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A common generalization of convolved (u, v)-Lucas first and second kinds p-polynomials

Adikanda Behera
Sambalpur University
Burla-768019, India
email: adikanda.math@gmail.com

Prasanta Kumar Ray
Sambalpur University
Burla-768019, India
email: prasantamath@suniv.ac.in

Abstract. In this note the convolved (u, v)-Lucas first kind and the convolved (u, v)-Lucas second kind p-polynomials are introduced and study some of their properties. Several identities related to the common generalization of convolved (u, v)-Lucas first and second kinds p-polynomials are also presented.

1 Introduction

Buschman [2] introduced the homogeneous linear second order difference equation with constant coefficients as

$$U_0;\ U_1;\ U_{n+1} = aU_n + bU_{n-1},\ \text{for } n \geq 1, \tag{1}$$

that generalizes almost all numbers and polynomials sequences. The Lucas sequence of first and second kinds $U = U(a, b)$ and $V = V(a, b)$ can be recovered from (1) by taking $U_0 = 0$, $U_1 = 1$ and $V_0 = 2, V_1 = a$ respectively. These two kinds sequences comprise Fibonacci numbers, generalized Fibonacci

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numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, balancing polynomials, chebyshev polynomials etc. The interested reader may look [1, 3, 4, 5] for a detail review.

Şahin and Ramírez [6] introduced the convolved (p,q)-Fibonacci polynomials (convolved generalized Lucas polynomials) by $g^{(r)}_{p,q}(t) = \left(1 - p(x)t - q(x)t^2\right)^{-r} = \sum_{n=0}^{\infty} F^{(r)}_{p,q,n+1}(x)t^n,\ r \in \mathbb{Z}^+$. In [7], Ye and Zhang gave a common generalization of convolved generalized Fibonacci and Lucas polynomials and are given by $\sum_{n=0}^{\infty} T^{(r,m)}_{h,n}(x)t^n = \frac{\left(h(x)+2t\right)^m}{\left(1-h(x)t-t^2\right)^r},\ r \geq m$ and $r, m \in \mathbb{Z}^+$. They obtained some recurrence relations and identities of these polynomials.

In this study we introduce convolved (u,v)-Lucas first kind and second kind p-polynomials and derive some of their identities. Further the common generalization of these two polynomials is presented and some related results are discussed.

2 Convolved (u,v)-Lucas first and second kinds p-polynomials

In this section, we introduce convolved (u,v)-Lucas first kind p-polynomials and convolved (u,v)-Lucas second kind p-polynomials and present some of their properties.

Definition 1 *Let p be any non-negative integer. The (u,v)-Lucas first kind p-polynomials $\{L^p_{u,v,j}(x)\}_{j\geqslant p+1}$ are defined recursively by*

$$L^p_{u,v,j}(x) = u(x)L^p_{u,v,j-1}(x) + v(x)L^p_{u,v,j-p-1}(x)$$

with initials $L^p_{u,v,0}(x) = 0$ and $L^p_{u,v,j}(x) = (u(x))^{j-1}$ for $j = 1 \dots p$ and $u(x)$ and $v(x)$ are polynomials with real coefficients.

Let $g^p_{u,v}(t)$ be the generating function of $L^p_{u,v,j+1}(x)$. Then it is easy to see $g^p_{u,v}(t) = \sum_{j=0}^{\infty} L^p_{u,v,j+1}(x)t^j = \frac{1}{1-u(x)t-v(x)t^{p+1}}$. The finding result is the criterion to define the convolved (u,v)-Lucas first kind p-polynomials.

Definition 2 *Let $u(x)$ and $v(x)$ be polynomials with real coefficients. Then the convolved (u,v)-Lucas first kind p-polynomials $\{L^{(p,r)}_{u,v,j}(x)\}_{j\in\mathbb{N}}$ for $p \geqslant 1$ are defined by*

$$g^{(p,r)}_{u,v}(t) = \sum_{j=0}^{\infty} L^{(p,r)}_{u,v,j+1}(x)t^j = \left(1 - u(x)t - v(x)t^{p+1}\right)^{-r},\ r \in \mathbb{Z}^+. \qquad (2)$$

Further simplification of relation (2) gives the following explicit formula

$$L^{(p,r)}_{u,v,j+1}(x) = \sum_{k=0}^{\lfloor \frac{j}{p+1} \rfloor} \frac{(r)_{j-pk}}{(j-(p+1)k)!k!} u^{j-(p+1)k}(x) v^k(x). \tag{3}$$

Consideration of formula (3) for different measures of (p, r) with $r = 4$ yield some values of convolved (u, v)-Lucas first kind p-polynomials which are listed in Table 1.

