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A note on logarithmically completely monotonic ratios of certain mean values

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Abstract. We offer a new, unitary proof of some generalizations of results from paper [2]. Our method leads to similar results for other special means, too.

1 Introduction

A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic (c.m. for short), if f has derivatives of all orders and satisfies

$$(-1)^n \cdot f^{(n)}(x) \geq 0 \text{ for all } x > 0 \text{ and } n = 0, 1, 2, \ldots. \quad (1)$$

J. Dubourdieu [3] pointed out that, if a non-constant function f is c.m., then strict inequality holds in (1). It is known (and called as Bernstein theorem) that f is c.m. iff f can be represented as

$$f(x) = \int_0^\infty e^{-xt} d\mu(t), \quad (2)$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [11]).

Completely monotonic functions appear naturally in many fields, like, for example, probability theory and potential theory. The main properties of

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these functions are given in [11]. We also refer to [4, 1, 2], where detailed lists of references can be found.

Let $a, b > 0$ be two positive real numbers. The power mean of order $k \in \mathbb{R} \setminus \{0\}$ of a and b is defined by

$$A_k = A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}.$$

Denote $A = A_1(a,b) = \dfrac{a+b}{2}$, $G = G(a,b) = A_0(a,b) = \lim_{k\to\infty} A_k(a,b) = \sqrt{ab}$ the arithmetic, resp. geometric means of a and b.

The identric, resp. logarithmic means of a and b are defined by

$$I = I(a,b) = \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)} \text{ for } a \neq b; \quad I(a,a) = a;$$

and

$$L = L(a,b) = \frac{b-a}{\log b - \log a} \text{ for } a \neq b; \quad L(a,a) = a.$$

Consider also the weighted geometric mean S of a and b, the weights being $a/(a+b)$ and $b/(a+b)$:

$$S = S(a,b) = a^{a/(a+b)} \cdot b^{b/(a+b)}.$$

As one has the identity (see [6])

$$S(a,b) = \frac{I(a^2, b^2)}{I(a,b)},$$

the mean S is connected with the identric mean I.

Other means which occur in this paper are

$$H = H(a,b) = A_{-1}(a,b) = \frac{2ab}{a+b}, \quad Q = Q(a,b) = A_2(a,b) = \sqrt{\frac{a^2+b^2}{2}},$$

as well as Seiffert's mean (see [10], [9])

$$P = P(a,b) = \frac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)} \text{ for } a \neq b, \quad P(a,a) = a.$$

In the paper [2] C.–P. Chen and F. Qi have considered the ratios

a) $\frac{A}{I}(x, x+1)$, b) $\frac{A}{G}(x, x+1)$, c) $\frac{A}{H}(x, x+1)$,

d) $\frac{I}{G}(x, x+1)$, e) $\frac{I}{H}(x, x+1)$, f) $\frac{G}{H}(x, x+1)$,

g) $\frac{A}{L}(x, x+1)$,

where $\frac{A}{I}(x, x+1) = \frac{A(x, x+1)}{I(x, x+1)}$ etc., and proved that the logarithms of the ratios $a) - f)$ are c.m., while the ratio from $g)$ is c.m.

In [2] the authors call a function f as logarithmically completely monotonic (l.c.m. for short) if the function $g = \log f$ is c.m. They notice that they proved earlier (in 2004) that if f is l.c.m., then it is also c.m. We note that this result has been proved already in paper [4]:

Lemma 1 *If f is l.c.m, then it is also c.m.*

The following basic property is well-known (see e.g. [4]):

Lemma 2 *If $a > 0$ and f is c.m., then $a \cdot f$ is c.m., too. The sum and the product of two c.m. functions is c.m., too.*

Corollary 1 *If k is a positive integer and f is c.m., then the function f^k is c.m., too.*

Indeed, it follows by induction from Lemma 2 that, the product of a finite number of c.m. functions is c.m., too.

Particularly, when there are k equal functions, Corollary 1 follows.

The aim of this note is to offer new proofs for more general results than in [2], and involving also the means S, P, Q.

2 Main results

First we note that, as one has the identity

$$H = \frac{G^2}{A},$$

we get immediately

$$\frac{A}{H} = \frac{A^2}{G^2}, \quad \frac{G}{H} = \frac{A}{G}$$

so that as

$$\log \frac{A}{H} = 2 \log \frac{A}{G} \quad \text{and} \quad \log \frac{G}{H} = \log \frac{A}{G},$$

by Lemma 2 the ratios $c)$ and $f)$ may be reduced to the ratio $a)$.

Similarly, as

$$\frac{I}{H} = \frac{A}{G} \cdot \frac{I}{G},$$

the study of ratio $e)$ follows (based again on Lemma 2) from the ratios $b)$ and $d)$.

As one has

$$\frac{A}{G} = \frac{A}{I} \cdot \frac{I}{G},$$

it will be sufficient to consider the ratios $a)$ and $d)$.

Therefore, in Theorem 1 of [2] we should prove only that $\frac{A}{I}(x, x+1)$ and $\frac{I}{G}(x, x+1)$ are l.c.m., and $\frac{A}{L}(x, x+1)$ is c.m.

A more general result is contained in the following:

Theorem 1 *For any $a > 0$ (fixed), the ratios*

$$\frac{A}{I}(x, x+a) \ \textit{and} \ \frac{I}{G}(x, x+a)$$

are l.c.m., and the ratio

$$\frac{A}{L}(x, x+a)$$

is c.m. function.

