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On relation-theoretic F−contractions and applications in F−metric spaces

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Abstract. The aim is to introduce some relation theoretic variants of F−contraction in an F−metric space endowed with a binary relation $\mathcal{R}$ and to prove results for its fixed point. In the sequel, several classes of contractions are sharpened, generalized, and improved. Numerical examples are presented to illustrate the theoretical conclusions. As applications of the main results, we solve a Dirichlet-Neumann initial value problem and two Dirichlet boundary value problems.

1 Introduction and preliminaries

One of the major directions to extend the metric fixed point theory is to generalize a certain mathematical structure or weaken some assumptions on the mapping. Fŕechet was the first to introduce the idea of metric spaces as a generalization of distance functions. Lately, Jleli and Samet [9] introduced

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$\mathtt{F}-$metric space utilizing a particular class of functions and compared it with existing generalizations of metric spaces in the literature. On another point of note, Turinici [22] investigated the order-theoretic fixed point, while Ran and Reurings [18] rediscovered an order-theoretic variant of the Banach contraction principle [4]. Recently, Tomar et al. [26] gave a novel response to the open question presented by Rhoades [17] on continuity at a fixed point while proving a fixed point of a set-valued map satisfying relation-theoretic contractions in a partial Pompeiu-Hausdorff metric space.

Following the works of Wardowski [29] and Cosentino and Vetro [7], we introduce a relation-theoretic variant of an $\mathtt{F}-$contraction and a Hardy- Rogers type $\mathtt{F}-$contraction in the framework of $\mathtt{F}-$metric spaces equipped with a binary relation to prove the existence and uniqueness of the fixed point. In the sequel, we obtained sharpened relation-theoretic variants of several theorems given by Chatterjea [6], Kannan [10], Reich [19], Wardowski [29], and so on. Further, examples are given to demonstrate that our results are authentic generalizations, extensions, and improvements of some celebrated and recent results present in the literature. Motivated by the importance of initial value and boundary value problems in the study of real-world problems (for instance numerical solution of LCR - circuit is useful in many engineering branches, boundary value problems of hanging cable problem plays a crucial role in designing crane lifts and booms) we solve, a Dirichlet-Neumann initial value problem and two Dirichlet boundary value problems, by applying our theoretical results. For more applications of fixed point techniques in real-world problems, one may refer to Tomar and Joshi [27].

2 Preliminaries

Let $\mathtt{F}$ be the set of functions $\mathfrak{f}:(0,\infty)\to\mathbb{R}$ so that :

$\mathtt{F}_1$: $0<\kappa<\xi \Longrightarrow \mathfrak{f}(\kappa)<\mathfrak{f}(\xi)$;

$\mathtt{F}_2$: for every sequence $\{s_n\}\subseteq(0,\infty)$, $s_n\to 0$ if and only if $\mathfrak{f}s_n\to-\infty$;

$\mathtt{F}_3$: there exists $l\in(0,1)$ such that $\lim_{s\to 0^+} s^l\mathfrak{f}s=0$.

Definition 1 [9] *Let* $D:\mathcal{U}\times\mathcal{U}\to[0,\infty)$ *be a given map on a non-empty set* $\mathcal{U}$ *and there exists* $\mathfrak{f}:(0,\infty)\to\mathbb{R}$ *satisfying* $\mathtt{F}_1$, $\mathtt{F}_2$, *and* $\alpha\in\mathbb{R}^+$ *in such a way that for* $s,\nu\in\mathcal{U}$*:*

(D_1) $D(s,\nu)=0\Longleftrightarrow s=\nu$*;*

(D_2) $D(s,v) = D(v,s)$;

(D_3) *for* $\{s_i\} \subseteq \mathcal{U}$, $i = 1,2,\dots,n$ *and* $(s_1,s_n) = (s,v)$, $D(s,v) > 0 \Longrightarrow f(D(s,v)) \leq f(\Sigma_{i=1}^{n-1} D(s_i,s_{i+1})) + \alpha$, $n \in \mathbb{N}, n \geq 2$.

Then, $(\mathcal{U}, D)$ *is an* F*−metric space.*

Example 1 *Let* $D : \mathcal{U} \times \mathcal{U} \to [0,\infty)$ *and* $\mathcal{U} = \mathbb{Z}$ *be defined as*

$$D(s,v) = \begin{cases} |s-v|, & (s,v) \in [-5,5] \times [-5,5] \\ \frac{|s-v|^2}{6+|s-v|}, & (s,v) \notin [-5,5] \times [-5,5] \end{cases},$$

for $f(t) = \log t + 1$, $t > 0$.

Evidently, D *satisfies* D_1 *and* D_2*. Now, take an arbitrary* $(s,v) \in \mathcal{U} \times \mathcal{U}$ *in such a way that* $D(s,v) > 0$*. For* $n \in \mathbb{N}, n \geq 2$, $\{s_i\} \subseteq \mathcal{U}$, $i = 1,2,\dots,n$ *and* $(s_1,u_n) = (s,v)$*. Let*

$$\mathcal{A} = \{k = 1,2,\dots,n-1 : (s_k,s_{k+1}) \in [-5,5] \times [-5,5]\},$$
$$\mathcal{B} = \{l = 1,2,\dots,n-1\} \setminus \mathcal{A}.$$

Now, we have

$$\Sigma_{i=1}^{n-1} D(s_i,s_{i+1}) = \Sigma_{k\in\mathcal{A}} D(s_k,s_{k+1}) + \Sigma_{l\in\mathcal{B}} D(s_l,s_{l+1}),$$
$$= \Sigma_{k\in\mathcal{A}} |s_{k+1}-s_k| + \Sigma_{l\in\mathcal{B}} \frac{|s_{l+1}-s_l|^2}{1+|s_{l+1}-s_l|}.$$

Next, we discuss two possible cases.

Case (i): Let $(s,v) \in [-5,5] \times [-5,5]$*. In this case*

$$\begin{aligned} D(s,v) &= |s-v| \\ &\leq \Sigma_{i=1}^{n-1} |s_{i+1}-s_i| \\ &= \Sigma_{k\in\mathcal{A}} |s_{k+1}-s_k| + \Sigma_{l\in\mathcal{B}} |s_l-s_{l+1}|. \end{aligned}$$

Since,

$$|s_{l+1}-s_l| \leq 2\frac{|s_{l+1}-s_l|^2}{6+|s_{l+1}-s_l|}, l \in \mathcal{B}.$$

Therefore,

$$D(s,v) \leq 2\left(\Sigma_{k\in\mathcal{A}} |s_{k+1}-s_k| + \Sigma_{l\in\mathcal{B}} \frac{|s_{l+1}-s_l|^2}{1+|s_{l+1}-s_l|}\right). \tag{1}$$

Case (ii): Let $(s,v) \notin [-5,5] \times [-5,5]$. *In this case*

$$\begin{aligned} D(s,v) &= \frac{|s-v|^2}{6+|s-v|} \\ &\leq 2\Sigma_{i=1}^{n-1} \frac{|s_{i+1}-s_i|^2}{6+|s_{i+1}-s_i|} \\ &= 2\Sigma_{k\in\mathcal{A}} \frac{|s_{k+1}-s_k|^2}{6+|s_{k+1}-s_k|} + 2\Sigma_{l\in\mathcal{B}} \frac{|s_{l+1}-s_l|^2}{1+|s_{l+1}-s_l|} \\ &\leq 2\Sigma_{k\in\mathcal{A}} |s_{k+1}-s_k| + 2\Sigma_{l\in\mathcal{B}} \frac{|s_{l+1}-s_l|^2}{1+|s_{l+1}-s_l|}. \end{aligned}$$

Therefore

$$D(s,v) \leq 2(\Sigma_{k\in\mathcal{A}} |s_{k+1}-s_k| + \Sigma_{l\in\mathcal{B}} \frac{|s_{l+1}-s_l|^2}{6+|s_{l+1}-s_l|}. \tag{2}$$

By combining (1) *and* (2)

$$\begin{aligned} & D(s,v) > 0 \Longrightarrow D(s,v) \leq 2\Sigma_{i=1}^{n-1} D(s_{i+1}-s_i), \\ & \Longrightarrow \log(D(s,v)) \leq \log(\Sigma_{i=1}^{n-1} D(s_{i+1}-s_i)) + \log 2, \\ & \Longrightarrow \log(D(s,v)) + 1 \leq \log(\Sigma_{i=1}^{n-1} D(s_{i+1}-s_i)) + 1 + \log 2. \end{aligned}$$

So, D *satisfies* D_3 *for* $\alpha = \log 2$. *Hence,* D *is an* F−*metric.*
However, D *is never a metric on* $\mathcal{U}$ *as it does not verify the triangle inequality. Here,*

$$D(6,10) = \tfrac{4^2}{6+4} = \tfrac{16}{10} \geq D(6,8) + D(8,10) = \tfrac{4}{6+2} + \tfrac{4}{6+2} = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

It is interesting to see that each metric is an F−*metric however, the reverse is not essentially correct implying thereby that* F−*metric is more predominant than the standard metric.*

Definition 2 [9] *Let* $\{s_n\}$ *be a sequence in an* F−*metric space* $(\mathcal{U}, D)$.

(i) $\{s_n\}$ *is* F−*convergent to* $s \in \mathcal{U}$, *if* $\lim_{n\to\infty} D(s_n, s) = 0$.

(ii) $\{s_n\}$ *is* F−*Cauchy, if* $\lim_{n,m\to\infty} D(s_n, s_m) = 0$.