Table 1: Convolved (u, v)-Lucas first kind p-polynomials

j	$(p,r)=(1,4)$	$(p,r)=(2,4)$	$(p,r)=(3,4)$	$(p,r)=(4,4)$
0	1	1	1	1
1	$4u(x)$	$4u(x)$	$4u(x)$	$4u(x)$
2	$10u^2(x)+4v(x)$	$10u^2(x)$	$10u^2(x)$	$10u^2(x)$
3	$20u^3(x) + 20u(x)v(x)$	$20u^3(x)+4v(x)$	$20u^3(x)$	$20u^3(x)$
4	$35u^4(x) + 60u^2(x)v(x) + 10v^2(x)$	$35u^4(x) + 20u(x)v(x)$	$35u^4(x)+4v(x)$	$35u^4(x)$
5	$56u^5(x) + 140u^3(x)v(x) + 60u(x)v^2(x)$	$56u^5(x) + 60u^2(x)v(x)$	$56u^5(x) + 20u(x)v(x)$	$56u^5(x)+4v(x)$
6	$84u^6(x) + 280u^4(x)v(x) + 210u^2(x))v^2(x)+ 20v^3(x)$	$84u^6(x) + 140u^3(x)v(x) + 10v^2(x)$	$84u^6(x) + 60u^2(x)v(x)$	$84u^6(x)) + 20u(x)v(x)$

Theorem 1 *The convolved* (u, v)*-Lucas first kind* p*-polynomials* $L^{(p,r)}_{u,v,j+1}(x)$ *satisfies the following relation*

$$u(x)L^{(p,r)}_{u,v,j-1}(x) + v(x)L^{(p,r)}_{u,v,j-p-1}(x) + L^{(p,r-1)}_{u,v,j}(x) = L^{(p,r)}_{u,v,j}(x), \tag{4}$$

with parameters $r > 1$ *and* $j > 1$.

Proof. Using the explicit formula (3) on the left-hand side of (4), we get

$$\frac{(r)_{j-2-pk}}{(j-2-(p+1)k)!k!} u^{j-1-(p+1)k}(x)v^k(x) + \frac{(r)_{j-p-2-pk}}{(j-p-2-(p+1)k)!k!}$$

$$
\begin{aligned}
&\times u^{j-p-2-(p+1)k}(x)v^{k+1}(x)+\frac{(r-1)_{j-1-pk}}{(j-1-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^{k}(x)\\
=&\frac{(r)_{j-2-pk}}{(j-2-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^{k}(x)+\frac{(r)_{j-2-pk}}{(j-1-(p+1)k)!(k-1)!}\\
&\times u^{j-1-(p+1)k}(x)v^{k}(x)+\frac{(r-1)_{j-1-pk}}{(j-1-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^{k}(x)\\
=&\frac{u^{j-1-(p+1)k}(x)v^{k}(x)}{(j-1-(p+1)k)!k!}\left[\frac{(j-1-(p+1)k)(r)_{j-1-pk}}{(r+j-2-pk)}+\frac{k(r)_{j-1-pk}}{(r+j-2-pk)}\right.\\
&\left.+\frac{(r-1)(r)_{j-1-pk}}{(r+j-2-pk)}\right]\\
=&\frac{(r)_{j-1-pk}}{(j-1-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^{k}(x)\\
=&L^{(p,r)}_{u,v,j}(x).
\end{aligned}
$$

This completes the proof. □

Theorem 2 *The following relation*

$$
\sum_{k=1}^{r}L^{(p,k)}_{u,v,j}(x)=\frac{1}{u(x)}\left[L^{(p,r)}_{u,v,j+1}(x)-v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p}(x)\right] \tag{5}
$$

holds for $j\geqslant 1$ *with* $L^{(p,0)}_{u,v,j+1}=0$.

