

Two applications of Grunsky coefficients in the theory of univalent functions

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Abstract. Let \mathcal{S} denote the class of functions f which are analytic and univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and normalized with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Using a method based on Grunsky coefficients we study two problems over the class \mathcal{S} : estimate of the fourth logarithmic coefficient and upper bound of the coefficient difference $|a_5| - |a_4|$.

1 Introduction and definitions

Let \mathcal{A} be the class of functions f which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad (1)$$

and let \mathcal{S} be the subclass of \mathcal{A} consisting of functions that are univalent in \mathbb{D} .

For $f \in \mathcal{S}$ the logarithmic coefficients, γ_n , are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n. \quad (2)$$

2010 Mathematics Subject Classification: 30C45, 30C50, 30C55

Key words and phrases: univalent functions, Grunsky coefficients, fourth logarithmic coefficient, coefficient difference

Relatively little exact information is known about these coefficients. The natural conjecture $|\gamma_n| \leq 1/n$, inspired by the Koebe function (whose logarithmic coefficients are $1/n$) is false even in order of magnitude (see Duren [2, Section 8.1]). For the class \mathcal{S} the sharp estimates of single logarithmic coefficients are known only for γ_1 and γ_2 , namely,

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e} = 0.635\dots,$$

and are unknown for $n \geq 3$. In [7] the authors gave the estimate $|\gamma_3| \leq 0.5566178\dots$ for the class \mathcal{S} . In this paper for the same class we give the estimation $|\gamma_4| \leq 0.51059\dots$. For the subclasses of univalent functions the situation is not a great deal better. Only the estimates of the initial logarithmic coefficients are available. For details see [1].

Another problem is finding sharp upper and lower bounds of the coefficient difference $|a_{n+1}| - |a_n|$ over the class of univalent functions. Since the Koebe function has coefficients $a_n = n$, it is natural to conjecture that $||a_{n+1}| - |a_n|| \leq 1$. But this is false even when $n = 2$, due to Fekete and Szegő ([3]) who obtained the sharp bounds

$$-1 \leq |a_3| - |a_2| \leq \frac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029\dots,$$

where λ_0 is the unique solution of the equation $4\lambda = e^\lambda$ on the interval $(0, 1)$. Hayman in [4] showed that if $f \in \mathcal{S}$, then $||a_{n+1}| - |a_n|| \leq C$, where C is an absolute constant and the best estimate of C is $3.61\dots$ ([5]). In the case when $n = 3$ in [7], the authors improved this to $1.751853\dots$. In this paper we also consider the difference $|a_5| - |a_4|$.

For the study of the problems defined above we will use method based on Grunsky coefficients. In the proofs we will use mainly the notations and results given in the book of N. A. Lebedev ([6]).

Here are basic definitions and results.

Let $f \in \mathcal{S}$ and let

$$\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are so called Grunsky's coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients we have the next Grunsky's inequality ([2, 6]):

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \quad (3)$$

where x_p are arbitrary complex numbers such that last series converges.

Further, it is well-known that if f given by (1) belongs to \mathcal{S} , then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \dots \quad (4)$$

belongs to the class \mathcal{S} . So, for the function f_2 we have the appropriate Grunsky's coefficients of the form $\omega_{2p-1,2q-1}^{(2)}$ and inequality (3) reaches the form:

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \quad (5)$$

Here, and further in the paper we omit the upper index "(2)" in $\omega_{2p-1,2q-1}^{(2)}$ if compared with Lebedev's notation.

From inequality (5), when $x_{2p-1} = 0$ and $p = 3, 4, \dots$, we have

$$\begin{aligned} & |\omega_{11}x_1 + \omega_{31}x_3|^2 + 3|\omega_{13}x_1 + \omega_{33}x_3|^2 + 5|\omega_{15}x_1 + \omega_{35}x_3|^2 \\ & + 7|\omega_{17}x_1 + \omega_{37}x_3|^2 \leq |x_1|^2 + \frac{|x_3|^2}{3}. \end{aligned} \quad (6)$$