(iii) *If the* F−*Cauchy sequence in* $\mathcal{U}$ *is* F−*convergent to any point in* $\mathcal{U}$, *then* $(\mathcal{U}, D)$ *is* F−*complete.*

A binary relation denoted by $\mathcal{R}$ is a subset of $\mathcal{U} \times \mathcal{U}$, where $\mathcal{U}$ is a non-empty set. If $(s, v) \in \mathcal{R}$, then s is related to v. Throughout the paper $(\mathcal{U}, D)$ is an $\mathtt{F}-$metric space, $\mathcal{M}$ is a self map on $\mathcal{U}$, $\mathtt{F}(\mathcal{M})$ is the set of all fixed points of $\mathcal{M}$, and $\mathcal{U}(\mathcal{M}, \mathcal{R})$ is the collection of points $s \in \mathcal{U}$ in such a way that $(s, \mathcal{M}s) \in \mathbb{R}$.

Definition 3 [15] *$\mathcal{R}$ is complete if $s, v \in \mathcal{U}, [s, v] \in \mathcal{R}$, (that is, either $(s, v) \in \mathcal{R}$ or $(v, s) \in \mathcal{R}$).*

Definition 4 [14] *The symmetric closure $\mathcal{R}^s$ is the smallest symmetric relation containing $\mathcal{R}$, that is, $\mathcal{R}^s = \mathcal{R} \bigcup \mathcal{R}^{-1}$.*

Definition 5 [13] *$\mathcal{R}$ is $\mathcal{M}-$closed if $(s, v) \in \mathcal{R} \Longrightarrow (\mathcal{M}s, \mathcal{M}v) \in \mathcal{R}$, $s, v \in \mathcal{U}$.*
It is equivalent to saying that $\mathcal{M}$ is nondecreasing [20].

Definition 6 [1] *A sequence $\{s_n\}$ in $\mathcal{U}$ is $\mathcal{R}-$preserving if $(s_n, s_{n+1}) \in \mathcal{R}$, $n \in \mathbb{N}_0$.*

Definition 7 [3] *$\mathcal{R}$ is $D-$ self-closed, if $\mathcal{R}-$preserving sequence $\{s_n\}$ so that $[s_n, s_{n+1}] \in \mathcal{R}$, $n \in \mathbb{N}$, and $s_n \to s$, there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying $[s_{n_k,s}] \in \mathcal{R}$, $k \in \mathbb{N}$.*
$(\mathcal{U}, D, \mathcal{R}^s)$ is regular if and only if $\mathcal{R}^s$ is $D-$ self -closed.

Definition 8 [21] *A subset $\mathcal{D}$ of $\mathcal{U}$ is $\mathcal{R}^s-$directed if for $s, v \in \mathcal{D}$, there exists $z \in \mathcal{U}$ satisfying $(s, z) \in \mathcal{R}^s$ and $(v, z) \in \mathcal{R}^s$.*

Definition 9 [12] *Let $s, v \in \mathcal{U}$. Then, a finite sequence $\{w_0, w_1, w_2, \ldots, w_k\}$ in $\mathcal{U}$ is a path of length $k \in \mathbb{N}$ joining s to v in $\mathcal{R}$ if $w_0 = s, w_k = v$ and $(w_i, w_{i+1}) \in \mathcal{R}$, $0 \leq i \leq k-1$.*
Noticeably, a path of length k includes $k+1$ elements of $\mathcal{U}$ that are not essentially distinct. The family of paths in $\mathcal{R}$ from s to v is denoted by $\Gamma(s, v, \mathcal{R})$.

Definition 10 [1] *Let $\mathcal{V} \subseteq \mathcal{U}$. Then, the restriction of $\mathcal{R}$ to $\mathcal{V}$ is the set $\mathcal{R} \cap \mathcal{V} \times \mathcal{V}$ (that is, $\mathcal{R}|_{\mathcal{V}} = \mathcal{R} \cap \mathcal{V} \times \mathcal{V}$). Actually, $\mathcal{R}|_{\mathcal{V}}$ is a relation on $\mathcal{V}$ induced by $\mathcal{R}$.*

In order to establish a relation theoretic variant of the main result of Wardowski [29] and Cosentino and Vetro [7], we recall some necessary notions for our main results, namely $\mathcal{R}-$completeness [2], $\mathcal{R}-$continuity [2], and regularity [21] in the environment of an $\mathtt{F}-$metric space (Tomar and Joshi [28]).

Definition 11 [28] *A relational* F*−metric space* $(\mathcal{U}, D, \mathcal{R})$ *is* $\mathcal{R}$*−complete if every* $\mathcal{R}$*−preserving Cauchy sequence in* $\mathcal{U}$ *converges to a point in* $\mathcal{U}$*.*

Every complete F*−metric space is* $\mathcal{R}$*−complete however reverse is not essentially true.*

Example 2 *Let* $\mathcal{U} = (0, 1]$ *and* $D : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ *be defind as*

$$D(s, \nu) = \begin{cases} a^{(|s-\nu|)}, & s \neq \nu \\ 0, & s = \nu \end{cases} , a > 0 \tag{3}$$

with $\mathfrak{f}(t) = -\frac{1}{t} + t, t > 0$*,* $\alpha = 1$ *and a binary relation* $\mathcal{R} = \{(s, \nu) : s \leq \nu\}$*. Noticeably,* $(\mathcal{U}, D)$ *is an* F*−metric space and is neither a standard metric space nor any variant of standard metric space. Furthermore,* $(\mathcal{U}, D)$ *is* $\mathcal{R}$*−complete but not complete, as the Cauchy sequence* $\left\{\frac{1}{n}\right\}$ *in* $\mathcal{U}$ *converges to* $0 \notin \mathcal{U}$*.*

Definition 12 [28] *A self map* $\mathcal{M}$ *in a relational* F*−metric space* $(\mathcal{U}, D, \mathcal{R})$ *is* $\mathcal{R}$*−continuous at* s *if for* $\mathcal{R}$*−preserving sequence* $\{s_n\}$ *with* $s_n \to s$*, we have* $\mathcal{M}s_n \to \mathcal{M}s$*.*

Moreover, $\mathcal{M}$ *is* $\mathcal{R}$*−continuous if it is* $\mathcal{R}$*−continuous at each point of* $\mathcal{U}$*.*

Example 3 *Let* $\mathcal{U} = [0, 5]$ *be equipped with an* F*−metric as in* (3)*. Define a binary relation* $\mathcal{R} = \{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0), (2, 3), (2, 2), (3, 4), (4, 5), (3, 3)\}$ *on* $\mathcal{U}$ *and a discontinuous map* $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ *as*

$$\mathcal{M}s = \begin{cases} 0, & s \in [0, 1) \\ 2, & s \in [1, 3] \\ 3, & s \in (3, 5] \end{cases}.$$

Consider an $\mathcal{R}$*−preserving sequence* $\{s_n\}$ *in such a way that* $s_n \to u$*, so that* $(s_n, s_{n+1}) \in \mathcal{R}$ *,* $n \in \mathbb{N}_0$*. Since,* $(s_n, s_{n+1}) \in \{(0, , 0), (0, 1), (1, 0), (0, 2), (2, 0), (2, 2), (3, 3)\}$ *and* $(s_n, s_{n+1}) \notin \{(2, 3), (3, 4), (4, 5)\}$ *,* $n \in \mathbb{N}_0$*, which gives rise to* $\{s_n\} \subseteq \{0, 1, 2, 3\}$*. So* $[s_n, s] \in \mathcal{R}$*, that is, for every* $\mathcal{R}$*−preserving sequence* $s_n \to s \in \{0, 1, 2, 3\}$*,* $\mathcal{M}s_n \to \mathcal{M}s \in \{0, 1, 2, 3\}$*. Hence,* $\mathcal{M}$ *is* $\mathcal{R}$*−continuous.*

Definition 13 [28] *Let* $\mathcal{R}^s$*−symmetric closure of* $\mathcal{R}$*. Then,* $(\mathcal{U}, D, \mathcal{R}^s)$ *is regular if for a sequence* $\{s_n\} \subseteq \mathcal{U}$*,* $(s_n, s_{n+1}) \in \mathcal{R}^s$*,* $n \in \mathbb{N}$*, and* $s_n \to s$*, there exists a subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *in such a way that* $(s_{n_k}, s) \in \mathcal{R}^s$*,* $k \in \mathbb{N}_0$*.*

3 Main results

Following Wardowski [29], we introduce F− contraction in an F−metric space endowed with binary relation $\mathcal{R}$ and establish a relation theoretic variant of the main result of Wardowski [29].

Definition 14 *A self map $\mathcal{M}$ is a relational F−contraction if there exist $\mathfrak{f} \in \mathcal{F}$ and $\mu > 0$ satisfying*

$$s, \nu \in \mathcal{U}, D(\mathcal{M}s, \mathcal{M}\nu) > 0 \Longrightarrow \mu + \mathfrak{f}(D(\mathcal{M}s, \mathcal{M}\nu)) \leq \mathfrak{f}(D(s, \nu)), \ (s, \nu) \in \mathcal{R}. \quad (4)$$

Now, we state the first result of this section.