Proof. Taking summation over 1 to r in relation (4), we have

$$
\begin{aligned}
\sum_{k=1}^{r}L^{(p,k)}_{u,v,j}(x)&=u(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-1}(x)+v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p-1}(x)+\sum_{k=1}^{r}L^{(p,k-1)}_{u,v,j}(x)\\
&=u(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-1}(x)+v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p-1}(x)+\sum_{k=1}^{r-1}L^{(p,k)}_{u,v,j}(x).
\end{aligned}
$$

It follows that

$$
L^{(p,r)}_{u,v,j}(x)=u(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-1}(x)+v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p-1}(x),
$$

we get the desired result by replacing $j+1$ instead of j. □

Theorem 3 *For* $j \geqslant p+1$ *and* $L^{(p,r)}_{u,v,-j}(x) = 0$, *the polynomial* $L^{(p,r)}_{u,v,j+1}(x)$ *holds the following relation*

$$\begin{aligned} u(x)\sum_{i=0}^{j-1} L^{(p,r)}_{u,v,i+1}(x) + \sum_{i=0}^{j} L^{(p,r-1)}_{u,v,i+1}(x) \\ = \big(1 - v(x)\big)\sum_{i=0}^{j-p-1} L^{(p,r)}_{u,v,i+1}(x) + \sum_{i=j-p}^{j} L^{(p,r)}_{u,v,i+1}(x). \end{aligned} \tag{6}$$

Proof. Consider $j = 1, 2, \ldots$ in relation (4), follows

$$\begin{aligned} &u(x)L^{(p,r)}_{u,v,0}(x) + v(x)L^{(p,r)}_{u,v,-p}(x) + L^{(p,r-1)}_{u,v,1}(x) = L^{(p,r)}_{u,v,1}(x)\\ &u(x)L^{(p,r)}_{u,v,1}(x) + v(x)L^{(p,r)}_{u,v,-p+1}(x) + L^{(p,r-1)}_{u,v,2}(x) = L^{(p,r)}_{u,v,2}(x)\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &u(x)L^{(p,r)}_{u,v,p}(x) + v(x)L^{(p,r)}_{u,v,0}(x) + L^{(p,r-1)}_{u,v,p+1}(x) = L^{(p,r)}_{u,v,p+1}(x)\\ &u(x)L^{(p,r)}_{u,v,p+1}(x) + v(x)L^{(p,r)}_{u,v,1}(x) + L^{(p,r-1)}_{u,v,p+2}(x) = L^{(p,r)}_{u,v,p+2}(x)\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &u(x)L^{(p,r)}_{u,v,j-1}(x) + v(x)L^{(p,r)}_{u,v,j-p-1}(x) + L^{(p,r-1)}_{u,v,j}(x) = L^{(p,r)}_{u,v,j}(x)\\ &u(x)L^{(p,r)}_{u,v,j}(x) + v(x)L^{(p,r)}_{u,v,j-p}(x) + L^{(p,r-1)}_{u,v,j+1}(x) = L^{(p,r)}_{u,v,j+1}(x). \end{aligned}$$

Summation of these equalities yields the desired result. □

Definition 3 *Let* p *be any non-negative integer and* $u(x)$ *and* $v(x)$ *are polynomials of real coefficients. Then the* (u,v)*-Lucas second kind* p*-polynomials* $\{M^{p}_{u,v,j}(x)\}_{j\geqslant p+1}$ *are defined recursively by*

$$M^{p}_{u,v,j}(x) = u(x)M^{p}_{u,v,j-1}(x) + v(x)M^{p}_{u,v,j-p-1}(x),$$

with initials $M^{p}_{u,v,0}(x) = (p+1)\frac{M_1(x)}{u(x)}$ *and* $M^{p}_{u,v,j}(x) = M_1(x)u^{j-1}(x)$ *for* $j = 1 \ldots p$ *and* $M_1(x)$ *is the first term of Lucas second kind like polynomial sequences.*

Many well-known polynomial sequences are special cases of (u,v)-Lucas second kind p-polynomials. For example, for $p = 1$, when $\big(u(x), v(x)\big) = (x, 1)$ and $M_1 = x$, $\big(u(x), v(x)\big) = (2x, 1)$ and $M_1 = 2x$, $\big(u(x), v(x)\big) = (6x, -1)$ and $M_1 = 3x$, $\big(u(x), v(x)\big) = (1, 2x)$ and $M_1 = 1$, $\big(u(x), v(x)\big) = (3x, -2)$

and $M_1 = 3x$ etc. the (u, v)-Lucas second kind p-polynomials turn into classical Lucas polynomials, Pell-Lucas polynomials, Lucas-balancing polynomials, Jacobsthal-Lucas polynomials, Fermat-Lucas polynomials respectively.

