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Subclass of analytic functions with negative coefficients related with Miller-Ross-type Poisson distribution series

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Abstract. The purpose of the present paper is to find a necessary and sufficient condition for Miller-Ross-type Poisson distribution series to be in the class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$ of analytic functions with negative coefficients .Also, we investigate several inclusion properties of the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}^\tau(A,B)$ associated of the operator $\mathbb{I}^m_{\nu,c}$ defined by this distribution. Further, we consider an integral operator related to Miller-Ross-type Poisson distribution series. Several corollaries and consequences of the main results are also considered.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}: |z|<1\}$ given by the series expansion

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which are univalent in $\mathbb{U}$. Also, let $\mathcal{S}^*$ and $\mathcal{K}$ be the usual subclasses of functions whose members are univalent starlike and univalent convex in $\mathbb{U}$, respectively.

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Further, let $\mathcal{T}_\delta$ be a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n e^{i\delta} \geq 0, \ \ |\delta| < \pi/2, z \in \mathbb{U}. \tag{2}$$

Very recently, Frasin et al. [18] obtained necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in the class $\mathcal{W}_\delta(\alpha, \gamma, \beta)$ defined as follows:

Definition 1 *For $\gamma, \beta \geq 0$, $0 \leq \alpha < \cos\delta, |\delta| < \pi/2$ and function $f \in \mathcal{T}_\delta$ is said to be in the class $\mathcal{W}_\delta(\alpha, \gamma, \beta)$ if it satisfies the analytic criteria*

$$\Re\{e^{i\delta}[(1-\gamma+2\beta)\frac{f(z)}{z} + (\gamma - 2\beta)f'(z) + \beta z f''(z)]\} > \alpha, \quad (z \in \mathbb{U}). \tag{3}$$

Remark 1 *The class $\mathcal{W}_0(\alpha, \gamma, \beta)$ is a subclass of the class $\mathcal{W}_\beta(\alpha, \gamma)$ which is defined by Ali et al. [3] (see also [33]). In particular, the class $\mathcal{W}_0(\alpha, \gamma, 0) = \mathcal{Q}_\gamma(\alpha)$ was studied by Ding et al. [11], the classes $\mathcal{W}_\delta(\alpha, 1, 0) = \mathcal{S}(\delta, \alpha)$ and $\mathcal{W}_\delta(\alpha, 0, 0) = \mathcal{T}(\delta, \alpha)$ were introduced and studied by Sudharasan et al. [36].*

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A, B)$, $\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [12].

The distributions of random variables have generated a great deal of interest in recent years. Their probability density functions, in a real variable x and a complex variable z, have played an important role in statistics and probability theory. For this reason, distributions have been studied extensively. Many kinds of distributions appeared from real life situations like Binomial distribution, Poisson distribution, geometric distribution, hyper geometric distribution and negative binomial distribution.

Let $\mathbb{E}_{\nu,c}(z)$ be the Miller-Ross function [25] defined by

$$\mathbb{E}_{\nu,c}(z) = z^\nu \sum_{n=0}^{\infty} \frac{(cz)^n}{\Gamma(n+\nu+1)}, \quad (\nu, c, z \in \mathbb{C}\ ). \tag{4}$$

Also, let $E_{\sigma,\mu}(z)$ be the two parameters Mittag-Leffler function [42] defined by

$$E_{\sigma,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n + \mu)}, \quad (z, \sigma, \mu \in \mathbb{C}, \mathrm{Re}(\sigma) > 0, \ \mathrm{Re}(\mu) > 0 \ ). \tag{5}$$

If $\mu = 1$, from (5) we obtain the one parameter Mittag-Leffler function [26]

$$E_{\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n + 1)}, \quad (z, \sigma \in \mathbb{C}, \mathrm{Re}(\sigma) > 0). \tag{6}$$

Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found in [6, 8, 13, 20, 23].

From (4) and (5), the Miller-Ross function may be written as

$$\mathbb{E}_{\nu,c}(z) = z^{\nu} E_{1,1+\nu}(cz).$$

Very recently, Şeker et al. [38] introduced a power series whose coefficients are Miller-Ross-type Poisson distribution as follows

$$\mathbb{F}^{m}_{\nu,c}(z) := z + \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)} z^n, \ z \in \mathbb{U}, \tag{7}$$

where $\nu > -1$, $c > 0$.