Theorem 1 *Let $\mathcal{M}$ be a self map in an F−metric space $(\mathcal{U}, D)$ endowed with a binary relation $\mathcal{R}$, satisfying:*

(i) *$\mathcal{R}$ is $\mathcal{M}$-closed.*

(ii) *$\mathcal{U}[\mathcal{M}, \mathcal{R}]$ is non-empty.*

(iii) *there exist $\mathcal{V} \subseteq \mathcal{U}$ in such a way that $\mathcal{M}(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$ and $(\mathcal{V}, D)$ is $\mathcal{R}$-complete.*

(iv) *either $\mathcal{R}|_{\mathcal{V}}$ is D−self closed or $\mathcal{M}$ is $\mathcal{R}$-continuous.*

(v) *$\mathcal{M}$ is a relational F−contraction.*
Then, $\mathcal{M}$ has a fixed point.
Additionally, if

(vi) *$\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s$−connected.*

Then, $\mathcal{M}$ has a unique fixed point in $\mathcal{U}$ and for each $s_0 \in \mathcal{U}$, the sequence $\{s_n\} \subseteq \mathcal{U}$, $s_{n+1} = \mathcal{M}s_n, n \in \mathbb{N}_0$, is F−convergent to a fixed point.

Proof. Define the Picard sequence $\{s_n\} \subseteq \mathcal{U}$ by $s_{n+1} = \mathcal{M}s_n$, $n \in \mathbb{N}_0$, with initial point $s_0 \in \mathcal{U}[\mathcal{M}, \mathcal{R}]$ as $\mathcal{U}[\mathcal{M}, \mathcal{R}] \neq \phi$. If $s_n = s_{n+1}$, then s_n is the fixed point of $\mathcal{M}$. If $s_n \neq s_{n+1}, D(s_n, s_{n+1}) > 0$. Since, $(s_0, \mathcal{M}s_0) \in \mathcal{R}$ and $\mathcal{R}$ is $\mathcal{M}$-closed, $(\mathcal{M}s_0, \mathcal{M}^2 s_0), (\mathcal{M}^2 s_0, \mathcal{M}^3 s_0), \ldots, (\mathcal{M}^n s_0, \mathcal{M}^{n+1} s_0) \ldots, \in \mathcal{R}$. So

$$(s_n, s_{n+1}) \in \mathcal{R}, \ n \in \mathbb{N}_0.$$

Therefore, the sequence $\{s_n\}$ is $\mathcal{R}$−preserving. Using inequality (4), we get

$$\mu + \mathfrak{f}(D(s_n, s_{n+1})) = \mu + \mathfrak{f}(D(\mathcal{M}s_{n-1}, \mathcal{M}s_n)) \leq \mathfrak{f}(D(s_{n-1}, s_n)), n \in \mathbb{N}_0,$$

that is,

$$\mathfrak{f}(D(\mathcal{M}s_{n-1}, \mathcal{M}s_n)) \leq \mathfrak{f}(D(s_{n-1}, s_n)) - \mu.$$

Following similar steps

$$\mathfrak{f}(D(\mathcal{M}s_{n-1}, \mathcal{M}s_n)) \leq \mathfrak{f}(D(s_{n-2}, s_{n-1})) - 2\mu,\ n \in \mathbb{N}_0.$$

Hence, in general

$$\mathfrak{f}(D(\mathcal{M}s_{n-1}, \mathcal{M}s_n)) \leq \mathfrak{f}(D(s_0, s_1)) - n\mu \to -\infty,\ \text{as } n \to \infty. \tag{5}$$

Now, by $\mathtt{F}_2$, $\lim_{n\to\infty} D(\mathcal{M}s_{n-1}, \mathcal{M}s_n) = 0$, that is, $\lim_{n\to\infty} D(s_n, s_{n+1}) = 0$. Exploiting $\mathtt{F}_3$, there exists $l \in (0, 1)$ in such a way that $(D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_n, s_{n+1})) \to 0$, $n \in \mathbb{N}_0$. So

$$\begin{aligned}
&(D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_n, s_{n+1})) - (D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_0, s_1))\\
&\leq (D(s_n, s_{n+1}))^l (\mathfrak{f}(D(s_0, s_1) - n\mu) - (D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_0, s_1))\\
&= -n\mu(D(s_n, s_{n+1}))^l < 0.
\end{aligned}$$

Now, as $n \to \infty$, $n(D(s_n, s_{n+1}))^l \to 0$ or $\lim_{n\to\infty} n^{\frac{1}{l}}(D(s_n, s_{n+1})) = 0$, which implies that the series $\Sigma_{n=1}^{\infty} D(s_n, s_{n+1}) < \infty$. Hence, $\{s_n\}$ is an $\mathcal{R}-$preserving F−Cauchy sequence. Since, $\{s_n\} \subseteq \mathcal{M}(\mathcal{U}) \subseteq \mathcal{V}$, $\{s_n\}$ is also $\mathcal{R}-$preserving Cauchy sequence in $\mathcal{Y}$. Because $(\mathcal{V}, D)$ is $\mathcal{R}-$complete, there exists $s \in \mathcal{V}$ so that $s_n \to y$.

If $\mathcal{M}$ is $\mathcal{R}-$continuous, then $s_{n+1} = \mathcal{M}s_n \to \mathcal{M}s$. So $\mathcal{M}s = s$, that is, s is fixed point of $\mathcal{M}$.

Also, if $\mathcal{R}|_{\mathcal{V}}$ is D−self closed, then for $\mathcal{R}$-preserving sequence $\{s_n\}$ in $\mathcal{V}$ with $s_n \to s$, there is a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ so that $[s_{n_k}, s] \in \mathcal{R}|_{\mathcal{V}} \subseteq \mathcal{R}$, $k \in \mathbb{N}_0$ and $s_{n_k} \to s$. Now, for any $\mathfrak{f} \in \mathtt{F}$ and using D_3

$$\begin{aligned}
\mathfrak{f}(D(s, \mathcal{M}s)) &\leq \mathfrak{f}(D(s, \mathcal{M}s_{n_k}) + D(\mathcal{M}s_{n_k}, \mathcal{M}s)) + \alpha\\
&= \mathfrak{f}(D(s, s_{n_{k+1}}) + D(s_{n_{k+1}}, \mathcal{M}s)) + \alpha \to -\infty \text{ as } n \to \infty, \quad \text{using } \mathtt{F}_2,
\end{aligned}$$

a contradiction. Hence, $D(s, \mathcal{M}s) = 0$, that is, $\mathcal{M}s = s$.
Hence, s is a fixed point of $\mathcal{M}$.

If $\mathtt{F}(\mathcal{M})$ is singleton then there is nothing to prove. Let, if feasible s and s^* be two different fixed points of $\mathcal{M}$, that is, $\mathcal{M}s = s$ and $\mathcal{M}s^* = s^*$. So

$$s,\ s^* \in \mathcal{M}(\mathcal{U}) \ \text{ and } \ \mathcal{M}^n s = s, \ \ \mathcal{M}^n s^* = s^*, \ \ n \in \mathbb{N}_0.$$

Since, $\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s-$connected, so there exists a path $\{z_0,\ z_1,\ z_2,\ \ldots,\ z_k\}$ (say) of finite length k in $\mathcal{R}^s$ from s to s^* so that $z_0 = s,\ z_k = s^*$ and $(z_i, z_{i+1}) \in \mathcal{R}^s$, $i = 0, 1, 2, \ldots, k-1$. Since, $\mathcal{R}$ is $\mathcal{M}$-closed, $(\mathcal{M}^n z_i, \mathcal{M}^n z_{i+1}) \in \mathcal{R}^s$, $0 \le i \le k-1$ and $n \in \mathbb{N}_0$. Now, $D(s, s^*) > 0$, implies

$$\begin{aligned} \mathfrak{f}(D(s,s^*)) &= \mathfrak{f}(D(\mathcal{M}^n z_0, \mathcal{M}^n z_k)) \\ &\le \mathfrak{f}(D(\mathcal{M}^{n-1} z_0, \mathcal{M}^{n-1} z_k)) - \mu \\ &\le \mathfrak{f}(D(\mathcal{M}^{n-2} z_0, \mathcal{M}^{n-2} z_k)) - 2\mu \\ &\quad \vdots \ldots \\ &\le \mathfrak{f}(D(z_0, z_k)) - n\mu \to -\infty \quad \text{as} \quad n \to \infty, \end{aligned}$$

a contradiction. So $\mathcal{D}(s, s^*) = 0$, that is, $s = s^*$. □

Corollary 1 *Theorem 1 is true if* (vi) *is substituted by any one of the subsequent hypotheses:*

(u_1) $\mathcal{R}|_{\mathcal{M}(\mathcal{U})}$ *is complete.*

(u_2) $\mathcal{M}(\mathcal{U})$ *is* $\mathcal{R}^s-$*directed.*

(u_3) $\Gamma(s, \nu, \mathcal{R}^s) \neq \phi$.

Proof. If (u_1) holds, that is, $\mathcal{R}|_{\mathcal{M}(\mathcal{U})}$ is complete, then for $s,\ \nu \in \mathcal{M}(\mathcal{U})$, $[s, \nu] \in \mathcal{R}|_{\mathcal{M}(\mathcal{U})}$, implies that $\{s,\ \nu\}$ is a path of length 1 in $\mathcal{R}^s|_{\mathcal{M}(\mathcal{U})}$ from s to ν. Consequently, $\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s-$connected. Hence, Theorem 1 concludes the result.

In case (u_2) holds, that is, $\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s-$directed, then for $s,\ \nu \in \mathcal{M}(\mathcal{U})$, there exist $z \in \mathcal{U}$ so that $(s, z) \in \mathcal{R}^s$ and $(\nu, z) \in \mathcal{R}^s$, which implies that $\{s,\ z,\ \nu\}$ is a path of length 2 in $\mathcal{R}^s$ from s to ν. Thus $\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s-$connected. Consequently, Theorem 1 concludes the result.

