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A note on nil-clean rings

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Abstract. We study a special kind of nil-clean rings, namely those nil-clean rings whose nilpotent elements are difference of two "left-right symmetric" idempotents, and prove that in some various cases they are strongly π-regular. We also show that all nil-clean rings having cyclic unit 2-groups are themselves strongly nil-clean of characteristic 2 (and thus they are again strongly π-regular).

1 Introduction and background

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0 of R. Our terminology and notations are mainly standard being in agreement with [9]. Exactly, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R and $J(R)$ the Jacobson radical of R.

A ring R is called *von Neumann regular* or just *regular* for short if, for any element $r \in R$, there is an element $a \in R$ such that $r = rar$. In the case when $a = 1$, we have that $r = r^2$ and these rings are known to be *boolean*. Generalizing regularity, a ring R is called *π-regular* if, for each $r \in R$, there are $i \in \mathbb{N}$ and $b \in R$ both depending on r such that $r^i = r^i b r^i$. Likewise, a ring R is called *strongly π-regular* if, for every $r \in R$, there exist $j \in \mathbb{N}$ and

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$c \in R$ both depending on r with $r^j = r^{j+1}c$. It is well known that strongly π-regularity implies π-regularity, while the converse is wrong as some critical examples show (see, e.g., [9]).

On the other hand, referring to [7] for a more account, we shall say that a ring is *nil-clean* provided each its element is a sum of a nilpotent and an idempotent. If these two elements commute, the nil-clean ring is said to be *strongly nil-clean*. While nil-clean rings are not completely characterized up to an isomorphism yet, this was successfully done in [4] by proving that a ring R is strongly nil-clean if, and only if, the quotient ring $R/J(R)$ is boolean and $J(R)$ is nil.

That is why, classifying the structure of some special types of nil-clean rings will be of some interest and importance. Our workable purpose here is to examine those nil-clean rings whose nilpotents are differences of two (special) idempotents. Specifically, we shall prove that in Theorem 1 presented below that every nil-clean ring having only nilpotents which are difference of two special (so-called "left-right symmetric") idempotents is strongly π-regular. This contrasts an example due to Šter in [11] who constructed a nil-clean ring of unbounded index of nilpotence which is not strongly π-regular. Note that by an appeal to [6, Corollary 3.12] nil-clean rings of bounded index of nilpotence are always strongly π-regular. We also consider the challenging question of when a nil-clean ring with finite (in particular, cyclic) unit group is strongly nil-clean. It is necessarily such a group to be consisting only of elements of order being a power of 2, and the ring will be of characteristic 2 too.

2 Main results

We separate our chief results into two subsections as follows:

2.1 Nil-clean rings with nilpotents as a sum of two special idempotents

We start our assertions with the next one.

Proposition 1 *If R is a nil-clean ring such that each nilpotent is a difference of two commuting idempotents, then R is a boolean ring.*

Proof. We first claim that such a ring R is of characteristic 2. Indeed, as $2 \in \mathrm{Nil}(R)$ (see, e.g., [7]), one writes that $2 = e - f$ for some $e, f \in \mathrm{Id}(R)$. Hence, it easily follows that $ef = fe$ even not assuming this a priory and, therefore, $2^3 = (e-f)^3 = e-f = 2$. This means that $6 = 0$, i.e., $2 = 0$ because $3 \in U(R)$ and the claim is sustained.

Moreover, we assert that R has to be abelian, that is, all its idempotents are central. In fact, given an arbitrary $a \in R$ and an arbitrary $e \in Id(R)$, one sees that $ea(1-e) \in Nil(R)$ and thus $ea(1-e) = e_1 + e_2$ for some $e_1, e_2 \in Id(R)$ with $e_1e_2 = e_2e_1$. Squaring this, it follows at once that $0 = e_1 + e_2$ since $2 = 0$ which yields $ea = eae$. Similarly, one derives that $ae = eae$ by looking at the element $(1-e)ae$, which allows us to conclude that $ae = ea$, as asserted.

We next arrive at the fact that R is semi-primitive, which is equivalent to $J(R) = \{0\}$. To verify this, given any element $z \in J(R)$, one may write that $z = e-f$ for some $e, f \in Id(R)$ with $ef = fe$ since $J(R)$ is nil (see, for instance, [7]). Now, taking into account that $2 = 0$, we find that $z^2 = z$ whence $z(z-1) = 0$ ensuring that $z = 0$ because $z-1 \in U(R)$. Thus R is semi-primitive, as claimed.

Furthermore, we may apply either [4] or [7] to get the desired boolean property of R. □

It was established in [8, Proposition 1] that any nilpotent matrix over a field is a difference of two idempotent matrices (for another approach see [10] as well). This major statement allows us to extract the following assertion, independently proved also in [10] and partially in [3].