As it has been shown in [6, p.57], if f is given by (1) then the coefficients a_2 , a_3 , a_4 and a_5 are expressed by Grunsky's coefficients $\omega_{2p-1,2q-1}$ of the function f_2 given by (4) in the following way:

$$\begin{aligned} a_2 &= 2\omega_{11}, \\ a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\ a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3, \\ a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4, \\ 0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}, \\ 0 &= \omega_{17} - \omega_{35} - \omega_{11}\omega_{33} - \omega_{13}^2 + \frac{1}{3}\omega_{11}^4. \end{aligned} \quad (7)$$

We note that in the cited book of Lebedev there is a typing mistake for the coefficient a_5 . Namely, instead of the term $5\omega_{13}^2$ there stays $5\omega_{15}^2$.

We now give upper bound of the fourth logarithmic coefficient over the class \mathcal{S} .

Theorem 1 Let $f \in \mathcal{S}$ and be given by (1). Then

$$|\gamma_4| \leq 0.51059 \dots$$

Proof. From (1) and (2), after differentiating and comparing coefficients, we receive

$$\gamma_4 = \frac{1}{2} \left(a_5 - a_2 a_4 - \frac{1}{2} a_3^2 + a_2^2 a_3 - \frac{1}{4} a_2^4 \right),$$

or by using the relation (7):

$$\gamma_4 = \frac{1}{2} \left(2\omega_{35} + 3\omega_{13}^2 + 4\omega_{11}\omega_{33} + 4\omega_{11}^2\omega_{13} - \frac{5}{6}\omega_{11}^4 \right). \quad (8)$$

If we combine the two last relations from (7), then we have

$$\omega_{33} = \omega_{15} - \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3 \quad (9)$$

and

$$\omega_{35} = \omega_{17} - \omega_{11}\omega_{33} - \omega_{13}^2 + \frac{1}{3}\omega_{11}^4 = \omega_{17} - \omega_{11}\omega_{15} + \omega_{11}^2\omega_{13} - \omega_{13}^2. \quad (10)$$

Using the relations (8), (9) and (10), after some calculations, we get

$$\gamma_4 = \omega_{17} + \omega_{11}\omega_{15} + \omega_{11}^2\omega_{13} + \frac{1}{2}\omega_{13}^2 + \frac{1}{4}\omega_{11}^4.$$

Therefore,

$$\begin{aligned} |\gamma_4| &\leq |\omega_{17}| + |\omega_{11}||\omega_{15}| + |\omega_{11}|^2|\omega_{13}| + \frac{1}{2}|\omega_{13}|^2 + \frac{1}{4}|\omega_{11}|^4 \\ &:= \varphi(|\omega_{11}|, |\omega_{13}|, |\omega_{15}|, |\omega_{17}|). \end{aligned} \quad (11)$$

Now, choosing $x_1 = 1$ and $x_3 = 0$ in (6) we receive

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 + 7|\omega_{17}|^2 \leq 1,$$

and also

$$|\omega_{11}| \leq 1, \quad |\omega_{11}|^2 + 3|\omega_{13}|^2 \leq 1, \quad |\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 \leq 1.$$

The above inequalities imply

$$\begin{aligned} |\omega_{13}| &\leq \frac{1}{\sqrt{3}} \sqrt{1 - |\omega_{11}|^2}, \\ |\omega_{15}| &\leq \frac{1}{\sqrt{5}} \sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2}, \\ |\omega_{17}| &\leq \frac{1}{\sqrt{7}} \sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2}. \end{aligned} \quad (12)$$

Using (11) and (12) we conclude that it remains to find $\max \varphi_1$, where

$$\varphi_1(x, y, z, t) = \frac{1}{4}x^4 + \frac{1}{2}y^2 + x^2y + xz + t,$$

where (x, y, z, t) is in the four dimensional hypercube Ω described with

$$\begin{aligned} 0 &\leq x = |\omega_{11}| \leq 1, \\ 0 &\leq y = |\omega_{13}| \leq \frac{1}{\sqrt{3}} \sqrt{1 - x^2}, \\ 0 &\leq z = |\omega_{15}| \leq \frac{1}{\sqrt{5}} \sqrt{1 - x^2 - 3y^2}, \\ 0 &\leq t = |\omega_{17}| \leq \frac{1}{\sqrt{7}} \sqrt{1 - x^2 - 3y^2 - 5z^2}. \end{aligned} \quad (13)$$