Again, if (u_3) holds, that is, $\Gamma(s, \nu, \mathcal{R}^s) \neq \phi$, then for $s,\ \nu \in \mathcal{M}(\mathcal{U})$ there exists a path $\{z_0, z_1, z_2, \ldots, z_k\}$ (say) of finite length k in $\mathcal{R}^s$ from s to ν so that $z_0 = s,\ z_k = \nu$. Hence, $\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s-$connected. Consequently, again Theorem 1 concludes the result. □

Example 4 *Consider an* F*−metric space defined on a set* $\mathcal{U} = [2, 5)$ *such as*

$$D(s, \nu) = \begin{cases} 3^{|s-\nu|}, & s \neq \nu \\ 0, & s = \nu \end{cases},$$

with $\mathfrak{f}(t) = -\frac{1}{t}$ *and* $\alpha = 1$. *Clearly,* $(\mathcal{U}, D)$ *is not a standard metric space. Now, let a relation* $\mathcal{R}$ *on* $\mathcal{U}$ *be defined as*

$$\mathcal{R}(s, \nu) = \{(2,3), (2,2), (2,4), (3,2), (3,3)\},$$

whose symmetric closure is

$$\mathcal{R}^s = \{(2,3), (3,2), (2,2), (2,4), (4,2), (3,3)\}.$$

Take $\mathcal{V} = [2,4) \subseteq \mathcal{U}$. *Let* $\mathcal{M}(s) = \begin{cases} 2, & s \in [2,4) \\ 3, & u \in [4,5) \end{cases}$. *Here,* $\mathcal{M}(\mathcal{U}) = \{2,3\}$ *and* $\mathcal{M}(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$. *Notice that* $(\mathcal{V}, D)$ *is not complete but* $\mathcal{R}-$*complete and*

$$(s_n, s_{n+1}) \in \{(2,3), (3,2), (2,2), (3,3)\} \text{ and } (s_n, s_{n+1}) \notin \{(2,4)\}, n \in \mathbb{N}_0,$$

that is, $s_n \subseteq \{2,3\}$, *which is a finite set. So every* $\mathcal{R}-$*preserving sequence* $\{s_n\} \to s \in \{2,3\}$. *Clearly,* $\mathcal{R}$ *is* $\mathcal{M}-$*closed ,* $\mathcal{M}$ *is* $\mathcal{R}-$*continuous and* $\Gamma(s, \nu, \mathcal{R}^s)$ *is non-empty,* $s, \nu \in \mathcal{M}(\mathcal{U})$. *For* $\mathfrak{f}(t) = \log t \in \mathtt{F}$, *one may verify the hypotheses of Theorem 1 for* $\mu \in (0, \log 3)$ *and* $\mathcal{M}$ *has a unique fixed point at* $s = 2$. *Further, there exists a sequence* $\{s_n\} \subseteq \mathcal{U}, s_n = \frac{2n}{n+1}, n \in \mathbb{N}$, *which* $\mathcal{M}-$*converges to* 2.

Remark 1 (i) *It is worth mentioning here that Theorem 1 is an authentic extension and improvement of Alnasera et al. [3] to a relational* $\mathtt{F}-$*contraction without assuming the completeness of the whole space. Rather, we used a relatively weaker notion namely:* $\mathcal{R}-$*completeness of any subspace of the whole space. Further, we replaced the continuity of the map with* $\mathcal{R}-$*continuity,* $D-$*selfclosedness of* $\mathcal{R}$ *with* $D-$*selfclosedness of* $\mathcal{R}|_{\mathcal{V}}$, *and nonemptiness of family of paths in* $\mathcal{R}$ *with* $\mathcal{R}^s$ *connectedness of range space. Notice that,* $(\mathcal{U}, D)$ *is neither complete nor* $\mathcal{R}-$*complete in Example 4. The underlying binary relation* $\mathcal{R}$ *is none of reflexive, symmetric, or transitive. Consequently, it is none of near-order, preorder, strict order, partial order, or tolerance.*

(ii) *Further, Theorem 1 is an extended, improved, sharpened, and a generalized variant of the main result of Wardowski [29]. For more work on* $\mathtt{F}-$*contractions one may refer to Tomar et al. [23]-[24], Tomar and Sharma [23].*

Following Cosentino and Vetro [7], we introduce a relational Hardy-Rogers type $\mathtt{F}-$contraction in an $\mathtt{F}$-metric space and establish relation theoretic variant of Cosentino and Vetro [7].

Definition 15 *A self-map $\mathcal{M}$ is a relational $\mathtt{F}-$contraction of Hardy-Rogers type if there exist $\mathfrak{f} \in \mathtt{F}$ and $\mu > 0$ satisfying $s, v \in \mathcal{U}, D(\mathcal{M}s, \mathcal{M}v) > 0$, implies that*

$$\begin{aligned} \mu + \mathfrak{f}(D(\mathcal{M}s, \mathcal{M}v)) \leq \mathfrak{f}(&\mathfrak{a}D(s, v) + \mathfrak{b}D(s, \mathcal{M}s) + \mathfrak{c}D(v, \mathcal{M}v) \\ &+ \mathfrak{d}(s, \mathcal{M}v) + \mathfrak{e}D(v, \mathcal{M}s)), \end{aligned} \tag{6}$$

$(s, v) \in \mathcal{R}$, $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + 2\mathfrak{d} < 1$ *and* $\mathfrak{e} \geq 0$.

Theorem 2 *Theorem 1 remains correct even if $\mathcal{M}$ is a relational $\mathtt{F}-$contraction of Hardy-Rogers type and $\mathfrak{a} + 2\mathfrak{c} + \mathfrak{d} + \mathfrak{e} \leq 1$ (in place of (v)).*

Proof. Define a Picard sequence $\{s_n\} \subseteq \mathcal{U}$, $s_{n+1} = \mathcal{M}s_n$, $n \in \mathbb{N}_0$, with initial point $s_0 \in \mathcal{U}[\mathcal{M}, \mathcal{R}]$ as $\mathcal{U}[\mathcal{M}, \mathcal{R}] \neq \phi$. Since, $(s_0, \mathcal{M}s_0) \in \mathcal{R}$ and $\mathcal{M}$-closed, $(\mathcal{M}s_0, \mathcal{M}^2 s_0), (\mathcal{M}^2 s_0, \mathcal{M}^3 s_0), \ldots, (\mathcal{M}^n s_0, \mathcal{M}^{n+1} s_0), \ldots, \in \mathcal{R}$. So $(s_n, s_{n+1}) \in \mathcal{R}$, $n \in \mathbb{N}_0$, that is, the sequence $\{s_n\}$ is $\mathcal{R}-$preserving. Using (6),

$$\begin{aligned} \mu + \mathfrak{f}(D(s_n, s_{n+1})) &= \mu + \mathfrak{f}(D(\mathcal{M}s_{n-1}, \mathcal{M}s_n)) \\ &\leq \mathfrak{f}(\mathfrak{a}D(s_{n-1}, s_n) + \mathfrak{b}D(s_{n-1}, \mathcal{M}s_{n-1}) + \mathfrak{c}D(s_n, \mathcal{M}s_n) \\ &+ \mathfrak{d}D(s_{n-1}, \mathcal{M}s_n) + \mathfrak{e}D(s_n, \mathcal{M}s_{n-1})) \\ &= \mathfrak{f}(\mathfrak{a}D(s_{n-1}, s_n) + \mathfrak{b}D(s_{n-1}, s_n) + \mathfrak{c}D(s_n, s_{n+1}) \\ &+ \mathfrak{d}D(s_{n-1}, s_{n+1}) + \mathfrak{e}D(s_n, s_n)) \\ &\leq \mathfrak{f}(\mathfrak{a}D(s_{n-1}, s_n) + \mathfrak{b}D(s_{n-1}, s_n) + \mathfrak{c}D(s_n, s_{n+1}) \\ &+ \mathfrak{d}[D(s_{n-1}, s_n) + D(s_n, s_{n+1})]) \\ &= \mathfrak{f}((\mathfrak{a} + \mathfrak{b} + \mathfrak{d})D(s_{n-1}, s_n) + (\mathfrak{c} + \mathfrak{d})D(s_n, s_{n+1})). \end{aligned}$$

Using $\mathtt{F}_1$,
$D(s_n, s_{n+1}) \leq (\mathfrak{a} + \mathfrak{b} + \mathfrak{d})D(s_{n-1}, s_n) + (\mathfrak{c} + \mathfrak{d})D(s_n, s_{n+1})$,
$(1 - \mathfrak{c} - \mathfrak{d})D(s_n, s_{n+1}) \leq (\mathfrak{a} + \mathfrak{b} + \mathfrak{d})D(s_{n-1}, s_n)$,
$D(s_n, s_{n+1}) \leq (\frac{\mathfrak{a}+\mathfrak{b}+\mathfrak{d}}{1-\mathfrak{c}-\mathfrak{d}})D(s_{n-1}, s_n) = D(s_n, s_{n-1}), n \in \mathbb{N}_0$.
Consequently,
$\mu + \mathfrak{f}(D(s_n, s_{n+1})) \leq \mathfrak{f}(D(s_{n-1}, s_n))$,
$\mathfrak{f}\big(D(s_n, s_{n+1})\big) \leq \mathfrak{f}(D(s_{n-1}, s_n)) - \mu$.
Following similar steps, we get,
$\mathfrak{f}(D(s_n, s_{n+1})) \leq \mathfrak{f}(D(s_{n-1}, s_{n-2})) - 2\mu$.
Hence, in general,
$\mathfrak{f}(D(s_n, s_{n+1})) \leq \mathfrak{f}(D(s_0, s_1)) - n\mu \to -\infty$, as $n \to \infty$.
Now, using $\mathtt{F}_2$,

$$D(s_n, s_{n+1}) < \epsilon, \text{ that is, } \lim_{n\to\infty} D(s_n, s_{n+1}) = 0. \tag{7}$$