Let $h^p_{u,v}(t)$ denotes the generating function of $M^p_{u,v,j+1}(x)$. Then

$$h^p_{u,v}(t) = \sum_{j=0}^{\infty} M^p_{u,v,j+1}(x)t^j = \frac{M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p}{1 - u(x)t - v(x)t^{p+1}}.$$

The generating function $h^p_{u,v}(t)$ is more precious to define convolved (u, v)-Lucas second kind p-polynomials.

Definition 4 *Let p be any positive integer. Then the convolved (u, v)-Lucas second kind p-polynomials $\{M^{(p,r)}_{u,v,j}(x)\}_{j\in\mathbb{N}}$ are defined by*

$$h^{(p,r)}_{u,v}(t) = \sum_{j=0}^{\infty} M^{(p,r)}_{u,v,j+1}(x)t^j = \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1 - u(x)t - v(x)t^{p+1}\right)^r}, \; r \in \mathbb{Z}^+. \quad (7)$$

Expression (7) reduces to the explicit formula

$$M^{(p,r)}_{u,v,j+1}(x) = \sum_{k=0}^{\min\{r,j\}} \sum_{i=0}^{\lfloor \frac{j-pk}{p+1} \rfloor} \binom{r}{k} \frac{(r)_{j-pk-pi}}{i!(j-pk-(p+1)i)!} M_1^r(x)(p+1)^k v^{k+i}(x) \times u^{j-(p+1)k-(p+1)i}(x). \quad (8)$$

Consideration of formula (8) for different measures of (p, r) gives some values of convolved (u, v)-Lucas second kind p-polynomials which are listed in Table 2.

Theorem 4 *The convolved (u, v)-Lucas second kind p-polynomials $M^{(p,r)}_{u,v,j+1}(x)$ satisfies following relation*

$$\begin{aligned} M^{(p,r)}_{u,v,j}(x) =& u(x)M^{(p,r)}_{u,v,j-1}(x) + v(x)M^{(p,r)}_{u,v,j-1-p}(x) \\ &+ M_1(x)M^{(p,r-1)}_{u,v,j}(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,r-1)}_{u,v,j-p}(x), \end{aligned} \quad (9)$$

with parameters $r > 1$ and $j > 1$.

Table 2: Convolved (u,v)-Lucas second kind p-polynomials

j	$(p,r)=(2,1)$	$(p,r)=(2,2)$	$(p,r)=(2,3)$	$(p,r)=(2,4)$
0	$M_1(x)$	$M_1^2(x)$	$M_1^3(x)$	$M_1^4(x)$
1	$M_1(x)u(x)$	$2M_1^2(x)u(x)$	$3M_1^3(x)u(x)$	$4M_1^4(x)u(x)$
2	$M_1(x)u^2(x)+\frac{3M_1(x)v(x)}{u(x)}$	$3M_1^2(x)u^2(x)+\frac{6M_1^2(x)v(x)}{u(x)}$	$6M_1^3(x)u^2(x)+\frac{9M_1^3(x)v(x)}{u(x)}$	$10M_1^4(x)u^2(x)+\frac{12M_1^4(x)v(x)}{u(x)}$
3	$M_1(x)u^3(x)+4M_1(x)v(x)$	$4M_1^2(x)u^3(x)+14M_1^2(x)v(x)$	$10M_1^3(x)u^3(x)+30M_1^3(x)v(x)$	$20M_1^4(x)u^3(x)+52M_1^4(x)v(x)$
4	$M_1(x)u^4(x)+5M_1(x)u(x)v(x)$	$5M_1^2(x)u^4(x)+24M_1^2(x)u(x)v(x)+\frac{9M_1^2(x)v^2(x)}{u^2(x)}$	$15M_1^3(x)u^4(x)+66M_1^3(x)u(x)v(x)+\frac{27M_1^3(x)v^2(x)}{u^2(x)}$	$35M_1^4(x)u^4(x)+140M_1^4(x)u(x)\times v(x)+\frac{54M_1^4(x)v^2(x)}{u^2(x)}$
5	$M_1(x)u^5(x)+6M_1(x)u^2(x)v(x)+\frac{3M_1(x)v^2(x)}{u(x)}$	$6M_1^2(x)u^5(x)+36M_1^2(x)u^2(x)\times v(x)+\frac{30M_1^2(x)v^2(x)}{u(x)}$	$21M_1^3(x)u^5(x)+120M_1^3(x)u^2(x)\times v(x)+\frac{108M_1^3(x)v^2(x)}{u(x)}$	$56M_1^4(x)u^5(x)+300M_1^4(x)u^2(x)\times v(x)+\frac{264M_1^4(x)v^2(x)}{u(x)}$

