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Characterization of spectral elements in non-archimedean Banach algebras

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Abstract. Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e over an algebraically closed field. In this paper, we give a generalization of results of the paper [2] and we establish a new necessary and sufficient condition on the resolvent of an element $a \in \mathcal{A}$ such that for all $n \in \mathbb{N}$, $\|a^n\| \leq 1$.

1 Introduction and preliminaries

Throughout this paper, $\mathcal{A}$ is a non-archimedean Banach algebra with unit e $\Big(\|e\| = 1\Big)$ over a non trivially complete non-archimedean valued field $\mathbb{K}$ which is also algebraically closed with valuation $|\cdot|$, $\mathbb{Q}_p$ is the field of p-adic numbers equipped with p-adic valuation $|.|_p$ and $\mathbb{Z}_p$ denotes the ring of p-adic integers of $\mathbb{Q}_p$. For more details, we refer to [6] and [8]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$ ([6]). Let $r > 0$ and let Ω_r be the clopen ball of $\mathbb{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$. A non-archimedean normed algebra is a non-archimedean normed space with linear associative multiplication satisfying for all $a, b \in \mathcal{A}$, $\|ab\| \leq \|a\|\|b\|$. A non-archimedean complete normed algebra is

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called a non-archimedean Banach algebra, moreover, if there is $e \in \mathcal{A}$ such that for all $a \in \mathcal{A}$, $ae = ea = a$ and $\|e\| = 1$, $\mathcal{A}$ is said to be a non-archimedean Banach algebra with unit e. For more details, we refer to [1], [3], [8] and [10]. We have the following lemma.

Lemma 1 **([8])** *Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e, let $a \in \mathcal{A}$ such that $\|a\| < 1$, then $e - a$ is invertible in $\mathcal{A}$ and $(e-a)^{-1} = \sum_{k=0}^{\infty} a^k$.*

let $a \in \mathcal{A}$, we set $\sigma(a) = \{\lambda \in \mathbb{K} : a - \lambda e \text{ is not invertible}\}$.

Definition 1 **([9])** *Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e. Set $r(a) = \inf_n \|a^n\|^{\frac{1}{n}} = \lim_n \|a^n\|^{\frac{1}{n}}$, a is said to be a spectral element if $\sup\{|\lambda| : \lambda \in \sigma(a)\} = r(a)$. For $a \in \mathcal{A}$, set*

$$U_a = \{\lambda \in \mathbb{K} : (e - \lambda a)^{-1} \textit{ exists in } \mathcal{A}\}.$$

$\Big(U_a$ *is open and* $0 \in U_a\Big)$ *and*

$$C_a = \{\alpha \in \mathbb{K} : B(0, |\beta|) \subset U_a \textit{ for some } \beta \in \mathbb{K},\ |\beta| > |\alpha|\}.$$

We generalize the Proposition 6.6 of [9] as follows.

Proposition 1 *[9] Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e, then the following are equivalent.*

(i) a is a spectral element.

(ii) For all $\lambda \in C_a$, $(e - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$.

(iii) For each $\alpha \in C_a^*$, the function $\lambda \mapsto (e - \lambda a)^{-1}$ is analytic on $B(0, |\alpha|)$.

2 Main results

In the rest of this paper, for an element $a \in \mathcal{A}$ such that for all $n \in \mathbb{N}$, $\|a^n\| \leq 1$, we assume that $U_a = \Omega_1$ where for all $\lambda \in U_a$, $R(\lambda, a) = (e - \lambda a)^{-1}$.

Proposition 2 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let a be a spectral element such that $\sup_{n \in \mathbb{N}} \|a^n\| \leq 1$. Then,*

$$\textit{for all } \lambda \in C_a,\ \|R(\lambda, a)\| \leq 1.$$

Proof. From Proposition 1, for each $\lambda \in C_a$, $\lim_{n\to\infty} |\lambda|^n \|a^n\| = 0$, then

$$\begin{aligned} \|R(\lambda, a)\| &= \left\| \sum_{n=0}^{\infty} \lambda^n a^n \right\| \\ &\leq \max_{n\in\mathbb{N}} |\lambda^n| \\ &= 1. \end{aligned}$$