Since φ_1 is an increasing function of t on the interval $(0, +\infty)$, we realize that it reaches its maximal value for $t = t_0 = \frac{1}{\sqrt{7}} \sqrt{1 - x^2 - 3y^2 - 5z^2}$, i.e.,

$$\max\{\varphi_1(x, y, z, t) : (x, y, z, t) \in \Omega\} = \max\{\psi_1(x, y, z) : (x, y, z) \in \Omega_1\},$$

where $\psi_1(x, y, z) \equiv \varphi_1(x, y, z, t_0)$ and

$$\Omega_1 = \left\{ (x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{\sqrt{3}} \sqrt{1 - x^2}, 0 \leq z \leq \frac{1}{\sqrt{5}} \sqrt{1 - x^2 - 3y^2} \right\}.$$

The system of equations

$$\begin{cases} \frac{\partial \psi_1}{\partial x} = x^3 + 2xy + z - \frac{x/\sqrt{7}}{\sqrt{1 - x^2 - 3y^2 - 5z^2}} = 0 \\ \frac{\partial \psi_1}{\partial y} = x^2 + y - \frac{3y/\sqrt{7}}{\sqrt{1 - x^2 - 3y^2 - 5z^2}} = 0 \\ \frac{\partial \psi_1}{\partial z} = x - \frac{5z/\sqrt{7}}{\sqrt{1 - x^2 - 3y^2 - 5z^2}} = 0 \end{cases}$$

in the interior of Ω_1 , has a unique solution,

$$x_0 = 0.81907\dots, \quad y_0 = 0.233235\dots, \quad z_0 = 0.126778\dots,$$

(obtained with Wolfram's Mathematica) with $\psi_1(x_0, y_0, z_0, t_0) = \psi_1(x_0, y_0, z_0) = 0.51059\dots$ which will turn out to be the maximal value of φ_1 on Ω and an upper bound for $|\gamma_4|$.

Now we will study the behaviour of ψ_1 on the boundaries of Ω_1 .

For $x = 0$, we have that $\psi_1(0, y, z) = \frac{1}{2}y^2 + \frac{1}{\sqrt{7}}\sqrt{1-3y^2-5z^2}$ for $0 \leq y \leq \frac{1}{\sqrt{3}}$ and $0 \leq z \leq \frac{1}{\sqrt{5}}\sqrt{1-3y^2}$ has maximal value $\frac{1}{\sqrt{7}} = 0.37796\dots$ attained when $y = z = 0$.

Next, for $x = 1$, we have necessarily $y = z = 0$, which leads to a maximal value $\psi_1(1, 0, 0) = 1/4 = 0.25$.

For the case $y = 0$, we have $\psi_1(x, 0, z) = \frac{1}{4}x^4 + xz + \frac{1}{\sqrt{7}}\sqrt{1-x^2-5z^2}$, with $0 \leq x \leq 1$ and $0 \leq z \leq \frac{1}{\sqrt{5}}\sqrt{1-x^2}$. Further, for the solution (x_1, z_1) of the system of equations

$$\begin{cases} \frac{\partial \psi_1(x_1, 0, z_1)}{\partial x} = x_1^3 + z_1 - \frac{x_1/\sqrt{7}}{\sqrt{-x_1^2 - 5z_1^2 + 1}} = 0 \\ \frac{\partial \psi_1(x_1, 0, z_1)}{\partial z} = x_1 - \frac{5z_1/\sqrt{7}}{\sqrt{-x_1^2 - 5z_1^2 + 1}} = 0 \end{cases},$$

we have

$$-5z_1^2 - 5x_1^3z_1 + x_1^2 = 0,$$

leading further to

$$z_1 = \frac{1}{10} \left(-5x_1^3 + \sqrt{5}\sqrt{5x_1^6 + 4x_1^2} \right).$$

Finally,

$$\begin{aligned} \psi_1(x, 0, z) &\leq \psi(x_1, 0, z_1) \\ &= -\frac{x_1^4}{4} + \frac{1}{2}\sqrt{x_1^6 + \frac{4x_1^2}{5}}x_1 + \frac{1}{\sqrt{14}} \left[\sqrt{\left(-5x_1^4 + \sqrt{5}\sqrt{x_1^2(5x_1^4 + 4)}x_1 - 4 \right) x_1^2 + 2} \right] \end{aligned}$$