Proof. Using (7) on the right-hand side of relation (9), we get

$$\begin{aligned}
&u(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^2+v(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\
&\quad\times t^{p+2}+M_1(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t+(p+1)\frac{v(x)}{u(x)}M_1(x)\\
&\quad\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t^{p+1}\\
&=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t\left[u(x)t+v(x)t^{p+1}\right]\\
&\quad+\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t\left[M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right]\\
&=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t\left[\frac{u(x)t+v(x)t^{p+1}}{1-u(x)t-v(x)t^{p+1}}+1\right]
\end{aligned}$$

$$= \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1 - u(x)t - v(x)t^{p+1}\right)^r}t,$$

which proves the result. □

Theorem 5 *The polynomial* $M^{(p,r)}_{u,v,j+1}(x)$ *obey the following relation*

$$\begin{aligned} M^{(p,r)}_{u,v,j}(x) &= u(x)\sum_{k=1}^{r} M^{(p,k)}_{u,v,j-1}(x) + v(x)\sum_{k=1}^{r} M^{(p,k)}_{u,v,j-1-p}(x) \\ &\quad + \left(M_1(x) - 1\right)\sum_{k=1}^{r-1} M^{(p,k)}_{u,v,j}(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)\sum_{k=1}^{r-1} M^{(p,k)}_{u,v,j-p}(x), \end{aligned} \tag{10}$$

with parameters $r > 1$, $j \geqslant 2$, $p \geqslant 1$ *and* $M^{(p,0)}_{u,v,j+1}(x) = 0$.

Proof. Consider $r = 1, 2, \ldots$ in relation (9) which follows

$$\begin{aligned} M^{(p,1)}_{u,v,j}(x) &= u(x)M^{(p,1)}_{u,v,j-1}(x) + v(x)M^{(p,1)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,0)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,0)}_{u,v,j-p}(x) \\ M^{(p,2)}_{u,v,j}(x) &= u(x)M^{(p,2)}_{u,v,j-1}(x) + v(x)M^{(p,2)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,1)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,1)}_{u,v,j-p}(x) \\ M^{(p,3)}_{u,v,j}(x) &= u(x)M^{(p,3)}_{u,v,j-1}(x) + v(x)M^{(p,3)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,2)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,2)}_{u,v,j-p}(x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M^{(p,r)}_{u,v,j}(x) &= u(x)M^{(p,r)}_{u,v,j-1}(x) + v(x)M^{(p,r)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,r-1)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,r-1)}_{u,v,j-p}(x). \end{aligned}$$

Summation of these bunch equalities yields the desired result. □

In order to verify the result (10), assume $j = 5$ with $(p, r) = (2, 3)$, gives

$$M^{(2,3)}_{u,v,5} = u(x)\sum_{k=1}^{3} M^{(2,k)}_{u,v,4}(x) + v(x)\sum_{k=1}^{3} M^{(2,k)}_{u,v,2}(x)$$

$$+\big(M_1(x)-1\big)\sum_{k=1}^{2}M_{u,v,5}^{(2,k)}(x)+3\frac{v(x)}{u(x)}M_1(x)\sum_{k=1}^{2}M_{u,v,3}^{(2,k)}(x).$$

Simplification of right-hand side gives

$$\begin{aligned}&u(x)\big[M_{u,v,4}^{(2,1)}(x)+M_{u,v,4}^{(2,2)}(x)+M_{u,v,4}^{(2,3)}(x)\big]+v(x)\big[M_{u,v,2}^{(2,1)}(x)+M_{u,v,2}^{(2,2)}(x)\\&+M_{u,v,2}^{(2,3)}(x)\big]+\big(M_1(x)-1\big)\big[M_{u,v,5}^{(2,1)}(x)+M_{u,v,5}^{(2,2)}(x)\big]\\&+3\frac{v(x)}{u(x)}M_1(x)\big[M_{u,v,3}^{(2,1)}(x)+M_{u,v,3}^{(2,2)}(x)\big]=15M_1^3(x)u^4(x)\\&+66M_1^3(x)u(x)v(x)+27M_1^3(x)\frac{v^2(x)}{u^2(x)}=M_{u,v,5}^{(2,3)}(x),\end{aligned}$$

and the result is verified.