Exploiting F_3 , there exists $l \in (0,1)$ so that $(D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_n, s_{n+1})) \to 0$ as $D(s_n, s_{n+1}) \to 0$ or $\lim_{n\to\infty}(D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_n, s_{n+1})) = 0$, $n \in \mathbb{N}_0$ holds.

$$\begin{aligned}
&(D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_n, s_{n+1})) - (D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_0, s_1)) \\
&\le (D(s_n, s_{n+1}))^l (\mathfrak{f}(D(s_0, s_1)) - n\mu) - (D(s_n, s_{n+1}))^l \mathfrak{f}(D(s_0, s_1)) \\
&= -n\mu(D(s_n, s_{n+1}))^l < 0.
\end{aligned}$$

Now, as $n \to \infty, n(D(s_n, s_{n+1}))^l \to 0$ or $\lim_{n\to\infty} n^{\frac{1}{l}}(D(s_n, s_{n+1}))\mathfrak{f}(D(s_0, s_1)) = 0$, which implies that the series $\Sigma_{n=1}^{\infty} D(s_n, s_{n+1}) < \infty$. Hence, $\{s_n\}$ is an $\mathcal{R}$—preserving F—Cauchy sequence. Since, $\{s_n\} \subseteq \mathcal{M}(\mathcal{U}) \subseteq \mathcal{V}$, $\{s_n\}$ is also $\mathcal{R}$—preserving Cauchy sequence in $\mathcal{V}$. As $(\mathcal{V}, D)$ is $\mathcal{R}$—complete, there exists $s \in \mathcal{V}$ so that $s_n \to s$.

If $\mathcal{M}$ is $\mathcal{R}$—continuous, then $s_{n+1} = \mathcal{M}s_n \to \mathcal{M}s$. So $\mathcal{M}s = s$, that is, s is fixed point of $\mathcal{M}$.

Also, if $\mathcal{R}|_{\mathcal{V}}$ is D—self closed, then for $\mathcal{R}$-preserving sequence $\{s_n\}$ in $\mathcal{V}$ and $s_n \to s$, we have a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ so that $[s_{n_k}, s] \in \mathcal{R}|_{\mathcal{V}} \subseteq \mathcal{R}$, $k \in \mathbb{N}_0$ and $s_{n_k} \to s$. Now,

$$\begin{aligned}
\mathfrak{g}(D(s, \mathcal{M}s)) &\le \mathfrak{g}(D(s, \mathcal{M}s_{n_k}) + D(\mathcal{M}s_{n_k}, \mathcal{M}s)) + \alpha \\
&= \mathfrak{g}(D(s, s_{n_{k+1}}) + D(s_{n_{k+1}}, \mathcal{M}s)) + \alpha \to -\infty \quad \text{as} \quad n \to \infty, \quad \text{using} \quad F_2,
\end{aligned}$$

a contradiction. Hence, $d(s, \mathcal{M}s) = 0$, that is, $\mathcal{M}s = s$. Hence, s is a fixed point of $\mathcal{M}$.

If $F(\mathcal{M})$ is singleton then there is nothing to prove. Let, if feasible s and s^* be two different fixed points of $\mathcal{M}$, that is, $\mathcal{M}s = s$ and $\mathcal{M}s^* = s^*$. So

$$s, \ s^* \in \mathcal{M}(\mathcal{U}) \quad \text{and} \quad \mathcal{M}^n s = s, \quad \mathcal{M}^n s^* = s^*, \quad n \in \mathbb{N}_0.$$

Since, $\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s$—connected, so there exists a path $\{z_0, \ z_1, \ z_2, \ \ldots, \ z_k\}$ (say) of finite length k in $\mathcal{R}^s$ from s to s^* so that $z_0 = s, \ z_k = s^*$ and $(z_i, z_{i+1}) \in \mathcal{R}^s$, $i = 0, 1, 2, \ldots, k-1$. Since, $\mathcal{R}$ is $\mathcal{M}$-closed, $(\mathcal{M}^n z_i, \mathcal{M}^n z_{i+1}) \in \mathcal{R}^s$, $0 \le i \le k-1$ and $n \in \mathbb{N}_0$. Now, $D(s, s^*) > 0$, implies

$$\begin{aligned}
\mathfrak{f}(D(s, s^*)) = \mathfrak{f}(D(\mathcal{M}^n z_0, \mathcal{M}^n z_k)) \qquad & \\
\le \mathfrak{f}(&\mathfrak{a}D(\mathcal{M}^{n-1} z_0, \mathcal{M}^{n-1} z_k) + \mathfrak{b}(D(\mathcal{M}^{n-1} z_0, \mathcal{M}^n z_0) \\
&+ \mathfrak{c}D(\mathcal{M}^{n-1} z_k, \mathcal{M}^n z_k) \\
+ \mathfrak{d}(&D(\mathcal{M}^{n-1} z_0, \mathcal{M}^n z_k)) + \mathfrak{e}(D(\mathcal{M}^{n-1} z_k, \mathcal{M}^n z_0))) - \mu \\
\le \mathfrak{f}(&D(\mathcal{M}^{n-1} z_0, \mathcal{M}^{n-1} z_k)) - \mu, \quad \text{since} \quad (\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + 2\mathfrak{d}) < 1.
\end{aligned}$$

Following, similar steps we get

$$\mathfrak{f}(D(s,s^*)) \leq (\mathcal{M}^{n-2}z_0, \mathcal{M}^{n-2}z_k) - 2\mu.$$

In general,

$$\mathfrak{f}(D(s,s^*)) \leq \mathfrak{f}(D(z_0,z_k)) - n\mu \to -\infty \text{ as } n \to \infty,$$

a contradiction. So $D(s,s^*) = 0$, that is, $s = s^*$. Hence, $\mathcal{M}$ has a unique fixed point.

□

Example 5 *Let $\mathcal{U} = (0,5)$ and an* F*−metric be*

$$D(s,v) = \begin{cases} a^{|s-v|}, & s \neq v \\ 0, & s = v \end{cases}, \quad \text{where } a > 0$$

with $f(t) = -\frac{1}{t}$ and $\alpha = 1$. Define a binary relation on $\mathcal{U}$,

$$\mathcal{R} = \{(s,v) : s \leq v\},$$

whose symmetric closure is $\mathcal{R}^s = \mathcal{U} \times \mathcal{U}$ and a self-map

$$\mathcal{M}s = \begin{cases} \frac{s}{3} + 1, & s \in (0,3) \\ 2, & s \in [3,4) \\ \frac{s}{4} + 1, & s \in [4,5) \end{cases}.$$

Let $\mathcal{V} = \left(\frac{1}{2}, 3\right]$. Here, $\mathcal{M}(\mathcal{U}) = \left(1, \frac{9}{4}\right) \subseteq \mathcal{V}$, $\mathcal{V}$ is $\mathcal{R}-$complete but $\mathcal{U}$ is not $\mathcal{R}-$complete. Clearly, $\mathcal{R}$ is $\mathcal{M}-$closed and $\mathcal{M}$ is $\mathcal{R}-$continuous. Take any sequence $\{s_n\} \subseteq \mathcal{V}$, $s_n = \frac{3n}{2n+1}, n \in \mathbb{N}$. Here, $(s_n, s_{n+1}) \in \mathcal{R}^s|_{\mathcal{Y}}$, $n \in \mathbb{N}$ and there exists $s_n \to s \in \left(\frac{1}{2}, 3\right]$, $n \geq N \in \mathbb{N}$. So we choose a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ in such a way that $s_{n_k} \to s$, $k \in \mathbb{N}$. Therefore, $(s_{n_k}, s) \in \mathcal{R}^s|_{\mathcal{Y}}$, $k \in \mathbb{N}$, that is, $\mathcal{R}^s|_{\mathcal{V}}$ is $D-$self closed. Moreover, $\Gamma(s,v,\mathcal{R}^s)$ is non-empty. One may verify all the hypotheses of Theorem 2 for $\mu \in \left(0, \frac{15}{4}\right]$ and $s, v \in \mathcal{M}(\mathcal{U})$. Observe that $s = \frac{3}{2}$ is a unique fixed point of $\mathcal{M}$. Further, the sequence $\{s_n\}$ is F*−convergent to $\frac{3}{2}$.*

Corollary 2 *Conclusion of Theorem 2 is true if* (vi) *is substituted by any one of the subsequent hypotheses:*

(u_1) *$\mathcal{R}|_{\mathcal{M}(\mathcal{U})}$ is complete.*

(u_2) *$\mathcal{M}(\mathcal{U})$ is $\mathcal{R}^s-$directed.*

$(\mathfrak{u}_3)$ $\Gamma(s,\nu,\mathcal{R}^s) \neq \phi$.

Proof. The proof follows the pattern of Corollary 1. □

A relation theoretic variant of Kannan type F−contraction is given by putting $\mathfrak{a} = \mathfrak{d} = \mathfrak{e} = 0, \mathfrak{b} + \mathfrak{c} < 1$ and $\mathfrak{b} \neq 0$ in Theorem 2.

Corollary 3 *Theorem 2 remains correct even if* $(\nu)'$ *take the place of* (ν).