By the means of calculus of real functions of one real variables, one can verify that the last function attains its maximum for $x = 0.80210\dots$ and $z = 0.183847\dots$, and that maximum is $0.414666\dots$

The case $y = \frac{1}{\sqrt{3}}\sqrt{1-x^2}$, leads to $z = 0$, and further to the function

$$\psi_1\left(x, \frac{1}{\sqrt{3}}\sqrt{1-x^2}, 0\right) = \frac{x^4}{4} + \frac{x^2\sqrt{1-x^2}}{\sqrt{3}} + \frac{1}{6}(1-x^2)$$

with maximum $0.4000\dots$ for $x = 0.8874\dots$ and $y = 0.2661\dots$

For $z = 0$, we have $\psi_1(x, y, 0) = \frac{x^4}{4} + x^2y + \frac{y^2}{2} + \frac{1}{\sqrt{7}}\sqrt{-x^2 - 3y^2 + 1}$ and working in the similar way as in the case $y = 0$, we receive its maximum on $0 \leq x \leq 1$ and $0 \leq y \leq \frac{1}{\sqrt{3}}\sqrt{1-x^2}$ to be $0.4561\dots$ for $x = 0.8358\dots$ and $y = 0.2619\dots$

Finally, in a similar way as before, for the case $z = \frac{1}{\sqrt{5}}\sqrt{1-x^2-3y^2}$, by means of calculus, we can verify that the maximal value is $0.4570\dots$ obtained for $x = 0.864969\dots$ and $y = 0.239789\dots$ \square

We now give upper bound of $|a_5| - |a_4|$ over the class \mathcal{S} .

Theorem 2 *Let $f \in \mathcal{S}$ and be given by (1). Then*

$$|a_5| - |a_4| \leq 2.3297\dots$$

Proof. Since

$$\begin{aligned} |a_5| - |a_4| &\leq |a_5| - |\omega_{11}| |a_4| \leq |a_5 - \omega_{11}a_4| \\ &= \left| 2\omega_{35} + 6\omega_{11}\omega_{33} + 10\omega_{11}^2\omega_{13} + 5\omega_{13}^2 - \omega_{11}^4 \right|, \end{aligned}$$

after applying (9) and (10), and some calculations, we have

$$\begin{aligned} |a_5| - |a_4| &\leq \left| 2\omega_{17} + 4\omega_{11}\omega_{15} + 6\omega_{11}^2\omega_{13} + 3\omega_{13}^2 + \omega_{11}^4 \right| \\ &\leq 2|\omega_{17}| + 4|\omega_{11}||\omega_{15}| + 6|\omega_{11}|^2|\omega_{13}| + 3|\omega_{13}|^2 + |\omega_{11}|^4 \\ &:= \varphi_2(|\omega_{11}|, |\omega_{13}|, |\omega_{15}|, |\omega_{17}|), \end{aligned}$$

where

$$\varphi_2(x, y, z, t) = x^4 + 3y^2 + 6x^2y + 4xz + 2t,$$

with x, y, z, t , as well as their domain Ω , are given in (13) from the previous theorem.

Now, in a similar way as in the proof of the previous theorem we will find the maximal value of the function ψ_2 over the domain Ω .