Proof. The following series representations are well-known (see e.g. [6, 9]):

$$\log \frac{A}{G}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left(\frac{y-x}{y+x} \right)^{2k}, \tag{3}$$

$$\log \frac{I}{G}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k+1} \cdot \left(\frac{y-x}{y+x} \right)^{2k}. \tag{4}$$

By substraction, from (3) and (4) we get

$$\log \frac{A}{I}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \cdot \left(\frac{y-x}{y+x} \right)^{2k}, \tag{5}$$

where $\frac{A}{G}(x,y) = \frac{A(x,y)}{G(x,y)}$, etc.

By letting $y = x + a$ in (4), we get that

$$\log \frac{I}{G}(x, x+a) = \sum_{k=1}^{\infty} \frac{a^{2k}}{2k+1} \cdot \left(\frac{1}{2x+a}\right)^{2k}. \tag{6}$$

As $\frac{1}{2x+a}$ is c.m., by Corollary 1, $g(x) = \left(\frac{1}{2x+a}\right)^{2k}$ will be c.m., too. This means that

$$(-1)^n g^{(n)}(x) \geq 0 \text{ for any } x > 0,\ n \geq 0,$$

so by n times differentiation of the series from (6), we get that $\log \frac{I}{G}(x, x+a)$ is c.m., thus $\frac{I}{G}(x, x+a)$ is l.c.m.

The similar proof for $\frac{A}{I}(x, x+a)$ follows from the series representation (5).

Finally, by the known identity (see e.g. [6], [9])

$$\log \frac{I}{G} = \frac{A}{L} - 1 \tag{7}$$

we get the last part of Theorem 1. ■

Remark 1 *It follows from the above that* $\frac{A}{G}(x, x+a)$, $\frac{A}{H}(x, x+a)$, $\frac{I}{H}(x, x+a)$, $\frac{G}{H}(x, x+a)$ *are all l.c.m. functions.*

Theorem 2 *For any* $a > 0$, *the ratios*

$$\frac{\sqrt{2A^2+G^2}}{I\sqrt{3}}(x, x+a),\ \frac{\sqrt{2A^2+G^2}}{G\sqrt{3}}(x, x+a) \textit{ and } \frac{Q}{G}(x, x+a)$$

are l.c.m. functions.

Proof. In paper [8] it is proved that

$$\log \frac{\sqrt{2A^2+G^2}}{I\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left(\frac{1}{2k+1} - \frac{1}{3^k}\right) \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{8}$$

while in [9] that

$$\log \frac{\sqrt{2A^2+G^2}}{G\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left(1 - \frac{1}{3^k}\right) \cdot \left(\frac{y-x}{y+x}\right)^{2k}. \tag{9}$$

Letting $y = x + a$, by the method of proof of Theorem 1, the first part of Theorem 2 follows. Finally, the identity

$$\log \frac{Q}{G} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \left(\frac{y-x}{y+x}\right)^{4k-2} \tag{10}$$

appears in [9]. This leads also to the proof of l.c.m. monotonicity of the ratio $\frac{Q}{G}(x, x+a)$. ■

Theorem 3 *For any $a > 0$, the ratios*

$$\frac{L}{G}(x, x+a),\ -\frac{H}{L}(x, x+a) \text{ and } \frac{A}{P}(x, x+a)$$

are c.m. functions.

Proof. In [5] (see also [9] for a new proof) it is shown that

$$\frac{L}{G}(x, y) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \cdot \left(\frac{\log x - \log y}{2}\right)^{2k}. \tag{11}$$

Letting $y = x + a$ and remarking that the function $f(x) = \log(x+a) - \log x$ is c.m., by Corollary 1, and by differentiation of the series from (11), we get that $\frac{L}{G}(x, x+a)$ is c.m.

The identity

$$\log \frac{S}{I} = 1 - \frac{H}{L} \tag{12}$$

appears in [9]. Since we have the series representations (see [7], [9])

$$\log \frac{S}{G}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \left(\frac{y-x}{y+x}\right)^{2k} \tag{13}$$

and

$$\log \frac{S}{A}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{14}$$

by using relation (4), we get $\log \frac{S}{G} - \log \frac{I}{G} = \log \frac{S}{I}$, so

$$\log \frac{S}{I}(x, y) = \sum_{k=1}^{\infty} \frac{2}{4k^2 - 1} \cdot \left(\frac{y - x}{y + x}\right)^{2k}, \tag{15}$$

thus $\frac{S}{I}(x, x + a)$ is l.c.m., which by (12) implies that the ratio $-\frac{H}{L}$ is l.c.m. function.

Finally, Seiffert's identity (see [10], [9])

$$\log \frac{A}{P}(x, y) = \sum_{k=0}^{\infty} \frac{1}{4^k(2k + 1)} \cdot \binom{2k}{k} \cdot \left(\frac{y - x}{y + x}\right)^{2k}, \tag{16}$$

implies the last part of the theorem. ■

Remark 2 *By* (13), (14) *and* (15) *we get also that* $\frac{S}{G}(x, x + a)$, $\frac{S}{A}(x, x + a)$ *and* $\frac{S}{I}(x, x + a)$ *are l.c.m. functions.*

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