□

Proposition 3 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let a be a spectral element such that $\sup\limits_{n\in\mathbb{N}} \|a^n\| \leq 1$. Then, for all $\lambda, \mu \in C_a$,*

$$\lambda R(\lambda, a) - \mu R(\mu, a) = (\lambda - \mu) R(\lambda, a) R(\mu, a).$$

Proof. If $\lambda, \mu \in C_a$, then

$$\lambda R(\lambda, a)(e - \mu a) R(\mu, a) - \mu R(\lambda, a)(e - \lambda a) R(\mu, a) \tag{1}$$

and

$$\begin{aligned} (1) &= \lambda R(\lambda, a) R(\mu, a) - \lambda\mu R(\lambda, a) a R(\mu, a) - \mu R(\lambda, a) R(\mu, a) \\ &\quad + \lambda\mu R(\lambda, a) a R(\mu, a) \\ &= \lambda R(\lambda, a) R(\mu, a) - \mu R(\lambda, a) R(\mu, a) \\ &= (\lambda - \mu) R(\lambda, a) R(\mu, a). \end{aligned}$$

□

Proposition 4 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let a be a spectral element such that $\sup\limits_{n\in\mathbb{N}} \|a^n\| \leq 1$. Then for all $\lambda \in C_a$, $\|R(\lambda, a) - e\| \leq |\lambda|$.*

Proof. Since $a \in \mathcal{A}$ is a spectral element, we get for each $\lambda \in C_a$, $R(\lambda, a) = \sum_{n=0}^{\infty} \lambda^n a^n$. Then, for any $\lambda \in C_a$,

$$\begin{aligned} \|R(\lambda, a) - e\| &= \|\sum_{n=1}^{\infty} \lambda^n a^n\| && (2) \\ &\leq \sup_{n\geq 1} \|\lambda^n a^n\| && (3) \\ &\leq |\lambda|. && (4) \end{aligned}$$

□

Proposition 5 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit* e, *let* a *be a spectral element such that* $\sup_{n\in\mathbb{N}} \|a^n\| \leq 1$. *Then for any* $n \in \mathbb{N}, \alpha \in C_a^*$, $\lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, a) = \frac{n!(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n}.$$

Proof. From Proposition 3, for each $\lambda, \mu \in \Omega_{|\alpha|}$ with $\alpha \in C_a^*$,

$$\Big(\lambda e + (\mu - \lambda)e + (\lambda - \mu)R(\lambda, a)\Big)R(\mu, a) = \lambda R(\lambda, a). \tag{5}$$

Thus

$$\Big(e - \frac{1}{\lambda}(\mu - \lambda)(R(\lambda, a) - e)\Big)R(\mu, a) = R(\lambda, a). \tag{6}$$

The quantity in square brackets on the left of this equation is invertible for $|\lambda|^{-1}|\mu - \lambda|\|R(\lambda, a) - e\| < 1$. Then

$$R(\mu, a) = \sum_{n=0}^{\infty} \frac{(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n}(\mu - \lambda)^n. \tag{7}$$

But it follows by Proposition 1 that $R(\mu, a)$ is analytic on $B(\lambda, |\alpha|)$. Since $a \in \mathcal{A}$ is a spectral element, we get for all $\lambda, \mu \in \Omega_{|\alpha|}$, $R(\mu, a)$ can be written as follows:

$$R(\mu, a) = \sum_{n=0}^{\infty} \frac{R^{(n)}(\lambda, a)}{n!}(\mu - \lambda)^n.$$

Then, for any $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, a) = \frac{n!(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n}.$$

□

We have the following theorem.

Theorem 1 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit* e, *let* a *be a spectral element. Then for all* $n \in \mathbb{N}$, $\|a^n\| \leq 1$ *if and only if*

$$\left\|\Big(R(\lambda, a) - e\Big)^n R(\lambda, a)\right\| \leq |\lambda|^n, \tag{8}$$

for all $\lambda \in \Omega_{|\alpha|}$ *where* $\alpha \in C_a^*$ *and* $R(\lambda, a) = (e - \lambda a)^{-1}$.