We note that if we put $\nu = 0$ and $c = 1$ in (7), we get the Poisson distribution series introduced by Porwal [30].

Also, Şeker et al. [38] defined the series

$$\mathbb{K}^{m}_{\nu,c}(z) := 2z - \mathbb{F}^{m}_{\nu,c}(z) = z - \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)} z^n, \ z \in \mathbb{U}. \tag{8}$$

Very recently, Amourah et al. [4] considered the linear operator $\mathbb{I}^{m}_{\nu,c} : \mathcal{A} \to \mathcal{A}$ defined by the convolution or Hadamard product

$$\mathbb{I}^{m}_{\nu,c} f(z) := \mathbb{F}^{m}_{\nu,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)} a_n z^n, \ z \in \mathbb{U}, \tag{9}$$

where $\nu > -1$ and $c > 0$.

In recent years, several researchers have obtained some necessary and sufficient conditions for functions belong to certain classes of univalent function using distribution series such as Poisson distribution series [7, 16, 15, 29, 30, 32], Pascal distribution series [5, 10, 17, 19, 34], the confluent hypergeometric distribution series [9, 22, 24, 35, 37, 41], and the Mittag-Leffler-type Poisson distribution series [1, 21]. Motivated with the works mentioned, in the present paper we determine a necessary and sufficient condition for $\mathbb{K}^m_{\nu,c}$ to be in our class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$. Also, we investigate several inclusion properties of the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}^\tau(A,B)$ associated of the operator $\mathbb{I}^m_{\nu,c}$ defined by (9). Finally, we give sufficient conditions for the function f such that its image by the integral operator $\mathbb{G}^m_{\nu,c}f(z)=\int_0^z \frac{\mathbb{K}^m_{\nu,c}(t)}{t}dt$ belongs to the class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$.

2 Preliminary lemmas

Employing the same technique proved by Sekine [39] (see also, [14]), we can prove the following lemma.

Lemma 1 *A function $f\in\mathcal{T}_\delta$ of the form (2) is in the class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(n-1)\beta+(\gamma-2\beta)n+(1-\gamma+2\beta)]\,|a_n|\le\cos\delta-\alpha. \qquad (10)$$

for some $\gamma,\beta\ge 0$ and $0\le\alpha<\cos\delta, |\delta|<\pi/2$. The result (10) is sharp.
Furthermore, we also need the following result.

Lemma 2 *[12] If $f\in\mathcal{R}^\tau(A,B)$ is of the form (1) , then*

$$|a_n|\le(A-B)\frac{|\tau|}{n}, \qquad n\in\mathbb{N}-\{1\}.$$

The result is sharp for the function

$$f(z)=\int_0^z(1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}})dt, \qquad (z\in\mathbb{U};n\in\mathbb{N}-\{1\}).$$

Unless otherwise mentioned, we assume in the reminder of this paper that $\gamma,\beta\ge 0$ *and* $0\le\alpha<\cos\delta, |\delta|<\pi/2$.

3 Necessary and sufficient condition for $\mathbb{K}^m_{\nu,c}\in\mathcal{W}_\delta(\alpha,\gamma,\beta)$

Firstly, we obtain a necessary and sufficient condition for $\mathbb{K}^m_{\nu,c}$ to be in the class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$.

Theorem 1 *$\nu > -1$ and $c > 0$, then $\mathbb{K}^{m}_{\nu,c} \in \mathcal{W}_{\delta}(\alpha,\gamma,\beta)$ if and only if*

$$\frac{c}{\mathbb{E}_{\nu,c}(m)}\Big[\beta m^2\mathbb{E}_{\nu-1,c}(m) + (\gamma-2\beta\nu)m\mathbb{E}_{\nu,c}(m) \\ + (1+\nu(\beta\nu+\beta-\gamma))\mathbb{E}_{\nu+1,c}(m)\Big] \leq \cos\delta - \alpha. \tag{11}$$