$(\nu)'$ $D(\mathcal{M}s, \mathcal{M}\nu) > 0$ *implies that*

$$\mu + \mathfrak{f}(D(\mathcal{M}s, \mathcal{M}\nu)) \leq \mathfrak{f}(\mathfrak{b}D(s, \mathcal{M}s) + \mathfrak{c}D(\nu, \mathcal{M}\nu)), \tag{8}$$

$s, \nu \in \mathcal{U}$, $(s,\nu) \in \mathcal{R}$ *and* $\mathfrak{b} + \mathfrak{c} < 1, \mathfrak{c} \leq \frac{1}{2}$.

A relation theoretic variant of Chatterjea type F−contraction is given on substituting $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = 0$ and $\mathfrak{d} = \frac{1}{2}$ in Theorem 2.

Corollary 4 *Theorem 2 remains correct even if* $(\nu)''$ *take the place of* (ν).

$(\nu)''$ $D(\mathcal{M}s, \mathcal{M}\nu) > 0$ *implies that*

$$\mu + \mathfrak{f}(D(\mathcal{M}s, \mathcal{M}\nu)) \leq \mathfrak{f}(\mathfrak{d}D(s, \mathcal{M}\nu) + \mathfrak{e}D(\nu, \mathcal{M}s)), \tag{9}$$

$s, \nu \in \mathcal{U}$, $(s,\nu) \in \mathcal{R}$, $\mathfrak{d} \leq \frac{1}{2}$ *and* $\mathfrak{d} + \mathfrak{e} < 1$.

A relation theoretic variant of Reich type F−contraction is given on substituting $\mathfrak{d} = \mathfrak{e} = 0$ in Theorem 2.

Corollary 5 *Theorem 2 remains correct even if* $(\nu)'''$ *take the place of* (ν).

$(\nu)'''$ $D(\mathcal{M}s, \mathcal{M}\nu) > 0$ *implies that*

$$\mu + \mathfrak{f}(D(\mathcal{M}s, \mathcal{M}\nu)) \leq \mathfrak{f}(\mathfrak{a}D(s,\nu) + \mathfrak{d}D(s, \mathcal{M}s) + \mathfrak{e}D(\nu, \mathcal{M}\nu)), \tag{10}$$

$s, \nu \in \mathcal{U}$, $(s,\nu) \in \mathcal{R}$ *and* $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1$.

Similarly, by taking $\mathfrak{a} = 1$, $\mathfrak{b} = \mathfrak{c} = \mathfrak{d} = \mathfrak{e} = 0$ in Theorem 2, we obtain relation theoretic variant of Theorem 1 of Wardowski [29], that is, our first result - Theorem 1.

Remark 2 (i) *If in Corollaries 3, 4, and 5 either* $\mathcal{Y}$ *is* $\mathcal{R}^s$*-directed or* $\mathcal{R}$ *is* F*-complete or* $\Gamma(s,\nu,\mathcal{R}^s) \neq \phi$, *take the place* (νi) *then also the above consequences hold.*

(ii) *If we assume,* $\mathcal{R} = \mathcal{U} \times \mathcal{U}$ *in the above results, extensions, and improvements of results in metric and ordered metric spaces for a discontinuous self-map are obtained. As a consequence, our results subsume the majority of celebrated and contemporary results existing in the literature. It is fascinating to see that relation-theoretic contractions are comparatively weaker than standard contractions since these hold only for the elements in the relation under consideration (see, Examples 4 and 5).*

(iii) *Since, an* F*−metric is more predominant than the standard metric and* F*−contraction is a genuine generalization of Banach contraction. Our results improve and extend the classical Banach contraction principle* [4], *Alam and Imdad* [1]-[2], *Alnaser et al.* [3], *Ćirić* [5], *Chatterjea* [6], *Cosentino and Vetro* [7], *Hardy and Rogers* [8], *Jleli and Samet* [9], *Kannan* [10], *Nieto and Rodŕiguez-López* [16], *Petruşel et al.* [17], *Reich* [19], *Tomar and Joshi* [28], *Wardowski* [29], *and so on without using the continuity of the underlying map and the completeness of space.*

4 Applications

As an application to our outcomes, we solve an initial value and two boundary value problems of second-order differential equations. Let $\mathcal{I} = [0, 1]$ and $\mathcal{U} = \mathbf{C}[\mathcal{I}, \mathcal{R}]$ denotes the set of all continuous functions on $[0, 1]$. Define F−metric as

$$D(s, \nu) = \begin{cases} \exp(\|s - \nu\|_\infty), & s \neq \nu, \\ 0, & s = \nu. \end{cases}, \quad s, \nu \in \mathcal{U},$$

with $\mathfrak{f}(t) = -\frac{1}{t}$, $\alpha = 1$ and norm $\|s - \nu\|_\infty = \sup |s - \nu|$.

Firstly, LCR−circuit was used in the 1890s in spark-gap radio transmitters to permit the receiver to be tuned to the transmitter. These days LCR−circuit is being used as an oscillator circuit in television sets and Radio receivers for tuning to choose a narrow frequency range from nearby radio wave (as a tuned circuit) and in the high-pass filter, low-pass filter, band-pass filter, or band-stop filter (as a second-order circuit), oscillators voltage multiplier pulse discharge circuit, and so on. Inspired by these applications, we apply Theorem 1 to solve the Dirichlet-Neumann initial value problem of the LCR−Circuit.

Theorem 3 *Consider an* LCR*−circuit (a coil of inductance* L*, a capacitor of capacitance* C *and a resistor of resistance* R*) and an* AC *voltage source connected in series. If* q *is a charge on the capacitor,* I *is the current passing* *through the circuit at time* t. *Then, using Kirchhoff's law, the voltage equation is*

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \phi(t, q(t)), \tag{11}$$

with Dirichlet-Neumann initial condition $q(0) = 0, q^{'}(0) = I_0$, *where* ϕ *is applied voltage.*
Let there exists $\mu > 0$ *so that for* $q_1, q_2 \in C([0,T], \mathbb{R})$, $q_1 \geq q_2$ *and* $\phi : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is an increasing map and*

$$|\phi(t, q_1(t)) - \phi(t, q_2(t))| \leq \tau^2(|q_1 - q_2| - \mu), \tag{12}$$

$\mathfrak{a} + \mathfrak{b} + \mathfrak{c} < 1$, $T > 0$. *If* q *satisfies all the hypotheses of Theorem 1, then the initial value problem (11) has a solution* $q^* \in C([0,T], \mathbb{R})$.

Proof. The problem is comparable to the subsequent integral equation

$$q(t) = \int_0^t G(t, \xi)\phi(\xi, q(\xi))d\xi, \quad t \in [0,T]$$

and the Green function

$$G(t, \xi) = \begin{cases} (t - \xi) \exp \tau(t - \xi), & 0 \leq \xi \leq t \leq T \\ 0, & 0 \leq t \leq \xi \leq T \end{cases}.$$

Define a map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ in $(\mathcal{U}, D)$ by

$$\mathcal{M}q(t) = \int_0^t G(t, \xi)\phi(\xi, q(\xi))d\xi. \tag{13}$$

and a binary relation

$$\mathcal{R} = \{(q_1, q_2) \in \mathcal{U} \times \mathcal{U} : q_1(t)q_2(t) \geq 0 \text{ with } (q_1 - q_2)(t) \geq 0, \forall t \in I\}.$$

Now, $q \in C([0,T], \mathbb{R})$ is a solution to the initial value problem (11) if and only if it is a fixed point of $\mathcal{M}$.

(i) Now, select an $\mathcal{R}$−preserving Cauchy sequence $\{q_n\}$ in such a way that $q_n \to q$. Then, we must have $q_n(t)q_{n+1}(t) \geq 0$ and $q_n(t) \geq q_{n+1}(t)$, $t \in I$ and $n \in \mathbb{N}_0$. There arise two cases : either $q_n(t) \geq 0$ or $q_n(t) \leq 0$, $t \in I$. If $q_n(t) \geq 0$, then, $t \in I$, gives a sequence of non-negative real numbers converging to $q(t)$. So $q(t) \geq 0$, that is, $(q_n, q) \in \mathcal{R}$, $t \in I$ and $n \in \mathbb{N}_0$. Thus $\mathcal{R}$ is D−self-closed.

(ii) For $(q_1, q_2) \in \mathcal{R}$, that is, $q_1(t) \geq q_2(t)$, since ϕ is increasing and $G(t,\xi) \geq 0$, $(t,\xi) \in I \times I$, we have

$$\begin{aligned}\mathcal{M}q_1(t) &= \int_0^1 G(t,\xi)\phi(\xi, q(\xi))d\xi \\ &\geq \int_0^1 G(t,\xi)\phi(\xi, q_2(\xi))d\xi \\ &= \mathcal{M}q_2(t),\ t \in I,\end{aligned}$$

that is, $(\mathcal{M}q_1, \mathcal{M}q_2) \in \mathcal{R}$ and $\mathcal{R}$ is $\mathcal{M}-$closed. Clearly, for $q(t) \geq 0, \mathcal{M}q(t) \geq 0, t \in I$, that is, $(q, \mathcal{M}q) \in \mathcal{R}$. Hence, $\mathcal{U}[\mathcal{M}, \mathcal{R}]$ is non-empty.