3 Common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials

In this section we give the common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials and obtain some recurrence relations of these polynomials.

Definition 5 *Let p, r and m be all positive integers. Then the common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials $\{E_{u,v,j}^{(p,r,m)}(x)\}_{j\in\mathbb{N}}$ are defined by*

$$\sum_{j=0}^{\infty}E_{u,v,j}^{(p,r,m)}(x)t^j=\frac{\big(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\big)^m}{\big(1-u(x)t-v(x)t^{p+1}\big)^r},\ \ r\geqslant m. \tag{11}$$

Assumption of $m=0$ and $m=r$ reduces the expression (11) to convolved (u,v)-Lucas first kind p-polynomials and convolved (u,v)-Lucas second kind p-polynomials respectively.

Theorem 6 *The common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials has the following explicit formula*

$$\begin{aligned}E_{u,v,j}^{(p,r,m)}(x)=&\sum_{k=0}^{\min\{m,j\}}\sum_{i=0}^{\lfloor\frac{j-pk}{p+1}\rfloor}\binom{m}{k}\frac{(r)_{j-pk-pi}}{(j-pk-(p+1)i)!i!}M_1^m(x)(p+1)^kv^{k+i}(x)\\&\times u^{j-(p+1)k-(p+1)i}(x).\end{aligned}$$

Proof. We run the proof by taking right-hand side of the expression (11)

$$\begin{aligned}
&\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\
&=\sum_{k=0}^{m}\binom{m}{k}M_1^{m-k}(x)\left((p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^k\sum_{j=0}^{\infty}\binom{-r}{j}(-t)^j\left(u(x)+v(x)t^p\right)^j\\
&=\sum_{k=0}^{m}\binom{m}{k}M_1^{m-k}(x)(p+1)^k\frac{v^k(x)}{u^k(x)}M_1^k(x)t^{pk}\sum_{j=0}^{\infty}\frac{(r)_j}{j!}t^j\sum_{i=0}^{j}\binom{j}{i}u^{j-i}(x)v^i(x)t^{pi}\\
&=\sum_{j=0}^{\infty}\sum_{k=0}^{\min\{m,j\}}\sum_{i=0}^{\lfloor\frac{j-pk}{p+1}\rfloor}\binom{m}{k}M_1^m(x)(p+1)^k\frac{v^k(x)}{u^k(x)}t^{pk}\frac{(r)_{j-pk-pi}}{(j-pk-pi)!}t^{j-pk-pi}\\
&\quad\times\frac{(j-pk-pi)!}{\left(j-pk-(p+1)i\right)!i!}u^{j-pk-(p+1)i}(x)v^i(x)t^{pi}\\
&=\sum_{j=0}^{\infty}\sum_{k=0}^{\min\{m,j\}}\sum_{i=0}^{\lfloor\frac{j-pk}{p+1}\rfloor}\binom{m}{k}M_1^m(x)(p+1)^k v^{k+i}(x)u^{j-(p+1)k-(p+1)i}(x)\\
&\quad\times\frac{(r)_{j-pk-pi}}{\left(j-pk-(p+1)i\right)!i!}t^j.
\end{aligned}$$

Comparing the left-hand side of expression (11), we get the required result. □

Theorem 7 *The common generalization of convolved* (u,v)*-Lucas first and second kinds* p*-polynomials obeys the following relations*

(i) $E_{u,v,j}^{(p,r,m)}(x)=u(x)E_{u,v,j-1}^{(p,r,m)}(x)+v(x)E_{u,v,j-(p+1)}^{(p,r,m)}(x)+E_{u,v,j}^{(p,r-1,m)}(x)$;