Proof. Since $\mathbb{K}^{m}_{\nu,c}$ is defined by (8), in view of Lemma 1 it is sufficient to show that

$$\Psi = \sum_{n=2}^{\infty}[n(n-1)\beta + (\gamma-2\beta)n + (1-\gamma+2\beta)]\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\frac{1}{\mathbb{E}_{\nu,c}(m)} \leq \cos\delta - \alpha, \tag{12}$$

or, equivalently

$$\Psi = \sum_{n=2}^{\infty}[\beta n^2 + n(\gamma-3\beta) + (1-\gamma+2\beta)]\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\frac{1}{\mathbb{E}_{\nu,c}(m)} \leq \cos\delta - \alpha. \tag{13}$$

Writing

$$n^2 = (\nu+n-1)(\nu+n-2) + (3-2\nu)(\nu+n-1) + (1-\nu)^2$$

and

$$n = (\nu+n-1) + (1-\nu)$$

in (13), we have

$$\Psi = \frac{1}{\mathbb{E}_{\nu,c}(m)}\Bigg[\beta\sum_{n=2}^{\infty}(\nu+n-1)(\nu+n-2)\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} \\ + (\gamma-2\beta\nu)\sum_{n=2}^{\infty}(\nu+n-1)\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} \\ + (1+\nu(\beta\nu+\beta-\gamma))\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\Bigg] \\ = \frac{1}{\mathbb{E}_{\nu,c}(m)}\Bigg[\beta\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu-2)} + (\gamma-2\beta\nu)\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu-1)} \\ + (1+\nu(\beta\nu-\beta-\gamma))\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\Bigg]$$

$$= \frac{c}{\mathbb{E}_{\nu,c}(m)} \left[\beta m \sum_{n=0}^{\infty} \frac{m^{\nu}(cm)^n}{\Gamma(n+\nu)} + (\gamma - 2\beta\nu) m \sum_{n=0}^{\infty} \frac{m^{\nu}(cm)^n}{\Gamma(n+\nu+1)} \right.$$
$$\left. + (1+\nu(\beta\nu+\beta-\gamma)) m \sum_{n=0}^{\infty} \frac{m^{\nu}(cm)^n}{\Gamma(n+\nu+2)} \right]$$
$$= \frac{c}{\mathbb{E}_{\nu,c}(m)} \left[\beta m^2 \mathbb{E}_{\nu-1,c}(m) + (\gamma - 2\beta\nu) m \mathbb{E}_{\nu,c}(m) \right.$$
$$\left. + (1+\nu(\beta\nu+\beta-\gamma)) \mathbb{E}_{\nu+1,c}(m) \right],$$

but this last expression is upper bounded by $\cos\delta - \alpha$ if and only if (11) holds. □

4 Inclusion relations

In this section we will prove the inclusion relations of the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}^{\tau}(A, B)$ associated of the operator $\mathbb{I}^m_{\nu,c}$ defined by (9).

Theorem 2 *Let* $\nu > -1$ *and* $c > 0$.

(i) If $f \in \mathcal{S}^*$ *and the inequality*

$$\frac{c}{\mathbb{E}_{\nu,c}(m)} \left[\beta m^3 \mathbb{E}_{\nu-2,c}(m) + (\gamma + 3\beta(1-\nu)) m^2 \mathbb{E}_{\nu-1,c}(m) \right.$$
$$\left. + ((\nu-1)(3\beta\nu - 2\gamma)+1) m \mathbb{E}_{\nu,c}(m) + (1-\nu)(\nu(\beta+\beta\nu-\gamma)+1) \mathbb{E}_{\nu+1,c}(m) \right]$$
$$\leq \cos\delta - \alpha. \tag{14}$$

is satisfied then $\mathbb{I}^m_{\nu,c} f \in \mathcal{W}_{\delta}(\alpha, \gamma, \beta)$.

(ii) If $f \in \mathcal{K}$ *and the inequality* (11) *is satisfied then* $\mathbb{I}^m_{\nu,c} f \in W_{\delta}(\alpha, \gamma, \beta)$.

