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On Fefferman's inequality. A simple proof

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Abstract. In this short note we shall give a simple proof of the so called Fefferman's inequality allowing the potential V belong to L_p with $1 < p < \infty$.

1 Introduction

In his celebrated paper Charles Fefferman [6] prove the inequality

$$\int_B |u(x)|^p |V(x)|\, dx \leq C \int_B |\nabla u(x)|^p\, dx \tag{1}$$

for all $u \in C_c^\infty$, in case $p = 2$, assuming the potential V belong to the class $L^{r,n-2r}$, with $1 < r \leq \frac{n}{2}$.

In latter work, Chiarenza and Frasca [3] extended Fefferman's result with a different proof, assuming the potential V in $L^{r,n-pr}$ with $1 < r \leq \frac{n}{p}$ and $1 < p < n$.

In [4] Danielli, Garofallo and Nhice introduced a suitable version of Morrey Spaces adapted to the Carnot-Carahéodory (C-C) metric and proved the same inequality with V in the Morrey Space $L^{1,\lambda}$ for $\lambda > 0$.

A different approach to inequality (1) was started by Schecter in [7] where he proved the inequality with $p = 2$ and V in the Stummed-Kato Class.

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At the beginning of the 21st century in [8] inequality (1) was proved with $1 < p < n$ and V in a more general class of potentials, namely non-linear Kato class for details in this class see [2]. In [5] inequality (1) was proved by replacing the gradient in the right hand side of (1) by energy associated to an arbitrary system of vector fields, and the function V was take in an appropriate Stummed-Kato class, defined via the Carnot-Carathéodory metric associated to the vector fields in a metric space.

In [1] inequality (1) was proved allowing $V \in A_1 \cap L_{\frac{n}{p}} \cap C_c^2$ with $1 < p < \frac{n}{p}$.

In section 2 of this note we shall prove (1) allowing $V \in L_p$ with $1 < p < \infty$.

2 Main result

After Fefferman gave the proof of (1) for $p = 2$, all subsequent authors who have proved (1) have used the following Lemma, which is the cornerstone in the proof of the aforementioned inequality (1) in that sense (1) deserve to have a name and so we will call it the workhorse Lemma. In order to make this note self contained we will give its proof as well.

Lemma 1 (The workhorse Lemma) *Let $u \in C^1(\mathbb{R}^n)$ suppose that u and its partial derivatives of first order are integrable on $\mathbb{R}^n$. Then*

$$|u(x)| \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy$$

for $x \in \mathbb{R}^n$ where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Proof. Observe first that

$$\frac{(x-y)\cdot \nabla u(y)}{|x-y|^n}$$

is integrable on $\mathbb{R}^n$ as function of y; actually for $r > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|(x-y)\cdot\nabla u(y)|}{|x-y|^n}\, dy &\leq \int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy + \int_{\mathbb{R}^n \setminus B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy \\ &\leq \sup_{y \in B_r(x)} |\nabla u(y)| \int_{B_r(x)} \frac{dy}{|x-y|^{n-1}} \\ &\quad + \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} |\nabla u(y)|\, dy < \infty. \end{aligned}$$

Next, since $u \in C_c^1(\mathbb{R}^n)$ we also have

$$u(x) = -\int_0^\infty \frac{\partial}{\partial r} u(x + rz)\, dr \tag{2}$$

where $z \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Integrating (2) over the whole unit sphere surface S^{n-1} yields

$$\begin{aligned}
\omega_{n-1} u(x) &= \int\limits_{S^{n-1}} u(x)\, d\sigma(z) \\
&= -\int\limits_{S^{n-1}} \int_0^\infty \frac{\partial}{\partial r} u(x + rz)\, dr d\sigma(z) \\
&= -\int\limits_{S^{n-1}} \int_0^\infty \nabla u(x + rz) \cdot z\, dr d\sigma(z) \\
&= -\int_0^\infty \int\limits_{S^{n-1}} \nabla u(x + rz) \cdot z\, dr d\sigma(z).
\end{aligned}$$

Changing variables $y = x + rz$, $d\sigma(z) = r^{n-1}\, d\sigma(y)$ and

$$z = \frac{y - x}{|x - y|} \quad \text{and} \quad r = |x - y|,$$

hence we get

$$\begin{aligned}
\omega_{n-1} u(x) &= -\int_0^\infty \int\limits_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{|x - y|^n}\, d\sigma(y) dr \\
&= \int\limits_{\mathbb{R}^n} \nabla u(y) \cdot \frac{x - y}{|x - y|^n}\, dy,
\end{aligned}$$

which implies that

$$|u(x)| \le \frac{1}{n\omega_n} \int\limits_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x - y|^{n-1}}\, dy,$$

as we wish to prove. □

Theorem 1 (Fefferman's inequality) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded set and* $V \in L_p(\Omega)$ *for* $1 \le p < \infty$*. Then*

$$\int\limits_{\Omega} |u(x)|^p |V(x)|\, dx \le C(n, p, q) \|V\|_{L_p(\Omega)} \int\limits_{\Omega} |\nabla u(x)|^p\, dx.$$

Proof. For any $u \in C_c^\infty(\mathbb{R}^n)$, let us consider a ball B such that $u \in C_c^\infty(B)$. By Lemma (2) and Hölder's inequality we have

$$\begin{aligned} |u(x)| \leq & C_n \left(\int_B |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_B \frac{dy}{|x-y|^{q(n-1)}} \right)^{\frac{1}{q}} \\ = & C_n C_q \left(\int_B |\nabla (y)|^p \, dy \right)^{\frac{1}{p}}. \end{aligned}$$

Thus

$$|u(x)|^p \leq (C_n C_q)^p \int_B |\nabla u(y)|^p \, dy. \tag{3}$$

Next, multiplying by $|V(x)|$ at both side of (3) and integrating with respect to x and invoking one more time the Hölder inequality we obtain

$$\begin{aligned} \int_B |u(x)|^p |V(x)| \, dx \leq & (C_n C_q)^p \int_B |V(x)| \left(\int_\Omega |\nabla u(y)|^p \, dy \right) dx \\ \leq & (C_n C_q)^p [m(B)]^{\frac{1}{q}} \left(\int_\Omega |V(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |\nabla u(y)|^p \, dy \right) \\ = & C(n,p,q) \|V\|_{L_p(\Omega)} \int_\Omega |\nabla u(y)|^p \, dy. \end{aligned}$$

Finally

$$\begin{aligned} \int_\Omega |u(x)|^p |V(x)| \, dx = & \int_B |u(x)|^p |V(x)| \, dx \\ \leq & C(n,p,q) \|V\|_{L_p(\Omega)} \int_\Omega |\nabla u(x)|^p \, dx, \end{aligned}$$

as we announced. □

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