The function φ_2 is an increasing one over the variable t , and therefore it reaches its maximal value for $t = t_0 = \frac{1}{\sqrt{7}}\sqrt{1-x^2-3y^2-5z^2}$. Using the notations

$$\psi_2(x, y, z) \equiv \varphi_2(x, y, z, t_0) = x^4 + 6x^2y + 4xz + 3y^2 + \frac{2\sqrt{-x^2-3y^2-5z^2+1}}{\sqrt{7}},$$

and Ω_1 as in the Theorem 1, again using Wolfram's Mathematica we obtain that the system of equations

$$\begin{cases} \frac{\partial \psi_2}{\partial x} = 4x^3 + 12xy + 4z - \frac{2x}{\sqrt{7}\sqrt{1-x^2-3y^2-5z^2}} = 0 \\ \frac{\partial \psi_2}{\partial y} = 6x^2 + y \left(6 - \frac{6}{\sqrt{7}\sqrt{1-x^2-3y^2-5z^2}} \right) = 0 \\ \frac{\partial \psi_2}{\partial z} = 4x - \frac{10z}{\sqrt{7}\sqrt{1-x^2-3y^2-5z^2}} = 0 \end{cases},$$

in the interior of Ω_1 has a unique solution,

$$x_0 = 0.82745\dots, \quad y_0 = 0.29092\dots, \quad z_0 = 0.098698\dots\dots,$$

such that $\varphi_2(x_0, y_0, z_0, t_0) = \psi_2(x_0, y_0, z_0) = 2.3297\dots$. At the end, this will turn out to be the maximal value of ψ_2 on Ω_1 and upper bound of $|a_5| - |a_4|$.

Now we will study the behaviour of ψ_2 on the boundaries of Ω_1 .

For $x = 0$, we receive $\psi_2(0, y, z) = 3y^2 + \frac{2\sqrt{-3y^2-5z^2+1}}{\sqrt{7}}$ which is a decreasing function of z (since z is positive), thus with the same maximal value as $\psi_2(0, y, 0) = 3y^2 + \frac{2\sqrt{1-3y^2}}{\sqrt{7}}$ which turns out to be 1.142857... for $y = 0.5345\dots$

If $x = 1$, then necessarily $y = z = 0$, and $\psi_2(1, 0, 0) = 1$.

For $y = 0$, we have $\psi_2(x, 0, z) = x^4 + 4xz + \frac{2\sqrt{1-x^2-5z^2}}{\sqrt{7}}$ which can be shown to have no critical points in the interior of

$$\left\{ (x, z) : 0 \leq x \leq 1, 0 \leq z \leq \frac{1}{\sqrt{5}}\sqrt{1-x^2} \right\},$$

and a maximal value 1.3614... obtained for $x_1 = 0.9181\dots$ and $z_1 = \frac{1}{\sqrt{5}}\sqrt{1-x_1^2} = 0.1772\dots$

For the case $y^* = \frac{1}{\sqrt{3}}\sqrt{1-x^2}$, we have $z = 0$, and further, the function

$$\psi_2(x, y^*, 0) = x^4 + \left(2\sqrt{3-3x^2} - 1\right)x^2 + 1$$

has maximal value $2.118588\dots$ for $x = 0.8427\dots$

Next, for $z = 0$ we receive

$$\psi_2(x, y, 0) = x^4 + 6x^2y + 3y^2 + \frac{2\sqrt{-x^2-3y^2+1}}{\sqrt{7}},$$

with a critical point (x_2, y_2) ($x = 0.83589\dots$ and $y = 0.3097\dots$) in the interior of

$$\left\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{\sqrt{3}}\sqrt{1-x^2}\right\},$$

such that $\psi_2(x_2, y_2, 0) = 2.162\dots$. The boundaries of the above domain are already discussed above.

Finally, if $z^* = \frac{1}{\sqrt{5}}\sqrt{-x^2-3y^2+1}$, one can verify that the function

$$\psi_2(x, y, z) = x^4 + 6x^2y + 3y^2 + \frac{4x\sqrt{-x^2-3y^2+1}}{\sqrt{5}},$$

has critical point (x_3, y_3) with $x_3 = 0.8338\dots$ and $y_3 = 0.2921\dots$, such that $\psi_2(x_3, y_3, z^*) = 2.287\dots$

All the above analysis brings us to the final conclusion that ψ_2 on Ω_1 has a maximal value $2.3297\dots$ obtained for $x = x_0$, $y = y_0$ and $z = z_0$. \square

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Received: January 19, 2022