Proof. (i) According to Lemma 1 it is sufficient to show that

$$\Phi = \sum_{n=2}^{\infty} [n(n-1)\beta + (\gamma - 2\beta)n + (1-\gamma+2\beta)] \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)} |a_n| \leq \cos\delta - \alpha. \tag{15}$$

If $f \in \mathcal{S}^*$ has the form (1), then the well-known inequality $|a_n| \leq n$ holds for all $n \geq 2$. Therefore

$$\Phi \leq \sum_{n=2}^{\infty} n[n(n-1)\beta + (\gamma - 2\beta)n + (1-\gamma+2\beta)] \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} \frac{1}{\mathbb{E}_{\nu,c}(m)}$$

$$= \sum_{n=2}^{\infty}[\beta n^3 + n^2(\gamma - 3\beta) + n(1-\gamma+2\beta)]\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\frac{1}{\mathbb{E}_{\nu,c}(m)}. \tag{16}$$

Writing

$$n^3 = (\nu+n-1)(\nu+n-2)(\nu+n-3) + (6-3\nu)(\nu+n-1)(\nu+n-2)$$
$$(3\nu^2 - 9\nu + 7)(\nu+n-1) + (1-\nu)^3,$$

$$n^2 = (\nu+n-1)(\nu+n-2) + (3-2\nu)(\nu+n-1) + (1-\nu)^2$$

and

$$n = (\nu+n-1) + (1-\nu)$$

in (16), we get

$$\Phi \le \frac{1}{\mathbb{E}_{\nu,c}(m)}\left[\beta\sum_{n=2}^{\infty}(\nu+n-1)(\nu+n-2)(\nu+n-3)\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\right.$$
$$+ (\gamma + 3\beta\,(1-\nu))\sum_{n=2}^{\infty}(\nu+n-1)(\nu+n-2)\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}$$
$$+ ((\nu-1)\,(3\beta\nu - 2\gamma) + 1)\sum_{n=2}^{\infty}(\nu+n-1)\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}$$
$$\left. +(1-\nu)\,(\nu\,(\beta+\beta\nu-\gamma)+1))\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\right]$$
$$= \frac{1}{\mathbb{E}_{\nu,c}(m)}\left[\beta\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu-3)} + (\gamma+3\beta\,(1-\nu))\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu-2)}\right.$$
$$+ ((\nu-1)\,(3\beta\nu-2\gamma)+1)\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu-1)}$$
$$\left. +(1-\nu)\,(\nu\,(\beta+\beta\nu-\gamma)+1))\sum_{n=2}^{\infty}\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\right]$$
$$= \frac{c}{\mathbb{E}_{\nu,c}(m)}\left[\beta m\sum_{n=0}^{\infty}\frac{m^{\nu}(cm)^{n}}{\Gamma(n+\nu-1)} + (\gamma+3\beta\,(1-\nu))\,m\sum_{n=0}^{\infty}\frac{m^{\nu}(cm)^{n}}{\Gamma(n+\nu)}\right.$$
$$+ ((\nu-1)\,(3\beta\nu-2\gamma)+1)\,m\sum_{n=0}^{\infty}\frac{m^{\nu}(cm)^{n}}{\Gamma(n+\nu+1)}$$

$$+(1-\nu)\left(\nu\left(\beta+\beta\nu-\gamma\right)+1\right)m\sum_{n=0}^{\infty}\frac{m^{\nu}(cm)^{n}}{\Gamma(n+\nu+2)}\Bigg]$$

$$=\frac{c}{\mathbb{E}_{\nu,c}(m)}\left[\beta m^{3}\mathbb{E}_{\nu-2,c}(m)+\left(\gamma+3\beta\left(1-\nu\right)\right)m^{2}\mathbb{E}_{\nu-1,c}(m)\right.$$

$$\left.+\left((\nu-1)\left(3\beta\nu-2\gamma\right)+1\right)m\mathbb{E}_{\nu,c}(m)+(1-\nu)\left(\nu\left(\beta+\beta\nu-\gamma\right)+1\right)\mathbb{E}_{\nu+1,c}(m)\right],$$

but this last expression is upper bounded by $\cos\delta-\alpha$ if and only if (14) holds.

(ii) If $f\in\mathcal{K}$ has the form (1), then the well-known inequality $|a_n|\leq 1$ holds for all $n\geq 2$. Therefore, it is sufficient to show that

$$\sum_{n=2}^{\infty}n[n(n-1)\beta+(\gamma-2\beta)n+(1-\gamma+2\beta)]\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)}\frac{1}{\mathbb{E}_{\nu,c}(m)}\leq\cos\delta-\alpha.$$

By a similar proof like those of Theorem 1, we get that $\mathbb{I}^{m}_{\nu,c}f\in\mathcal{W}_{\delta}(\alpha,\gamma,\beta)$ if (11) holds. □

Making use of Lemma 2, we prove the following result.