(ii) $E_{u,v,j}^{(p,r,m+1)}(x)=M_1(x)E_{u,v,j}^{(p,r,m)}(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)E_{u,v,j-p}^{(p,r,m)}(x)$;

(iii) $E_{u,v,j}^{(p,r,m)}(x)=\frac{1}{u(x)}\left\{\frac{j+1}{r-1}E_{u,v,j+1}^{(p,r-1,m)}(x)-\frac{p(p+1)M_1(x)m}{r-1}\frac{v(x)}{u(x)}E_{u,v,j+1-p}^{(p,r-1,m-1)}(x)\right.$
$\left.\qquad -(p+1)v(x)E_{u,v,j-p}^{(p,r,m)}(x)\right\}$.

Proof. Using the expression (11), we have

$$\sum_{j=0}^{\infty}E_{u,v,j}^{(p,r,m)}(x)t^j=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}$$

$$= \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m \left(u(x)t + v(x)t^{p+1}\right)}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} + \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}$$

$$= u(x)\frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t + v(x)\frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p+1} + \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}},$$

and the expression (i) follows. The proof of (ii) is analogous to (i). In order to proof (iii), we proceed as follows:

The common generalization of convolved (u, v)-Lucas first and second kinds p-polynomials can be written undoubtedly as

$$(r-1)\left(u(x) - (p+1)v(x)t^p\right)\frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}$$
$$= \frac{d}{dt}\frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}} - p(p+1)\frac{v(x)}{u(x)}M_1(x)m\frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{m-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t^{p-1},$$

and if and only if

$$(r-1)\left[u(x)E^{(p,r,m)}_{u,v,j}(x) + (p+1)v(x)E^{(p,r,m)}_{u,v,j-p}(x)\right] = (j+1)E^{(p,r-1,m)}_{u,v,j+1}(x) - p(p+1)\frac{v(x)}{u(x)}M_1(x)mE^{(p,r-1,m-1)}_{u,v,j+1-p}(x).$$

This follows the result. □

The following corollary is an immediate consequence of Theorem 7.

Corollary 1 *Let* r *and* m *be any positive integers with* $r \geq m$*. Then, the* *following relations*

$$\sum_{k=1}^{r-m}\left[u(x)E^{p,m+k,m}_{u,v,j-1}(x)+v(x)E^{(p,m+k,m)}_{u,v,j-(p+1)}(x)\right]=E^{(p,r,m)}_{u,v,j}(x)-E^{(p,m,m)}_{u,v,j}(x) \quad (12)$$

and

$$\sum_{k=0}^{m}M_1^k(x)E^{(p,r,m-k)}_{u,v,j-p}(x) = \frac{u(x)}{v(x)(p+1)M_1(x)}\left[E^{(p,r,m+1)}_{u,v,j}(x)-M_1^{m+1}(x)E^{(p,r,0)}_{u,v,j}(x)\right] \quad (13)$$

hold.

The following are some examples to understand the above corollary.

Example 1 *Consider* $r=4$ *and* $m=3$ *on the left of relation* (12), *gives*

$$\begin{aligned}
&\sum_{k=1}^{1}\left[u(x)E^{(p,3+k,3)}_{u,v,j-1}(x)+v(x)E^{(p,3+k,3)}_{u,v,j-(p+1)}(x)\right]\\
&=u(x)E^{(p,4,3)}_{u,v,j-1}(x)+v(x)E^{(p,4,3)}_{u,v,j-(p+1)}(x)\\
&=u(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}t\\
&\quad+v(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}t^{p+1}\\
&=\left(u(x)t+v(x)t^{p+1}-1\right)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}\\
&\quad+\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}\\
&=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}-\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^3}\\
&=E^{(p,4,3)}_{u,v,j}(x)-E^{(p,3,3)}_{u,v,j}(x).
\end{aligned}$$

Example 2 *Consider* $r = 5$ *and* $m = 2$ *in relation* (13), *we have*

$$\sum_{k=0}^{2} M_1^k(x) E_{u,v,j-p}^{(p,5,2-k)}(x) = \frac{u(x)}{v(x)(p+1)M_1(x)} \left[E_{u,v,j}^{(p,5,3)}(x) - M_1^3(x) E_{u,v,j}^{(p,5,0)}(x) \right]. \tag{14}$$