Lemma 1 *In regular rings all nilpotent elements are difference of two idempotents.*

Proof. Consulting with the main result from [1] which shows that, in an arbitrary ring, a nilpotent with all powers regular can be thought of as locally just a nilpotent matrix in Jordan or Weyr form. With this at hand, the aforementioned matrix result in [8] gives the desired presentation. □

Imitating [3], two idempotents e, f are called *left-right symmetric* if the two equalities $ef = e$ and $fe = f$ hold. It is evident that both e and f are somewhat "left-active" in the sense that they are "preserved on the left multiplication".

So, we have accumulated all the information necessary to establish the following.

Theorem 1 *Every nil-clean ring in which all nilpotents are difference of two left-right symmetric idempotents are strongly π-regular.*

Proof. We foremost assert that for such a ring R it must be that $char(R) = 2$. To see that, as $2 \in Nil(R)$ holds in view of [7], one writes that $2 = e_1 - e_2$ for two $e_1, e_2 \in Id(R)$. This surely means that e_1 and e_2 do commute, so that

$2^3 = (e_1 - e_2)^3 = e_1 - e_2 = 2$ whence $6 = 0$. Consequently, $2 = 0$ because $3 \in U(R)$, as asserted.

For such a ring R, given an arbitrary $q \in Nil(R)$, we write that $q = e - f = e + f$ for some two $e, f \in Id(R)$ with $ef = e$ and $fe = f$. We, therefore, obtain by squaring that $q^2 = 2q = 0$. Thus R is of bounded index of nilpotence and [6, Corollary 3.12] is a guarantor for the validity of our assertion that R is strongly π-regular. □

The given proof allows us to consider whether a more general situation in which we have slightly amended relationships between e and f, that are, $efe = e$ and $fef = f$. Certainly, $ef = e$ forces $efe = e$ as well as $fe = f$ forces $fef = f$. Furthermore, writing $q = e + f$ and squaring this, we deduce that $q^2 - q = ef + fe$. Again squaring the last equality, we derive that $q^4 + q^2 = (q^2 - q)^2 = efef + efe + fef + fefe = ef + e + f + fe = q^2$. Finally, $q^4 = 0$ and hence R is with bounded index of nilpotence, too.

We can now mention some constructions of nil-clean rings having only nilpotent elements which are difference of two idempotents.

Remark 1 *By what we have just previously shown, a crucial example of such a sort of nil-clean rings is any nil-clean ring which is simultaneously regular – in fact, such is, for instance, the ring $\mathbb{M}_n(\mathbb{Z}_2)$ for all $n \geq 1$ by an appeal to [2] and to the well-known fact from [9] that it is a regular ring because so is $\mathbb{Z}_2$. Indeed, this is not always possible as it was recently exhibited in [11] an ingenious example of a nil-clean ring of characteristic 2 which is not strongly π-regular as well as of a nil-clean ring of characteristic 4 which is not π-regular.*

An other interesting example of a nil-clean ring whose nilpotent elements are differences of two idempotents and which ring is not regular (due to the fact that it has a non-zero Jacobson radical) is the upper triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$, which fact we leave to the interested reader for a direct inspection. This ring is, however, strongly π-regular.

Moreover, the indecomposable nil-clean ring $\mathbb{Z}_4$ does not have the indicated above specific property of its nilpotents since $2 \neq 0$ in it.

We end our work in this subsection with the following challenging problem.

Problem 1 *Characterize nil-clean rings whose nilpotent elements are differences of two arbitrary idempotents.*

2.2 Nil-clean rings with cyclic unit group

In [5, p.81] it was asked of whether or not a clean ring with cyclic units is strongly clean. We shall resolve this question in the case of nil-clean rings (note that nil-clean rings are always clean and a *clean* ring is the one whose elements are sums of a unit and an idempotent; if these two elements commute, the clean ring is called *strongly clean*). It was established in [4, Corollary 4.10] that a nil-clean is strongly nil-clean if, and only if, its unit group is a 2-group.

We are now arriving at the following statement.

Theorem 2 *Suppose* R *is a nil-clean ring with cyclic* $U(R)$. *Then* R *is strongly nil-clean of characteristic* 2 *if, and only if,* $U(R)$ *is a* 2*-group.*

Proof. If we assume for a moment that $U(R) = \{1\}$, then $\mathrm{Nil}(R) = \{0\}$ as $1 + \mathrm{Nil}(R) \subseteq U(R)$, so that R must be boolean whence strongly nil-clean. So, we shall assume hereafter that $U(R) \neq \{1\}$.