Theorem 3 *Let $\nu>-1$ and $c>0$. If $f\in\mathcal{R}^{\tau}(A,B)$ and the inequality*

$$\frac{(A-B)c\,|\tau|}{\mathbb{E}_{\nu,c}(m)}\left[\beta m^{3}\mathbb{E}_{\nu-2,c}(m)+(1-\beta\nu)\,\mathbb{E}_{\nu+1,c}(m)\right]\leq\cos\delta-\alpha. \tag{17}$$

is satisfied then $\mathbb{I}^{m}_{\nu,c}f\in\mathcal{W}_{\delta}(\alpha,\gamma,\beta)$.

Proof. According to Lemma 1 it is sufficient to show that

$$\sum_{n=2}^{\infty}[n(n-1)\beta+(\gamma-2\beta)n+(1-\gamma+2\beta)]\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)}\,|a_n|\leq\cos\delta-\alpha.$$

Since $f\in\mathcal{R}^{\tau}(A,B)$, using Lemma 2 we have

$$|a_n|\leq\frac{(A-B)\,|\tau|}{n},\ n\in\mathbb{N}\setminus\{1\},$$

therefore

$$\sum_{n=2}^{\infty}([\beta n(n-1)+(\gamma-2\beta)n+(1-\gamma+2\beta)]\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)}\,|a_n|$$

$$\leq(A-B)\,|\tau|\left[\sum_{n=2}^{\infty}\left[\beta(n-1)+(\gamma-2\beta)+\frac{1}{n}(1-\gamma+2\beta)\right]\frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)}\right]$$

$$= \frac{(A-B)|\tau|}{\mathbb{E}_{\nu,c}(m)} \left[\beta \sum_{n=2}^{\infty} (\nu+n-1) \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} + (\gamma - \beta(\nu+2)) \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} \right.$$
$$\left. +(1-\gamma+2\beta) \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{n\Gamma(n+\nu)} \right]$$
$$\leq \frac{(A-B)|\tau|}{\mathbb{E}_{\nu,c}(m)} \left[\beta \sum_{n=2}^{\infty} (\nu+n-1) \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} + (\gamma - \beta(\nu+2)) \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} \right.$$
$$\left. +(1-\gamma+2\beta) \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)} \right]$$
$$= \frac{(A-B)c|\tau|}{\mathbb{E}_{\nu,c}(m)} \left[\beta m \sum_{n=0}^{\infty} \frac{m^{\nu}(cm)^{n}}{\Gamma(n+\nu-1)} + (1-\beta\nu)\, m \sum_{n=0}^{\infty} \frac{m^{\nu}(cm)^{n}}{\Gamma(n+\nu+2)} \right]$$
$$= \frac{(A-B)c|\tau|}{\mathbb{E}_{\nu,c}(m)} \left[\beta m^3 \mathbb{E}_{\nu-2,c}(m) + (1-\beta\nu)\, \mathbb{E}_{\nu+1,c}(m) \right].$$

But this last expression is upper bounded by $\cos\delta - \alpha$ if (17) holds, which completes our proof. □

5 An integral operator

Theorem 4 *Let* $\nu > -1$ *and* $c > 0$. *If integral operator* $\mathbb{G}^{m}_{\nu,c}$ *is given by*

$$\mathbb{G}^{m}_{\nu,c}(z) := \int_0^z \frac{\mathbb{K}^{m}_{\nu,c}(t)}{t} dt,\ z \in \mathbb{U}, \tag{18}$$

then $\mathbb{G}^{m}_{\nu,c} \in \mathcal{W}_\delta(\alpha,\gamma,\beta)$, *if and only if*

$$\frac{c}{\mathbb{E}_{\nu,c}(m)} \left[\beta m^3 \mathbb{E}_{\nu-2,c}(m) + (1-\beta\nu)\, \mathbb{E}_{\nu+1,c}(m) \right] \leq \cos\delta - \alpha. \tag{19}$$

