Acta Univ. Sapientiae, Mathematica, **16**, 1 (2024) 98–100

DOI: 10.47745/ausm-2024-0008

Embedding topological manifolds into L^p spaces

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Abstract. With a simple argument, we show as a main note that, for every given $1 \leq p \leq +\infty$, every locally compact second-countable Hausdorff space is topologically embeddable into some L^p space with respect to some finite nonzero Borel measure, where the embedding may be chosen so that its range is included in some open proper subset of the L^p space.

Throughout, a manifold is always assumed to be a topological manifold, i.e. a second-countable Hausdorff space where every point has some neighborhood homeomorphic to some (fixed) Euclidean space. And an embedding is always assumed to be a topological embedding, i.e. a homeomorphism acting between a topological space and a subspace of a topological space.

In addition to the existing embedding results for various types of manifolds, we wish to show with a simple elementary proof that, given any $1 \leq p \leq +\infty$, every manifold is embeddable into some L^p space with some additional properties.

Our main result is more general:

Theorem 1 *If $1 \leq p \leq +\infty$, then every locally compact second-countable Hausdorff space is embeddable into some L^p space such that i) the underlying measure may be chosen to be a finite nonzero Borel one, and ii) the embedding may be chosen so that its range is included in some open proper subset of the L^p space chosen in i).*

2010 Mathematics Subject Classification: embeddings; Lebesgue spaces; local compactness; topological manifolds
Key words and phrases: 57N35; 54B99; 57N99

Proof. Let M be a locally compact second-countable Hausdorff space. If M_∞ denotes the Alexandroff (one-point) compactification of M, possibly without denseness of M, then, since M is sigma-compact, the space M_∞ is in addition second-countable and hence metrizable by the usual Urysohn construction.

Upon choosing a metric d for M_∞, define for every $x \in M_\infty$ the continuous function $f_x : M_\infty \to \mathbb{R}, y \mapsto d(x, y)$; the functions f_x are evidently a version of the Kuratowski construction. Since M_∞ is compact, it suffices to work with f_x in the simplified form.

On the other hand, let μ be a weighted sum of Dirac measures (restricted to the Borel sigma-algebra of M_∞) over M_∞ concentrated respectively at the points of a chosen countable dense subset of M_∞ with the property that $\mu(M_\infty) = 1$; such a choice of μ is always possible by considering for example the coefficients $2^{-1}, 2^{-2}, \ldots$. Then μ is a finite nonzero Borel probability measure over M_∞.

Identify two functions in the real Banach space $L^p(\mu)$ that are μ-almost everywhere equal with each other. Then, as every f_x is bounded and hence lies in $L^p(\mu)$, the map $F : x \mapsto f_x$ is continuous with respect to the L^p-norm; indeed, if $1 \leq p < +\infty$ then

$$\left(\int_{M_\infty} |f_x - f_z|^p \, d\mu \right)^{1/p} \leq d(x, z)$$

for all $x, z \in M_\infty$, and

$$|f_x - f_z|_{L^\infty} \leq |d(x, z)|_{L^\infty} \leq d(x, z)$$

for all $x, z \in M_\infty$. Moreover, since $d(x, \cdot) = d(z, \cdot)$ implies $0 = d(z, x)$, and since the equivalence class of f_x is $\{f_x\}$ by the construction of μ for every $x \in M_\infty$, the map F is an injection; the compactness of M_∞ and the continuity of F then jointly imply that F is a closed map and hence embeds M_∞ into $L^p(\mu)$.

Since M is by construction a subspace of M_∞, the composition $\Phi : M \to L^p(\mu)$ of $F|_M$ circ the inclusion map $\mathrm{id}_{M_\infty}|_M$ serves as an embedding.

As M is by construction open in M_∞, with ∞ denoting the additional element of M_∞ there is some open $V \subset L^p(\mu)$ such that

$$\Phi^{1)}(M) = V \cap F^{1)}(M_\infty) = (V \cap \Phi^{1)}(M)) \cup (V \cap \{F(\infty)\}).$$

Since $V \cap \{F(\infty)\}$ is then empty, it follows that V is a proper subset of $L^p(\mu)$ and

$$\Phi^{1)}(M) \subset V.$$

The above argument proves for the noncompact case; by a manifest slight modification it also works for M compact (e.g., adjoining a single point to M as an isolated point). This completes the proof. □

Given the importance of L^2 spaces as Hilbert spaces, the case where $p = 2$ in Theorem 1 would be of particular interest.

We will use the phrase "locally Euclidean" in the following sense: A second-countable Hausdorff space is called locally Euclidean if and only if for every point of it there are some neighborhood of the point and some $n \in \mathbb{N}$ such that the neighborhood is homeomorphic to the Euclidean space $\mathbb{R}^n$.

Since every locally Euclidean space is evidently also locally compact, we summarize for ease of reference the intended corollaries in the following

Corollary 1 *Let $1 \leq p \leq +\infty$. Then every locally Euclidean space is embeddable into some L^p space in the way described in Theorem 1.*

In particular, every manifold, and hence every Euclidean space $\mathbb{R}^n$, is embeddable into some L^p space in the same way.

Received: April 22, 2021