(iii) For $q_1 \neq q_2$ and $(q_1, q_2) \in \mathcal{R}$,
$D(\mathcal{M}q_1, \mathcal{M}q_2) = \exp|\mathcal{M}q_1 - \mathcal{M}q_2|$
$= \exp\left|\int_0^t G(t,\xi)\phi(\xi, q(\xi))d\xi - \int_0^t G(t,\xi)\phi(\xi, q_2(\xi))\nu(\xi)d\xi\right|$
$\leq \exp\left|\int_0^t G(t,\xi)\tau^2(|q_1 - q_2| - \mu)\right| d\xi$
$\leq \exp(\tau^2(\|q_1 - q_2\|_\infty - \mu)\left|\int_0^t G(t,\xi)d\xi\right|$
$= \exp(\tau^2(\|q_1 - q_2\|_\infty - \mu)\left|\int_0^t (t-\xi)\exp\tau(t-\xi)d\xi\right|)$
$\leq \exp(\tau^2(\|q_1 - q_2\|_\infty - \mu)\left|\frac{-t}{\tau}e^{\tau t} + \frac{1}{\tau^2}(1 - e^{\tau t})\right|$
$\leq \exp(\tau^2(\|q_1 - q_2\|_\infty - \mu)\frac{1}{\tau^2})$.
Now, taking logarithm on both sides and $\mathfrak{f} = \ln t$
$\mathfrak{f}(D(\mathcal{M}q_1, \mathcal{M}q_2)) \leq \mathfrak{f}(D(q_1, q_2)) - \mu$ and $\mu > 0$.

If $\mathcal{U} = \mathcal{V} = \mathbf{C}(I)$, then $\mathcal{V}$ is $\mathcal{R}^s-$directed. Consequently, all the hypotheses of Theorem 1 are verified and hence $\mathcal{M}$ has a unique fixed point, which is a solution to the problem 13. □

Now, we solve the two-point boundary value problem arising in the vibrations of a vertically hanging heavy cable using Corollary 5.

Theorem 4 *Let a heavy cable AB of constant mass per unit length suspended at the top end A be hanging vertically. If the cable is displaced by a small initial displacement through AB′ in the vertical plane at any time* t*. The displacement of the cable is so small that each of its particles is assumed to move horizontally, then the equation of motion of vibration of the cable is:*

$$\frac{d}{dt}\left(t\frac{ds}{dt}\right) = \psi(t, s(t)), t \in I = [0, 1] \tag{14}$$

with Dirichlet boundary conditions $s(0) = 0, s(1) = 0$. *Let there exists a self map* $\mathcal{M}$ *on* $\mathcal{U}$ *and* $\mu > 0$, $s, v \in \mathcal{U}$ *with* $s \geq v$ *satisfying*

$$|\psi(t, s(t)) - \psi(t, (t))| \leq [\mathfrak{a}\,|s - v| + \mathfrak{b}\,|u - \mathcal{M}s| + \mathfrak{c}\,|v - \mathcal{M}v| - \mu], \qquad (15)$$

$\mathfrak{a} + \mathfrak{b} + \mathfrak{c} < 1$. *If* s *satisfies all the hypotheses of Corollary 5, then problem (14) has a solution* $s^* \in \mathcal{U}$.

Proof. The boundary value problem (14) is comparable to following the integral equation

$$s(t) = \int_0^1 G(t, \xi)\psi(\xi, s(\xi))d\xi, \quad t \in I, \qquad (16)$$

where the Green function

$$G(t, \xi) = \begin{cases} \ln t, & 0 \leq \xi \leq t \leq 1 \\ \ln \xi, & 0 \leq t \leq \xi \leq 1 \end{cases}$$

Define a map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ in $(\mathcal{U}, D)$ by

$$(\mathcal{M}s)t = \int_0^1 G(t, \xi)\psi(\xi, s(\xi))d\xi$$

and a binary relation

$$\mathcal{R} = \{(s, v) \in \mathcal{U} \times \mathcal{U} : s(t)v(t) \geq 0 \text{ with } (s - v)(t) \geq 0, \forall t \in I\}.$$

Now, $s \in \mathcal{U}$ is a solution to the boundary value problem (14) if and only if it is a fixed point of $\mathcal{M}$.

(i) Now, select an $\mathcal{R}-$preserving Cauchy sequence $\{s_n\}$ such that $s_n \to u$. Then, we must have $s_n(t)s_{n+1}(t) \geq 0$ and $s_n(t) \geq s_{n+1}(t)$, $t \in I$ and $n \in \mathbb{N}_0$. There arise two cases : either $s_n \geq 0$ or $s_n \leq 0$, $t \in I$. If $s_n \geq 0$, then $t \in I$, gives a sequence of non-negative real numbers converging to $s(t)$. So $s(t) \geq 0$, that is, $(s_n, z) \in \mathcal{R}$, $n \in \mathbb{N}_0$. Thus, $\mathcal{R}$ is $D-$self-closed.

(ii) For $(s, v) \in \mathcal{R}$, that is, $s(t) \geq v(t)$ and $G(t, \xi) \geq 0$, $(t, \xi) \in I \times I$,

$$\zeta \int_0^1 G(t, \xi)\psi(\xi, s(\xi))d\xi \geq \zeta \int_0^1 G(t, \xi)\psi(\xi, v(\xi))d\xi,$$
$$\text{that is,} \quad (\mathcal{M}s)t \geq (\mathcal{M}v)t, \quad \text{for all} \quad t \in I,$$

$(\mathcal{M}s, \mathcal{M}v) \in \mathcal{R}$, that is, $\mathcal{R}$ is $\mathcal{M}-$closed. Also, for $u(t) \geq 0, t \in I$, we have $\mathcal{M}u(t) \geq 0$, $t \in I$, that is, $(s, \mathcal{M}s) \in \mathcal{R}$. So $\mathcal{U}[\mathcal{M}, \mathcal{R}]$ is non-empty.

(iii) For $(s, \nu) \in \mathcal{R}$,

$$\begin{aligned}
&D(\mathcal{M}s, \mathcal{M}\nu) = \exp|\mathcal{M}s - \mathcal{M}\nu|, \text{ for } s \neq \nu \text{ and } t \in I \\
&= \exp|\mathcal{M}\nu - \mathcal{M}s| \\
&= \exp\left|\int_0^1 G(t,\xi)\psi(\xi, s(\xi))d\xi - \int_0^1 G(t,\xi)\psi(\xi,\nu(\xi))]d\xi\right| \\
&\leq \exp\left|\int_0^1 G(t,\xi)(\psi(\xi, s(\xi)) - \psi(\xi,\nu(\xi)))d\xi\right| \\
&\leq \exp\left|\int_0^1 G(t,\xi)[\mathfrak{a}|s-\nu| + \mathfrak{b}|s - \mathcal{M}s| + \mathfrak{c}|\nu - \mathcal{M}\nu| - \mu]d\xi\right| \\
&\leq \exp[\mathfrak{a}\|s-\nu\|_\infty + \lfloor \|s - \mathcal{M}s\|_\infty + \mathfrak{c}\|\nu - \mathcal{M}\nu\|_\infty - \mu]\left|\int_0^1 G(t,\xi)d\xi\right| \\
&= \exp[\mathfrak{a}\|s-\nu\|_\infty + \lfloor \|s - \mathcal{M}s\|_\infty + \mathfrak{c}\|\nu - \mathcal{M}\nu\|_\infty - \mu]\left|\int_0^t \ln t d\xi + \int_t^1 \ln \xi d\xi\right| \\
&= \exp[\mathfrak{a}\|s-\nu\|_\infty + \lfloor \|s - \mathcal{M}s\|_\infty + \mathfrak{c}\|\nu - \mathcal{M}\nu\|_\infty - \mu]\,|t-1| \\
&\leq \exp[[\mathfrak{a}\|s-\nu\|_\infty + \lfloor \|s - \mathcal{M}s\|_\infty + \mathfrak{c}\|\nu - \mathcal{M}\nu\|_\infty - \mu].
\end{aligned}$$

Now, taking logarithm on both sides and $\mathfrak{f} = \ln t$,
$\mathfrak{f}(D(\mathcal{M}s, \mathcal{M}\nu)) \leq \mathfrak{f}[\mathfrak{a}D(s,\nu) + \mathfrak{b}D(s, \mathcal{M}s) + \mathfrak{c}D(\nu, \mathcal{M}\nu)] - \mu$, since $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} < 1$.
If $\mathcal{U} = \mathcal{V} = \mathbf{C}(I)$, then $\mathcal{V}$ is $\mathcal{R}^s$−directed. Therefore, all the hypotheses of Corollary 5 are verified and consequently, $\mathcal{M}$ has a unique fixed point, which is a solution to (16). □

Theorem 5 *Consider a boundary value problem*

$$\frac{d^2u}{dt^2} = \phi(t, u(t)), t \in I \text{ and } \phi \in \mathcal{U}, \tag{17}$$

$s(0) = 0, s(1) = 0.$
If there exists $\mu > 0$ *in such a way that for* $s, \nu \in \mathbf{C}(I)$, $s \geq \nu$.

$$0 \leq [\phi(t,s) + \mu s] - [\phi(t,\nu) + \mu\nu] \leq 8|u - \nu| - \mu \tag{18}$$

and s *satisfies all the hypotheses of Theorem 1, then the boundary value problem (17) has a unique solution* $s^* \in \mathcal{U}$.