Proof. Suppose that for each $n \in \mathbb{N}, \|a^n\| \leq 1$, let $\alpha \in C_a^*$, from Proposition 1, $R(\lambda, a) = (e - \lambda a)^{-1} = \sum_{k=0}^{\infty} \lambda^k a^k$ is analytic on $\Omega_{|\alpha|}$. By Proposition 5, for any $n \in \mathbb{N}, \lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, a) = \frac{n!(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n} \tag{9}$$

and

$$R^{(n)}(\lambda, a) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\lambda^{k-n} a^k = \sum_{k=n}^{\infty} n! \binom{k}{n} \lambda^{k-n} a^k.$$

Hence for each $n \in \mathbb{N}$ and for all $\lambda \in \Omega_{|\alpha|}$,

$$\begin{aligned}
\left\| \frac{R^{(n)}(\lambda, a)}{n!} \right\| &= \left\| \sum_{k=n}^{\infty} \binom{k}{n} \lambda^{k-n} a^k \right\| && (10)\\
&\leq \sup_{k \geq n} \left| \binom{k}{n} \right| |\lambda|^{k-n} \|a^k\| && (11)\\
&\leq \sup_{k \geq n} |\lambda|^{k-n} \|a^k\| && (12)\\
&\leq 1. && (13)
\end{aligned}$$

Then, for any $n \in \mathbb{N}$ and $\lambda \in \Omega_{|\alpha|}$,

$$\left\| \frac{R^{(n)}(\lambda, a)}{n!} \right\| \leq 1. \tag{14}$$

From (9) and (14), we have for any $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$\|(R(\lambda, a) - e)^n R(\lambda, a)\| \leq |\lambda|^n. \tag{15}$$

Conversely, we assume that (8) holds. From a is spectral, we have for any $\lambda \in \Omega_{|\alpha|}, R(\lambda, a) = \sum_{n=0}^{\infty} \lambda^n a^n$. Put for any $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $S_k(\lambda) = \lambda^{-k}(R(\lambda, a) - e)^k R(\lambda, a)$, then for any $\lambda \in \Omega_{|\alpha|}, k \in \mathbb{N}$, $\|S_k(\lambda)\| \leq 1$. Since a and $R(\lambda, a)$ commute, we have

$$\begin{aligned}
S_k(\lambda) &= \lambda^{-k}\Big(\big(e - (e - \lambda a)\big) R(\lambda, a)\Big)^k R(\lambda, a), && (16)\\
&= \lambda^{-k}(\lambda a R(\lambda, a))^k R(\lambda, a), && (17)\\
&= a^k R(\lambda, a)^{k+1}. && (18)
\end{aligned}$$

Then for each $\lambda \in \Omega_{|\alpha|}$ and for all $k \in \mathbb{N}$,

$$\begin{aligned} \|a^k\| &= \|(e-\lambda a)^{k+1}S_k(\lambda)\|, && (19)\\ &\le \|(e-\lambda a)^{k+1}\|\|S_k(\lambda)\|, && (20)\\ &\le \left\|\sum_{j=0}^{k+1}\binom{k+1}{j}(-\lambda a)^j\right\|, && (21)\\ &\le \max\{1, \|\lambda a\|, \|\lambda^2 a^2\|, \cdots, \|\lambda^{k+1}a^{k+1}\|\}, && (22) \end{aligned}$$

for $\lambda \to 0$, we get for all $k \in \mathbb{N}, \|a^k\| \le 1$. □

We generalize the result of [4] in non-archimedean Banach algebra as follows.

Theorem 2 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let $a \in \mathcal{A}$ be a spectral element with $U_a = \Omega_1$, then for all $n \ge 1$, $\|a^n\| \le 1$ if and only if*

$$\left\|\Big(R(\lambda, a) - e\Big)^k\right\| \le |\lambda|^k, \tag{23}$$

for all $\lambda \in \Omega_{|\alpha|}$, $k \ge 1$ where $\alpha \in C_a^*$ and $R(\lambda, a) = (e - \lambda a)^{-1}$.

