau)^2}\\ &= (x-2)\frac{x^4+100x^3+1150x^2-2500x-31875}{(x^2+\tau)^2(x+\tau)^2}.\end{aligned}$$

The Descartes rule of signs implies that the equation $x^4 + 100x^3 + 1150x^2 - 2500x - 31875 = 0$ has no more than one positive root, thus it is easily seen that the equation $\nu_{10}'(x) = 0$ has exactly one root $x_0 = 5.13\ldots$. This means that ν_{10} is stictly decreasing on the interval $[3, x_0]$ and strictly increasing on $[x_0, \infty)$. Consequently $\min_{x\in[3,\infty)} \nu_{10}(x) = \nu_{10}(x_0) = 0.01\ldots > 0$, and this implies

$$\nu_{10}(x) > 0, \quad \text{for} \quad \text{all } x \in [3, \infty).$$

3 Proof of the main result

In this section we shall prove the main theorems.

Theorem 2 *Let the function* $g_{\alpha,\beta} : (0, \infty) \to \mathbb{R}$ *be defined by*

$$g_{\alpha,\beta}(x) = \begin{cases} \frac{\ln(x^2+\alpha)-\ln(x+\alpha)}{\ln(x^2+\beta)-\ln(x+\beta)}, & x \in (0,1)\cup(1,\infty)\\ \frac{1+\beta}{1+\alpha}, & x = 1.\end{cases} \tag{12}$$

If $\alpha > \beta > 0$, *then the mapping* $g_{\alpha,\beta}$ *is strictly increasing on the interval* $(0, \infty)$.

Proof. We will prove the theorem in two steps. Let $x_1 = \frac{\beta}{\beta+\sqrt{\beta^2+\beta}}$ be the positive root of the equation $x^2 + 2\beta x - \beta = 0$, and let $x_2 = \frac{\alpha}{\alpha+\sqrt{\alpha^2+\alpha}}$ be the positive root of $x^2 + 2\alpha x - \alpha = 0$.

In the first step let $x \in (0, 1)$. Since $\left(\frac{x^2+2\alpha x-\alpha}{x^2+2\beta x-\beta}\right)' = \frac{2(\alpha-\beta)(x-x^2)}{(x^2+2\beta x-\beta)^2} > 0$, $x \in (0, x_1) \cup (x_2, 1)$, it follows that the function $h : (0, \infty) \to \mathbb{R}$ defined by

$$h(x) = \frac{\big(\ln(x^2+\alpha)-\ln(x+\alpha)\big)'}{\big(\ln(x^2+\beta)-\ln(x+\beta)\big)'} = \frac{x^2+\beta}{x^2+\alpha}\cdot\frac{x+\beta}{x+\alpha}\cdot\frac{x^2+2\alpha x-\alpha}{x^2+2\beta x-\beta},$$

is strictly increasing on the intervals $(0, x_1)$ and $(x_2, 1)$, (because h is a product of positive strictly increasing functions). Now Lemma 1 implies that $g_{\alpha,\beta}$ is strictly increasing on $(0, x_1)$ and $(x_2, 1)$ too. On the other hand

$$g'_{\alpha,\beta}(x) = \frac{D(x)}{(\ln(x^2+\beta) - \ln(x+\beta))^2},$$

where

$$D(x) = \frac{x^2 + 2\alpha x - \alpha}{(x^2+\alpha)(x+\alpha)} \ln \frac{x^2+\beta}{x+\beta} - \frac{x^2+2\beta x - \beta}{(x^2+\beta)(x+\beta)} \ln \frac{x^2+\alpha}{x+\alpha}.$$

Since $\frac{x^2+2\alpha x-\alpha}{(x^2+\alpha)(x+\alpha)} \ln \frac{x^2+\beta}{x+\beta} > 0,\ x \in (x_1, x_2)$, and $\frac{x^2+2\beta x-\beta}{(x^2+\beta)(x+\beta)} \ln \frac{x^2+\alpha}{x+\alpha} < 0,\ x \in (x_1, x_2)$, it follows that $D(x) > 0,\ x \in (x_1, x_2)$, and consequently $g'(x) > 0,\ x \in (x_1, x_2)$.

We have deduced that $g_{\alpha,\beta}$ is a strictly increasing function on the intervals $(0, x_1)$, (x_1, x_2), and $(x_2, 1)$. The continuity of $g_{\alpha,\beta}$ implies that this function is strictly increasing on $(0, 1)$.

In the second step we prove that $g_{\alpha,\beta}$ is strictly increasing on $(1, \infty)$. We will prove that

$$D(x) > 0, \quad x \in (1, \infty). \tag{13}$$

Let $k : (0, \infty) \to \mathbb{R}$ be the function defined by $k(\tau) = \frac{\ln(x^2+\tau)-\ln(x+\tau)}{\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}}$. The following equivalence chain holds

$$g'_{\alpha,\beta}(x) > 0 \quad \Leftrightarrow D(x) > 0 \quad \Leftrightarrow k(\beta) > k(\alpha), \tag{14}$$

providing that $x \in (1, \infty)$, and $\alpha > \beta > 0$.