Firstly, to prove the "right-to-left" implication, assume that $U(R)$ is a cyclic 2-group. Thus, as commented above, it follows immediately from [4, Corollary 4.10] that R is strongly nil-clean. What remain to show is that $2 = 0$ holds in R. Indeed, since $2 \in \mathrm{Nil}(R)$, one observes that the infinite sequence $\{3, 5, 7, ..., 2k-1, 2k+1, ...\}$ will invert in R for any $k \in \mathbb{N}$. But as $U(R)$ is finite, there will exist a natural number k with $2k-1 = 2k+1$, so that $2 = 0$ is really fulfilled.

Secondly, the direct application of [4, Corollary 4.10] gives the "left-to-right" part, as desired. □

We finish our work in this subsection with the following useful comments which shed some further light on the explored theme.

Remark 2 *For nil-clean rings with finite unit group the above theorem is not longer true: in fact, as an example we can consider the* 2×2 *matrix ring* $\mathbb{M}_2(\mathbb{Z}_2)$ *which, in accordance with* [2]*, is nil-clean but surely not strongly nil-clean (however, it is strongly* π*-regular being finite). This suggests to extract even the more general claim that nil-clean rings with finite unit group are strongly* π*-regular of characteristic* 2*. In fact, as unipotents (= the sum of* 1 *and a nilpotent) are always units, it readily follows that the set of nilpotents is also finite and so the ring is with bounded index of nilpotence. We, therefore, can apply* [6, Corollary 3.12] *to get the wanted claim. That* $\mathrm{char}(R) = 2$ *follows now in the same manner as in the proof of Theorem 2.*

In closing, we pose a few intriguing problems of some interest and importance which immediately arise.

Problem 1. If R is a nil-clean ring with bounded $U(R)$, does it follow that R is (strongly) π-regular?

Problem 2. If R is a nil-clean ring of characteristic 2 and $U(R)$ is a p-group (or, respectively, a $2p$-group) for some prime p, is it true that R is (strongly) π-regular?

For eventual counterexamples in case we have dropped some of the requirements, see Examples 3.1 and 3.2 from [11].

In regard to both sections explored above, one may state the following:

Problem 3. Is any nil-clean ring R such that its nilpotents are differences of two idempotents always π-regular? In particular, if $J(R) = 0$, is then R necessarily von Neumann regular.

In fact, each such nil-clean ring is of characteristic 2. If the above question holds in the affirmative, this will be in sharp contrast to the recent example by Šter from [11] showing that there is a nil-clean ring which is not π-regular.

Letting $QNil(R)$ be the set of all quasi-nilpotent elements of the ring R, we note that both inclusions $Nil(R) \subseteq QNil(R)$ and $J(R) \subseteq QNil(R)$ hold. We thereby come in mind to our next question as follows:

Problem 4. Examine those (nil-clean) rings for which the equality $U(R) = 1 + QNil(R)$ is true.

Notice that the condition $U(R) = 1 + Nil(R) + J(R)$ obviously implies the condition $U(R) = 1 + QNil(R)$, as in the latter situation we shall say that the ring R has *quasi-nilpotent units*.

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References

[1] K. I. Beidar, K. C. O'Meara and R. M. Raphael, On uniform diagonalisation of matrices over regular rings and one-accesible regular algebras, *Commun. Algebra*, **32** (2004), 3543–3562.

[2] S. Breaz, G. Călugăreanu, P. Danchev and T. Micu, Nil-clean matrix rings, *Linear Algebra and Appl.*, (10) **439** (2013), 3115–3119.

[3] P. V. Danchev, Weakly exchange rings whose units are sums of two idempotents, *Vestnik St. Petersburg Univ., Math., Mech. & Astronomy*, (2) **6 (64)** (2019), 265–269.

[4] P. V. Danchev and T. Y. Lam, Rings with unipotent units, *Publ. Math. Debrecen*, (3-4) **88** (2016), 449–466.

[5] P. Danchev and J. Matczuk, $\mathfrak{n}$-Torsion clean rings, *Contemp. Math.*, **727** (2019), 71–82.

[6] P. Danchev and J. Šter, Generalizing π-regular rings, *Taiwanese J. Math.*, (6) **19** (2015), 1577–1592.

[7] A. J. Diesl, Nil clean rings, *J. Algebra*, **383** (2013), 197–211.

[8] R. E. Hartwig and M. S. Putcha, When is a matrix a difference of two idempotents, *Linear and Multilinear Algebra*, (4) **26** (1990), 267–277.

[9] T.-Y. Lam, *A First Course in Noncommutative Rings*, Second Edition, Graduate Texts in Math., Vol. 131, Springer-Verlag, Berlin-Heidelberg-New York, 2001.

[10] K. C. O'Meara, *Nilpotents often the difference of two idempotents*, private correspondence on draft privately circulated on March 2018.

[11] J. Šter, On expressing matrices over $\mathbb{Z}_2$ as the sum of an idempotent and a nilpotent, *Linear Algebra Appl.*, **544** (2018), 339–349.

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