Expansion of left side of (14) *gives*

$$\begin{aligned}
&E_{u,v,j-p}^{(p,5,2)}(x) + M_1(x) E_{u,v,j-p}^{(p,5,1)}(x) + M_1^2(x) E_{u,v,j-p}^{(p,5,0)}(x) \\
&= \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)} M_1(x) t^p\right)^2}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p + M_1(x) \\
&\quad \times \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)} M_1(x) t^p\right)}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p + M_1^2(x) \frac{1}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p \\
&= \frac{3M_1^2(x) + (p+1)^2 \frac{v^2(x)}{u^2(x)} M_1^2(x) t^{2p} + 3(p+1)\frac{v(x)}{u(x)} M_1^2(x) t^p}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p.
\end{aligned}$$

On the other hand, expansion of right side of (14) *gives*

$$\begin{aligned}
&\frac{u(x)}{v(x)(p+1)M_1(x)} \left[E_{u,v,j}^{(p,5,3)}(x) - M_1^3(x) E_{u,v,j}^{(p,5,0)}(x) \right] \\
&= \frac{u(x)}{v(x)(p+1)M_1(x)} \left[\frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)} M_1(x) t^p\right)^3}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} \right. \\
&\quad \left. - M_1^3(x) \frac{1}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} \right] \\
&= \frac{3M_1^2(x) + (p+1)^2 \frac{v^2(x)}{u^2(x)} M_1^2(x) t^{2p} + 3(p+1)\frac{v(x)}{u(x)} M_1^2(x) t^p}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p.
\end{aligned}$$

Theorem 8 *The following identity*

$$\begin{aligned}
E_{u,v,j+1-p}^{(p,r,m)}(x) &= \frac{u(x)}{v(x)(p+1)M_1(x)(mp-j)} \left[M_1(x)(j+1) E_{u,v,j+1}^{(p,r,m)}(x) \right. \\
&\quad \left. - ru(x) E_{u,v,j}^{(p,r+1,m+1)}(x) - r(p+1)v(x) E_{u,v,j-p}^{(p,r+1,m+1)}(x) \right]
\end{aligned}$$

holds for every non-negative integers r *and* m.

Proof. It is observed that

$$\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} = \frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} \times\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-m}.$$

Differentiating both sides gives

$$\begin{aligned}\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} &= \left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-m}\\ &\quad\times\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\ &\quad+(r-m)p(p+1)\frac{v(x)}{u(x)}M_1(x)t^{p-1}\\ &\quad\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r},\end{aligned}$$

and we have

$$\begin{aligned}\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} &= rp(p+1)\frac{v(x)}{u(x)}M_1(x)\\ &\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p-1}\\ &+ru(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}+r(p+1)v(x)\\ &\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}t^p.\end{aligned}$$

Now, we get

$$\begin{aligned}&\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-m}\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\ &+(r-m)p(p+1)\frac{v(x)}{u(x)}M_1(x)t^{p-1}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\end{aligned}$$

$$
\begin{aligned}
&= rp(p+1)\frac{v(x)}{u(x)}M_1(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p-1}+ru(x)\\
&\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}+r(p+1)v(x)\\
&\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}t^p.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\
&\quad-mp(p+1)\frac{v(x)}{u(x)}M_1(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p-1}\\
&=ru(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{m+1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}+r(p+1)v(x)\\
&\quad\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{m+1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}t^p.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)\frac{d}{dt}E_{u,v,j}^{(p,r,m)}(x)-mp(p+1)\frac{v(x)}{u(x)}\\
&\times M_1(x)E_{u,v,j+1-p}^{(p,r,m)}(x)=ru(x)E_{u,v,j}^{(p,r+1,m+1)}(x)+r(p+1)v(x)E_{u,v,j-p}^{(p,r+1,m+1)}(x).
\end{aligned}
$$

Further simplification gives

$$
\begin{aligned}
(p+1)\frac{v(x)}{u(x)}M_1(x)(j-mp)E_{u,v,j+1-p}^{(p,r,m)}(x)&=ru(x)E_{u,v,j}^{(p,r+1,m+1)}(x)+r(p+1)\\
&\quad\times v(x)E_{u,v,j-p}^{(p,r+1,m+1)}(x)-M_1(x)(j+1)E_{u,v,j+1}^{(p,r,m)}(x),
\end{aligned}
$$

and the result follows. □

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