Proof. According to (8) it follows that

$$\mathbb{G}^{m}_{\nu,c}(z) = z - \sum_{n=2}^{\infty} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)} \frac{z^n}{n},\ z \in \mathbb{U}.$$

Using Lemma 1, the function $\mathbb{G}^{m}_{\nu,c}(z)$ belongs to $\mathcal{W}_\delta(\alpha,\gamma,\beta)$ if and only if

$$\sum_{n=2}^{\infty} [n(n-1)\beta + (\gamma-2\beta)n + (1-\gamma+2\beta)] \frac{1}{n} \frac{m^{\nu}(cm)^{n-1}}{\Gamma(n+\nu)\mathbb{E}_{\nu,c}(m)} \leq \cos\delta - \alpha.$$

By a similar proof like those of Theorem 3 we get that $\mathbb{G}^{m}_{\nu,c} \in \mathcal{W}_\delta(\alpha,\gamma,\beta)$ if and only if (19) holds. □

6 Corollaries and consequences

By specializing the parameter $\beta = 0$ in Theorems 1-4,we obtain the following special cases for the subclass $\mathcal{QT}_\gamma(\alpha) := \mathcal{Q}_\gamma(\alpha) \cap \mathcal{T}_o$.

Corollary 1 *Let $\nu > -1$ and $c > 0$. Then $\mathbb{K}^m_{\nu,c} \in \mathcal{QT}_\gamma(\alpha)$ if and only if*

$$\frac{c}{\mathbb{E}_{\nu,c}(m)}\left[\gamma m \mathbb{E}_{\nu,c}(m) + (1-\nu\gamma)\,\mathbb{E}_{\nu+1,c}(m)\right] \le \cos\delta - \alpha. \tag{20}$$

Corollary 2 *Let $\nu > -1$ and $c > 0$.*

(i) If $f \in \mathcal{S}^*$ and the inequality

$$\frac{c}{\mathbb{E}_{\nu,c}(m)}\Big[\gamma m^2 \mathbb{E}_{\nu-1,c}(m) + (1-(2\gamma\nu-1))\, m\mathbb{E}_{\nu,c}(m) + (1-\nu\gamma(1-\nu))\,\mathbb{E}_{\nu+1,c}(m)\Big] \le \cos\delta - \alpha. \tag{21}$$

is satisfied then $\mathbb{I}^m_{\nu,c} f \in \mathcal{QT}_\gamma(\alpha)$.
(ii) If $f \in \mathcal{K}$ and the inequality (20) is satisfied then $\mathbb{I}^m_{\nu,c} f \in \mathcal{QT}_\gamma(\alpha)$.

Corollary 3 *Let $\nu > -1$ and $c > 0$. If $f \in \mathcal{R}^\tau(A, B)$ and the inequality*

$$\frac{(A-B)c\,|\tau|\,\mathbb{E}_{\nu+1,c}(m)}{\mathbb{E}_{\nu,c}(m)} \le \cos\delta - \alpha. \tag{22}$$

is satisfied then $\mathcal{I}^m_q f \in \mathcal{QT}_\gamma(\alpha)$.

Corollary 4 *Let $\nu > -1$ and $c > 0$. If the function $\mathbb{G}^m_{\nu,c}$ is given by (18), then $\mathbb{G}^m_{\nu,c} \in \mathcal{QT}_\gamma(\alpha)$ if and only if*

$$\frac{c\mathbb{E}_{\nu+1,c}(m)}{\mathbb{E}_{\nu,c}(m)} \le \cos\delta - \alpha. \tag{23}$$

Remark 2 *If we put $\nu = 0$ and $c = 1$ in Theorems 1-4, then we obtain the corresponding results of Poisson distribution series.*

Remark 3 *Specializing the parameter β and γ we can state various interesting inclusion results (as proved in above theorems) for the subclasses $\mathcal{S}(\delta, \alpha)$ and $\mathcal{T}(\delta, \alpha)$ as stated in Remark 1.*

7 Conclusions

In the present paper, we find a necessary and sufficient condition for Miller-Ross-type Poisson distribution series to be in the class $\mathcal{W}_\delta(\alpha, \gamma, \beta)$ of analytic functions with negative coefficients. Also, we investigate several inclusion properties of the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}^\tau(A, B)$ associated of the operator $\mathbb{I}^m_{\nu,c}$ defined by Miller-Ross-type Poisson distribution. Some interesting corollaries and applications of the results are also discussed. Making use of Miller-Ross-type Poisson distribution series (7) could inspire researchers to find new necessary and sufficient conditions and inclusion relations for this distribution series to be in different subclasses of analytic functions with negative coefficients defined in the open unit disk $\mathbb{U}$.