Consequently in order to prove that $g_{\alpha,\beta}$ is strictly increasing we have to show that if $x \in (1, \infty)$ is a fixed number, then k is strictly decreasing on $(0, \infty)$. We have

$$k'(\tau) = \frac{E(\tau)}{(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau})^2},$$

$$E(\tau) = \left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) + \left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \ln \frac{x^2+\tau}{x+\tau}.$$

It is easily seen that if $\tau \in (0, \infty)$ and $x \in (1, \infty)$, then $\frac{1}{x^2+\tau} - \frac{1}{x+\tau} < 0$, $\frac{2x}{x^2+\tau} - \frac{1}{x+\tau} > 0, \quad \ln \frac{x^2+\tau}{x+\tau} > 0$.

This second case has two sub-cases.

First suppose that $\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2} \le 0$, for some $x \in (1,\infty)$, $\tau \in (0,\infty)$. In this case we have $\left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) < 0$ and $\left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \ln \frac{x^2+\tau}{x+\tau} \le 0$. Thus it follows $E(\tau) < 0$, and so we get $k'(\tau) < 0$, and we are done.
Now we suppose $\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2} > 0$.
In this case we use the well-known inequality $t-1 \ge \ln t$, $\quad t \in (0,\infty)$. Putting $t = \frac{x^2+\tau}{x+\tau}$ we get $\ln \frac{x^2+\tau}{x+\tau} \le \frac{x^2-x}{x+\tau}$, for every $x \in (1,\infty)$, $\tau \in (0,\infty)$, and it follows that

$$E(\tau) = \left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) + \left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right)$$

$$\begin{aligned}\ln \frac{x^2+\tau}{x+\tau} &\le \left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) \\ &\quad + \left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \cdot \frac{x^2-x}{x+\tau} \\ &= \frac{(x-x^2)(x^2+2\tau x-\tau)}{(x^2+\tau)^2(x+\tau)^2} + \frac{[2x(x+\tau)^2-(x^2+\tau)^2](x^2-x)}{(x^2+\tau)^2(x+\tau)^3} \\ &= \frac{(x-x^2)(x^4-x^3+\tau x^2-\tau x)}{(x^2+\tau)^2(x+\tau)^3} < 0.\end{aligned}$$

Consequently, provided that x is fixed, $x \in (1,\infty)$, the inequality $k'(\tau) < 0$ holds for every $\tau \in (0,\infty)$. According to (14) it follows $g'_{\alpha,\beta}(x) > 0$, $x \in (1,\infty)$, and the proof is finished. □

Theorem 3 *If* $\tau = 25$, *then the mapping* u_τ *is strictly increasing on the interval* $(0,\infty)$, *where* u_τ *is defined by* (3).

Proof. Provided that $\tau = 25$, Lemma 2 implies that the function u_τ is strictly increasing on the interval $(0, x_1)$, where x_1 is the positive root of the equation $x^2 + 2\tau x - \tau = 0$.
Let $x_2 = 1.4616\ldots$ be the positive root of the equation $\psi(x) = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{(n+x)n} = 0$. If $\tau = 25$, then Lemma 3 implies that the function

$$v : (x_1,\infty) \to \mathbb{R}, \quad v(x) = \frac{-\frac{1}{x} - \gamma + \sum\limits_{n=1}^{\infty} \frac{x}{(n+x)n}}{\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}},$$

is strictly increasing on the interval (x_1, x_2).
Let $x_3 = 2.2324\ldots$ be the positive root of the equation $\psi(x) = \psi'(x)$. Since

Lemma 4 implies that ν is strictly increasing on the interval (x_2, x_3), it follows that ν is strictly increasing on (x_1, x_3). Now this result and Lemma 1 imply that the mapping u_τ is also strictly increasing on the intervals $(x_1, 1)$ and $(1, x_3)$.

Further we will prove that u_τ is strictly increasing on $(x_3, 3)$. We observe that if $x \in (x_3, 3)$ and $\tau = 25$ we can multiply the inequalities (6), (7), (8) and it follows that

$$\left(-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}\right) \ln \frac{x^2+\tau}{x+\tau} > \frac{x^2+2\tau x-\tau}{(x^2+\tau)(x+\tau)} \ln \Gamma(x), \tag{15}$$

and consequently we obtain

$$u_\tau'(x) = \frac{\left(-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}\right) \ln \frac{x^2+\tau}{x+\tau} - \frac{x^2+2\tau x-\tau}{(x^2+\tau)(x+\tau)} \ln \Gamma(x)}{\ln^2\left(\frac{x^2+\tau}{x+\tau}\right)} > 0,\ x \in (x_3, 3).$$

Summarizing, if $\tau = 25$, then we have proved that the function u_τ is strictly increasing on the intervals $(0, x_1)$, (x_1, x_3), $(x_3, 3)$. The continuity of u_τ implies that u_τ is strictly increasing on the interval $(0, 3)$.

We will prove in the followings that if $\tau = 25$, then u_τ is strictly increasing on $(3, \infty)$.

It is easily seen that multiplying the inequalities (9), (10), and (11) the inequality (15) follows in case $\tau = 25$ and $x \in (3, \infty)$. Thus we have $u_{25}'(x) > 0,\ x \in (3, \infty)$, and so u_{25} is strictly increasing on $(3, \infty)$. The continuity of u_{25} implies that this function is strictly increasing on $(0, \infty)$. □

Proof of Theorem 1.: From the equality

$$u_\tau(x) = u_{25}(x) \cdot g_{25,\tau}(x),$$

and from the results of Theorem 2. and Theorem 3. we infer that u_τ is strictly increasing on the interval $(0, \infty)$ in case of every given $\tau \in (0, 25]$. □

Other interesting results regarding the Γ function can be found in [1], [2], [5], [6] and [7].

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