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Automorphisms of Zappa-Szép product fixing a subgroup

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Abstract. In this paper, we have found the automorphism group of the Zappa-Szép product of two groups fixing a subgroup. We have computed these automorphisms for a subgroup of order p^2 of a group G which is the Zappa-Szép product of two cyclic groups in which one is of order p^2 and other is of order m.

1 Introduction

G. Zappa in [10], introduced the general product of two groups called the Zappa-Szép product. J. Szép studied such products in the series of papers (few of them are [6, 4, 5, 7]). Let H and K be two subgroups of a group G. Then G is called Zappa-Szép product of H and K if $G = HK$ and $H \cap K = \{1\}$ and is written as $G = H \bowtie K$. Moreover, any element $g \in G$ can be uniquely expressed as $g = hk$, where $h \in H$ and $k \in K$. So, for $kh \in G$, we must have elements $\sigma(k, h) \in H$ and $\theta(k, h) \in K$ such that $kh = \sigma(k, h)\theta(k, h)$. Thus we have matched pair of maps $\sigma : K \times H \to H$ and $\theta : K \times H \to K$ defined by $\sigma(k, h) = k \cdot h$ and $\theta(k, h) = k^h$ and satisfies the following conditions (See [2])

(C1) $1 \cdot h = h$ and $k^1 = k$,

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(C2) $k \cdot 1 = 1 = 1^h$,

(C3) $kk' \cdot h = k \cdot (k' \cdot h)$,

(C4) $(kk')^h = k^{k' \cdot h} k'^h$,

(C5) $k \cdot (hh') = (k \cdot h)(k^h \cdot h')$,

(C6) $k^{hh'} = (k^h)^{h'}$,

for all $h, h' \in H$ and $k, k' \in K$.

The automorphisms of the Zappa-Szép product of two groups is studied in [3] as the 2×2 matrices of maps satisfying some certain conditions. The terminology used in this paper is same as in [3]. In this paper, we have found the automorphism group of the Zappa-Szép product fixing the subgroup H as the 2×2 matrices of maps satisfying some certain conditions. As an application, we have computed the automorphism group fixing a subgroup of order p^2 of a group G which is the Zappa-Szép product of two cyclic groups in which one is of order p^2 and other is of order m. Throughout the paper, $\mathbb{Z}_n$ denotes the cyclic group of order n and $U(n)$ denotes the group of units of $\pmod{n}$. Also, $\mathrm{Aut}_H(G) = \{\phi \in \mathrm{Aut}(G) \mid \phi(H) = H\}$ denotes the group of all automorphisms of a group G fixing the subgroup H. Let a group U acts on a group V, then $\mathrm{Stab}_U(V)$ denotes the stabilizer of V in U.

2 Structure of automorphism group, $\mathrm{Aut}_H(G)$

Let $G = H \bowtie K$ be the Zappa-Szép product of two groups H and K. Let U, V and W be any groups. $\mathrm{Map}(U, V)$ denotes the set of all maps between the groups U and V. If $\phi, \psi \in \mathrm{Map}(U, V)$ and $\eta \in \mathrm{Map}(V, W)$, then $\phi + \psi \in \mathrm{Map}(U, V)$ is defined by $(\phi+\psi)(u) = \phi(u)\psi(u)$, $\eta\phi \in \mathrm{Map}(U, W)$ is defined by $\eta\phi(u) = \eta(\phi(u))$, $\phi \cdot \psi \in \mathrm{Map}(U, V)$ is defined by $\phi \cdot \psi(u) = \phi(u) \cdot \psi(u)$ and $\phi^\psi \in \mathrm{Map}(U, V)$ is defined by $\phi^\psi(u) = \phi(u)^{\psi(u)}$, for all $u \in U$.

Now, consider the set,

$$\mathcal{A}_H = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \mid \begin{matrix} \alpha \in \mathrm{Aut}(H), & \beta \in \mathrm{Map}(K, H), \\ \text{and} & \delta \in \mathrm{Map}(K, K) \end{matrix} \right\},$$

where α, β and δ satisfy the following conditions,

(A1) $\beta(kk') = \beta(k)(\delta(k) \cdot \beta(k'))$, (A2) $\delta(kk') = \delta(k)^{\beta(k')}\delta(k')$,

(A3) $\beta(k)(\delta(k)\cdot\alpha(h)) = \alpha(k\cdot h)\beta(k^h)$,

(A4) $\delta(k)^{\alpha(h)} = \delta(k^h)$,

(A5) For any $h'k' \in G$, there exists a unique $h \in H$ and $k \in K$ such that $h' = \alpha(h)\beta(k)$ and $k' = \delta(k)$.