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References

[1] M. Ahmad, B. Frasin, G. Murugusundaramoorthy, A. Al-khazaleh, An application of Mittag–Leffler-type Poisson distribution on certain subclasses of analytic functions associated with conic domains, *Heliyon*, 7 (2021) e08109.

[2] M. S. Ahmad, Q. Mehmood, W. Nazeer, A. U. Haq, An application of a hypergeometric distribution series on certain analytic functions. *Sci. Int.* 27, 2989–2992 (2015).

[3] R. M. Ali, A. Badghaish, V. Ravichandran, and A. Swaminathan, Starlikeness of integral transforms and duality, *J. Math. Anal. Appl.* 385(2) (2012), pp. 808–822.

[4] A. Amourah, B. Frasin, T. M. Seoudy, An Application of Miller–Ross-Type Poisson Distribution on Certain Subclasses of Bi-Univalent Functions Subordinate to Gegenbauer Polynomials, *Mathematics* 2022, 10, 2462. https://doi.org/10.3390/math10142462.

[5] A. Amourah, B. A. Frasin, M. Ahmad and F. Yousef, Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions, *Symmetry* 2022, 14, 147.

[6] A. A. Attiya, Some applications of Mittag-Leffler function in the unit disk, *Filomat* 30(7), 2075–2081 (2016).

[7] R. M. El-Ashwah, W. Y. Kota, Some condition on a Poisson distribution series to be in subclasses of univalent functions, *Acta Univ. Apulensis Math. Inform.*, 51(2017), 89–103.

[8] D. Bansal, J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, *Complex Var. Elliptic Equ.*, 61(3), 338–350(2016).

[9] N. E. Cho, S. Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, *Fract. Calc. Appl. Anal.*, 5(3)(2002), 303–313.

[10] S. M. El-Deeb, T. Bulboacă and J. Dziok, Pascal distribution series connected with certain subclasses of univalent functions, *Kyungpook Math. J.*, 59(2019), 301–314.

[11] S. S. Ding, Y. Ling and G. J. Bao, Some properties of a class of analytic functions, *J. Math. Anal. Appl.* no. 1, 195 (1995), 71–81.

[12] K. K. Dixit and S. K. Pal, On a class of univalent functions related to complex order, *Indian J. Pure Appl. Math.*, 26(9)(1995), 889–896.

[13] B. A. Frasin, Starlikeness and convexity of integral operators involving Mittag-Leffler functions, *TWMS J. App. Eng. Math.*, in press.

[14] B. A. Frasin, Comprehensive family of uniformly analytic functions, *Tamkang J. Math.*, 36, (2005), 243–254.

[15] B. A. Frasin and Mohammed M. Gharaibeh, Subclass of analytic functions associated with Poisson distribution series, *Afr. Mat.* 31, 1167–1173 (2020). https://doi.org/10.1007/s13370-020-00788-z.

[16] B. A. Frasin, On certain subclasses of analytic functions associated with Poisson distribution series, *Acta Univ. Sapientiae Math.*, 11(1)(2019) 78–86.

[17] B. A. Frasin, Subclasses of analytic functions associated with Pascal distribution series, *Adv. Theory Nonlinear Anal. Appl.*, 4 (2020) No. 2, 92–99.

[18] B. A. Frasin, G. Murugusundaramoorthy and S. Yalçin, Subclass of analytic functions associated with Pascal distribution series, *Bull. Transilv. Univ. Braşov*, Ser. III, Math. Inform. Phys. Vol. 13(62), No. 2 - 2020. Series III: Mathematics, Informatics, Physics, 521–528.