Proof.
The problem in equation (17) may be rewritten as

$$\begin{aligned}
&s''(t) + \beta s(t) = \phi(t, s(t)) + \beta s(t), \ t \in [0,1], \\
&s(0) = 0, s(1) = 0.
\end{aligned}$$

This is comparable to the integral equation

$$s(t) = \int_0^1 G(t,\xi)[\phi(\xi, s(\xi)) + \beta s(\xi)]d\xi, \text{ for } t \in I, \tag{19}$$

where the Green function

$$G(t,\xi) = \begin{cases} (1-t)\xi, & 0 \leq \xi \leq t \leq 1 \\ (1-\xi)t, & 0 \leq t \leq \xi \leq 1 \end{cases}$$

Define $\mathcal{M}:\mathcal{U} \to \mathcal{U}$ in an F−metric space $(\mathcal{U}, D)$ by

$$(\mathcal{M}s)t = \int_0^1 G(t,\xi)[\phi(\xi, s(\xi)) + \beta s(\xi)]d\xi$$

and a binary relation

$$\mathcal{R} = \{(s,\nu) \in \mathcal{U} \times \mathcal{U} : s(t)\nu(t) \geq 0 \text{ with } (s-\nu)(t) \geq 0, \forall t \in I\}.$$

Now, $s \in \mathcal{U}$ is a solution of (19) if and only if it is the solution of (17).

(i) Now, select an $\mathcal{R}$−preserving Cauchy sequence $\{s_n\}$ in such a way that $s_n \to u$. Then, we must have $s_n(t)s_{n+1}(t) \geq 0$ and $s_n(t) \geq s_{n+1}(t)$, $t \in I$ and $n \in \mathbb{N}_0$. There arise two cases : either $s_n(t) \geq 0$ or $s_n(t) \leq 0$, $t \in I$. If $s_n(t) \geq 0$, then, for $t \in I$, produces a sequence of non-negative real numbers converging to $s(t)$. Thus $s(t) \geq 0$, $t \in I$, that is, $(s_n, s) \in \mathcal{R}$, $t \in I$ and $n \in \mathbb{N}_0$. Hence, $\mathcal{R}$ is D−self-closed.

(ii) For $(s,\nu) \in \mathcal{R}$, that is, $s(t) \geq \nu(t)$ by (18), $\phi(t,\nu)+\beta\nu \leq \phi(t,s)+\beta s, \forall t \in I$ and $G(t,\xi) \geq 0$, $(t,\xi) \in I \times I$, we have

$$\begin{aligned} \mathcal{M}s(t) &= \int_0^1 G(t,\xi)[\phi(\xi, s(\xi)) + \beta s(\xi)]d\xi \\ &\geq \int_0^1 G(t,\xi)[\phi(\xi, \nu(\xi)) + \beta \nu(\xi)]d\xi \\ &= \mathcal{M}\nu(t), \quad t \in I, \end{aligned}$$

that is, $(\mathcal{M}s, \mathcal{M}\nu) \in \mathcal{R}$ and $\mathcal{R}$ is $\mathcal{M}$−closed. Clearly, for $s(t) \geq 0, \mathcal{M}s(t) \geq 0$, $t \in I$, that is, $(s, \mathcal{M}s) \in \mathcal{R}$ and $\mathcal{U}[\mathcal{M}, \mathcal{R}]$ is non-empty.

(iii) For $(u,\nu) \in \mathcal{R}$,

$$\begin{aligned} &D(\mathcal{M}s, \mathcal{M}\nu) = \exp|\mathcal{M}s - \mathcal{M}\nu|, \text{ for } s \neq \nu \text{ and } t \in I \\ &= \exp\left|\int_0^1 G(t,\xi)[\phi(\xi,u(\xi)) + \mu s(\xi)]d\xi - \int_0^1 G(t,\xi)[\phi(\xi,\nu(\xi)) + \mu\nu(\xi)]d\xi\right| \\ &\leq \exp\left|\int_0^1 G(t,\xi)[8\,|s-\nu| - \mu]\right| d\xi \\ &\leq \exp(8\,\|s-\nu\|_\infty - \mu)\int_0^1 G(t,\xi)d\xi \\ &= \exp(8\,\|s-\nu\|_\infty - \mu)[\int_0^t (1-t)\xi d\xi + \int_t^1 (1-\xi)t d\xi] \\ &\leq \exp(8\,\|s-\nu\|_\infty - \mu)\frac{t(1-t)}{2} \end{aligned}$$

$\leq \exp \frac{1}{8}(8 \|s - \nu\|_\infty - \mu)$.
Now, taking logarithm on both sides and $\mathfrak{f} = \ln t$
$\mathfrak{f}(D(\mathcal{M}s, \mathcal{M}\nu)) \leq \mathfrak{f}(D(s, \nu)) - \frac{\mu}{8}$ and $\mu > 0$.

Taking $\mathcal{U} = \mathcal{V} = \mathbf{C}(I)$, then $\mathcal{V}$ is $\mathcal{R}^s$−directed. Hence, all the hypotheses of Theorem 1 are verified and as a result, $\mathcal{M}$ has a unique fixed point, which is, a solution to problem (17).

□

5 Conclusion

We have created an environment for the existence and uniqueness of a fixed point for relation theoretic variants of F−contraction and Hardy-Rogers type F−contraction maps. Our theorems and corollaries are sharpened versions of the well-known results, wherein completeness and continuity are replaced by their $\mathcal{R}$−analogs, which are comparatively weaker notions. Application to real-world problems substantiates the utility of these extensions.

References

[1] A. Alam, M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.*, **17**(4), (2015), 693–702.

[2] A. Alam, M. Imdad, Relation-theoretic metrical coincidence theorems, *Filomat*, (2017), 4421–4439.

[3] L. A. Alnaser, D. Lateef, H. A. Fouad, J. Ahmad, Relation theoretic contraction results in F−metric spaces, *J. Nonlinear Sci. Appl.*, **12**, (2019), 337–344.

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équation intégrales, *Fund. Math.*, **3**, (1922), 133–181.

[5] Lj. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**, (1974), 267–273.

[6] S. K. Chatterjea, Fixed-point theorems, *Comptes Rendus de L'Academie Bulgare des Sciences*, **25**(6), (1972), 727–730.

[7] M. Cosentino, P. Vetro, Fixed point results for $\mathcal{F}$−contractive mappings of Hardy-Rogers-type, *Filomat*, **28**(4), (2014), 715–722.

[8] G. E. Hardy, T. D. Rogers, A generalization of a fixed point theorem of Reich, *Cand. Math. Bull.*, **16**(2), (1973), 201–206.

[9] M. Jleli, B. Samet, On a new generalization of metric spaces, *J. Fixed Point Theory Appl.*, **20**(3), (2018), 1–20.

[10] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60**, (1968), 71–76.

[11] W. Kirk, N. Shahzad, *Fixed point theory in distance spaces*, Springer, Cham, 2014.

[12] B. Kolman, R. C. Busby, S. Ross, *Discrete Mathematical Structures*, Third Edition, PHI Pvt. Ltd., New Delhi, (2000).

[13] M. A. Kutbi, A. Roldán, W. Sintunavarat, J. Martinez-Moreno, C. Roldán, F−closed sets and coupled fixed point theorems without the mixed monotone property, *Fixed Point Theory Appl.*, **2013**(1), (2013), 1–11.

[14] S. Lipschutz, *Schaum's outlines of theory and problems of set theory and related topics*, McGraw-Hill, New York, (1964).

[15] R. D. Maddux, *Relation algebras, studies in Logic and the Foundations of Mathematics*, Elsevier B. V., Amsterdam, **150**, (2006).

[16] J. J. Nieto, R. Rodŕiguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**(3), (2005), 223–239.

[17] A. Petruşel, G. Petruşel, Yi-Bin Xiao, J.-C. Yao, Fixed point theorems for generalized contractions with applications to coupled fixed point theory, *J. Nonlinear Convex Anal.*, **19**(1), (2018), 71–88.

[18] A. C. M. Ran, M. C. Reurings, A fixed point theorem in partially ordered set and some applications to matrix equation, *Proc. Amer. Math. Soc.*, **132**, (2004), 1435–1443.

[19] S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14**(1), (1971), 121–124.

[20] A. F. Roldán, Lòpez de Hierro A unified version of Ran and Reurings theorem and Nieto and Rodríguez-Lópezs theorem and low-dimensional generalizations, *Appl. Math. Inf. Sci.*, **10**(2), (2016), 383–393.

[21] B. Samet, M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, *Commun. Math. Anal.*, **13**(2), (2012), 82–97.

[22] M. Turinici, Ran-Reurings fixed point results in ordered metric spces, *Libertas Math.*, **31**, (2011), 49–55.

[23] A. Tomar, C. Giniswamy, Jeyanthi, and P. G. Maheshwari, On coincidence and common fixed point of six maps satisfying $\mathcal{F}$-contractions, *TWMS J. App. Eng. Math.*, **6**(2), (2016), 224–231.

[24] A. Tomar, C. Giniswamy, Jeyanthi, and P. G. Maheshwari, Coincidence and common fixed point of $\mathcal{F}$-contractions via $\mathsf{CLR}_{\mathsf{ST}}$ property, *Surv. Math. Appl.*, **11**, (2016), 21–31.

[25] A. Tomar and R. Sharma, Almost alpha-Hardy-Rogers-F-contractions and applications , *Armen. J. Math.*, **11**(11), (2019), 1–19.

[26] A. Tomar, M. Joshi, S. K. Padaliya, B. Joshi, A. Diwedi, Fixed point under set-valued relation-theoretic nonlinear contractions and application. *Filomat* **33**(14), (2019), 4655–4664.

[27] A. Tomar and M. C. Joshi, "*Fixed Point Theory & its Applications to Real World Problem*", Nova Science Publishers, New York, USA, 2021. ISBN978-1-53619-336-7

[28] A. Tomar, M. Joshi, Relation-theoretic nonlinear contractions in an $\mathsf{F}-$metric space and applications, *Rend. Circ. Mat. Palermo*, **70**(2), (2021), 835–852.

[29] D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.* **94**, (2012).

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