Proof. Assume that for any $n \in \mathbb{N}, \|a^n\| \le 1$, let $\alpha \in C_a^*$, then $R(\lambda, a) = (e-\lambda a)^{-1} = \sum_{k=0}^{\infty} \lambda^k a^k$ is analytic on $\Omega_{|\alpha|}$. Using $R(\lambda, a) - e = \lambda a R(\lambda, a)$ and Proposition 5, we have

$$(R(\lambda, a) - e)^{n+1} = \lambda a (R(\lambda, a) - e)^n R(\lambda, a) = \frac{\lambda^{n+1}}{n!} a R^{(n)}(\lambda, a)$$

and

$$R^{(n)}(\lambda, a) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\lambda^{k-n}a^k = \sum_{k=n}^{\infty} n!\binom{k}{n}\lambda^{k-n}a^k.$$

Thus

$$(R(\lambda, a) - e)^{n+1} = \sum_{k=n}^{\infty}\binom{k}{n}(\lambda a)^{k+1}.$$

Then for all $n \in \mathbb{N}$ and for any $\lambda \in \Omega_{|\alpha|}$,

$$\begin{aligned} \left\|(R(\lambda, a) - e)^{n+1}\right\| &= \left\|\sum_{k=n}^{\infty}\binom{k}{n}(\lambda a)^{k+1}\right\| \\ &\le \sup_{k \ge n}\left|\binom{k}{n}\right||\lambda|^{k+1}\|a^{k+1}\| \end{aligned}$$

$$\begin{aligned} &\leq \sup_{k\geq n} |\lambda|^{k+1}\|a^{k+1}\| \\ &\leq |\lambda|^{n+1}. \end{aligned}$$

Conversely, we assume that (23) holds. Since a is a spectral element, then for all $\lambda \in \Omega_{|\alpha|}, R(\lambda, a) = \sum_{n=0}^{\infty} \lambda^n a^n$. Put for any $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $S_k(\lambda) = \lambda^{-k-1}(R(\lambda, a) - e)^{k+1}$, then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $\|S_k(\lambda)\| \leq 1$. Since a and $R(\lambda, a)$ commute. From $R(\lambda, a) - e = \lambda a R(\lambda, a)$, we get $S_k(\lambda) = (aR(\lambda, a))^{k+1}$, hence:

$$a^{k+1} = (e - \lambda a)^{k+1} S_k(\lambda).$$

Then for all $\lambda \in \Omega_{|\alpha|}$ and for each $k \in \mathbb{N}$,

$$\begin{aligned} \|a^{k+1}\| &= \|(e-\lambda a)^{k+1} S_k(\lambda)\| \\ &\leq \|(e-\lambda a)^{k+1}\| \|S_k(\lambda)\| \\ &\leq \left\| \sum_{j=0}^{k+1} \binom{k+1}{j} (-\lambda a)^j \right\| \\ &\leq \max\{1, \|\lambda a\|, \|\lambda^2 a^2\|, \cdots, \|\lambda^{k+1} a^{k+1}\|\}, \end{aligned}$$

for $\lambda \to 0$, we get for any $k \in \mathbb{N}$, $\|a^{k+1}\| \leq 1$. □

References

[1] J. Alvarez, C^*-algebras of operators in non-Archimedean Hilbert spaces, *Comment. Math. Univ. Carolin.*, **33** (1992), 573–580.

[2] A. El amrani, J. Ettayb and A. Blali, P-adic discrete semigroup of contractions, *Proyecciones (Antofagasta)*, **40**(6) (2021), 1507–1519.

[3] A. Escassut, Ultrametric Banach algebras, *World Scientific Publishing*, 2003.

[4] J. Ettayb, Further results on p-adic semigroup of contractions, arXiv preprint arXiv:2211.12630 (2022).

[5] A. G. Gibson, A Discrete Hille-Yoshida-Phillips Theorem, *J. Math. Anal. Appl.*, **39** (1972), 761–770.

[6] N. Koblitz, p-*adic Analysis: a Short Course on Recent Work*, Cambridge University Press, Cambridge, (1980).

[7] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci, **44**, Springer-Verlag, 1983.

[8] A. C. M. van Rooij, *Non-Archimedean functional analysis*, Monographs and Textbooks in Pure and Applied Math., **51**, Marcel Dekker, Inc., New York, 1978.

[9] W. H. Schikhof, On p-adic compact operators, Tech. Report 8911, Departement of Mathematics, Catholic University, Nijmengen, The Netherlands, (1989), 1–28.

[10] W. H. Schikhof, Uniqueness of Banach algebra topology for a class of non-archimedean algebras, *Indag. Math. (N.S.)*, **87**(1) (1984), 47–49.

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