[19] B. A. Frasin, G. Murugusundaramoorthy and Tariq Al-Hawary, Uniformly convex spiral functions and uniformly spirallike functions associated with Pascal distribution series, *Mathematica Bohemica*, Vol. 147, No. 3, pp. 407–417, 2022, doi: 10.21136/MB.2021.0132-20.

[20] B. A. Frasin, T. Al-Hawary and F. Yousef, Some properties of a linear operator involving generalized Mittag-Leffler function, *Stud. Univ. Babeş-Bolyai Math.* 65(1), 67–75 (2020).

[21] B. A. Frasin, G. Murugusundaramoorthy and M. K. Aouf, Subclasses of analytic functions associated with Mittag-Leffler-type Poisson distribution, *Palestine J. Math.*, Vol. 11(1)(2022) , 496–503.

[22] B. A. Frasin, Tariq Al-Hawary and Feras Yousef, Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, Afr. Mat., 30(1–2)(2019), 223–230.

[23] M. Garg, P. Manohar and S. L. Kalla, A Mittag-Leffler-type function of two variables. *Integral Transforms Spec. Funct.* 24(11), 934–944 (2013).

[24] E. Merkes and B. T. Scott, Starlike hypergeometric functions, *Proc. Amer. Math. Soc.*, 12(1961), 885–888.

[25] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore (1993).

[26] G. M. Mittag-Leffler, Sur la nouvelle fonction $E(x)$, C. R. *Acad. Sci. Paris*, 137(1903), 554–558.

[27] G. Murugusundaramoorthy, Studies on classes of analytic functions with negative coefficients, Ph.D. Thesis, University of Madras, 1994.

[28] G. Murugusundaramoorthy, Subclasses of starlike and convex functions involving Poisson distribution series, *Afr. Mat.*, 28(2017), 1357–1366.

[29] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, *Hacettepe J. Math. Stat.*, 45(4)(2016), 1101–1107.

[30] S. Porwal, An application of a Poisson distribution series on certain analytic functions, *J. Complex Anal.*, (2014), Art. ID 984135, 1–3.

[31] S. Porwal, Mapping properties of generalized Bessel functions on some subclasses of univalent functions, *An. Univ. Oradea Fasc. Mat.*, 20(2)(2013), 51–60.

[32] S. Porwal and M. Kumar, A unified study on starlike and convex functions associated with Poisson distribution series, *Afr. Mat.*, 27(5)(2016), 1021–1027.

[33] C. Ramachandran L. Vanitha, Certain Aspect of Subordination for a class of analytic function, *International Journal of Mathematical Analysis*, 9 (20) 2015, 979 - 984.

[34] A. Shamakhi, B.A. Frasin and T. M. Seoudy, Subclass of analytic functions related with Pascal distribution series, *J. Math.*,Volume 2022, Article ID 8355285, 5 pages, https://doi.org/10.1155/2022/8355285.

[35] H. Silverman, Starlike and convexity properties for hypergeometric functions, *J. Math. Anal. Appl.*, 172(1993), 574–581.

[36] T. V. Sudharasan, K. G. Subramanian and P. Balasubrahmanyam, On two generalized classes of analytic functions with negative coefficient, *Soochow Journal of Mathematics*, 25(1) (1999), 11–17.

[37] S. Porwal, Confluent hypergeometric distribution and its applications on certain classes of univalent functions of conic regions, K*yungpook Math. J.* 58(3), 495–505 (2018).

[38] B. Şeker, S. S. Eker and B. Çekiç, On a subclass of analytic functions associated with Miller-Ross-type Poisson distribution series, submitted.

[39] T. Sekine, A generalization of certain class of analytic functions with negative coefficients, *Math. Japonica* 36(1991), no. 1, 13–19.

[40] H. M. Srivastava, B. A. Frasin and Virgil Pescar, Univalence of integral operators involving Mittag-Leffler functions, *Appl. Math. Inf. Sci.* 11, No. 3, 635–641 (2017).

[41] R. Themangani, S. Porwal and N. Magesh, Generalized hypergeometric distribution and its applications on univalent functions. *J. Inequal. Appl.* 2020, 249 (2020). https://doi.org/10.1186/s13660-020-02515-5.

[42] A. Wiman, Über die Nullstellun der Funcktionen $\mathsf{E}(\mathsf{x})$, *Acta Math.*, 29(1905), 217–134.

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