Then, the set $\mathcal{A}_H$ forms a group with the binary operation as the usual multiplication of matrices defined by

$$\begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha'\alpha & \alpha'\beta + \beta'\delta \\ 0 & \delta'\delta \end{pmatrix}.$$

Clearly, $\mathrm{Aut}_H(K)$ is a subgroup of $\mathrm{Aut}(G)$.

Theorem 1 *Let* $G = H \bowtie K$ *be the Zappa-Szép product of two groups* H *and* K*, and* $\mathcal{A}_H$ *be as above. Then there is an isomorphism of groups between* $\mathrm{Aut}_H(G)$ *and* $\mathcal{A}_H$ *given by* $\theta \longleftrightarrow \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$*, where* $\theta(h) = \alpha(h)$ *and* $\theta(k) = \beta(k)\delta(k)$*, for all* $h \in H$ *and* $k \in K$*.*

Proof.
Let $\theta \in \mathrm{Aut}_H(G)$ be defined by $\theta(h) = \alpha(h)$ and $\theta(k) = \beta(k)\delta(k)$, for all $h \in H$ and $k \in K$. Then $\alpha = \theta|_H$, so $\alpha \in \mathrm{Aut}(H)$. Now, for all $k, k' \in K$, $\theta(kk') = \theta(k)\theta(k') = (\beta(k)\delta(k))(\beta(k')\delta(k')) = \beta(k)(\delta(k)\cdot\beta(k'))\delta(k)^{\beta(k')}\delta(k')$. Thus, $\beta(kk')\delta(kk') = \beta(k)(\delta(k)\cdot\beta(k'))\,\delta(k)^{\beta(k')}\delta(k')$. Therefore, by uniqueness of representation, we have (A1) and (A2).

Now, $\theta(kh) = \theta((k\cdot h)(k^h)) = \theta(k\cdot h)\theta(k^h) = \alpha(k\cdot h)\beta(k^h)\delta(k^h)$. Also, $\theta(kh) = \theta(k)\theta(h) = \beta(k)\delta(k)\alpha(h) = \beta(k)(\delta(k)\cdot\alpha(h))\delta(k)^{\alpha(h)}$. Therefore, by the uniqueness, $\beta(k)(\delta(k)\cdot\alpha(h)) = \alpha(k\cdot h)\beta(k^h))$ and $\delta(k)^{\alpha(h)} = \delta(k^h)$, which proves (A3) and (A4). Finally, (A5) holds because θ is onto. Thus, to every $\theta \in \mathrm{Aut}_H(G)$ we can associate the matrix $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathcal{A}_H$. This defines a map $T : \mathrm{Aut}_H(G) \longrightarrow \mathcal{A}_H$ given by $\theta \longmapsto \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$. Now, if $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathcal{A}_H$ satisfying the conditions (A1) $-$ (A5), then we associate to it, the map $\theta : G \longrightarrow G$ defined by $\theta(h) = \alpha(h)$ and $\theta(k) = \beta(k)\delta(k)$, for all $h \in H$ and $k \in K$. Using (A1)$-$(A4), one can check that θ is an endomorphism of G. Also, by (A5), the map θ is onto. Now, let $hk \in \ker(\theta)$. Then $\theta(hk) = 1$. Therefore,

$\alpha(h)\beta(k)\delta(k) = 1$ and so, by the uniqueness of representation $\alpha(h)\beta(k) = 1$ and $\delta(k) = 1$. By (A5), δ is a bijection and so, $k = 1$. Thus using [3, Proposition 2.1, p. 3], $\beta(k) = 1$ which further implies that $\alpha(h) = 1$. Again, by (A5), α is a bijection so, $h = 1$. Therefore, $h = 1 = k$ and so, $\ker(\theta) = \{1\}$. Thus, θ is one-one and hence, $\theta \in \mathrm{Aut}_H(G)$. Thus, T is a bijection. Let α, β and δ be the maps associated with θ and α', β' and δ' be the maps associated with θ'. Now, for all $h \in H$ and $k \in K$, we have $\theta'\theta(h) = \alpha'\alpha(h)$ and $\theta'\theta(k) = \theta'(\beta(k)\delta(k)) = \alpha'(\beta(k))\beta'(\delta(k))\delta'(\delta(k)) = (\alpha'\beta + \beta'\delta)(k)\delta'(\delta(k))$.
Therefore, if we write hk as $\begin{pmatrix} h \\ k \end{pmatrix}$, then $\theta'\theta(h) = \begin{pmatrix} \alpha'\alpha \\ 0 \end{pmatrix}\begin{pmatrix} h \\ 1 \end{pmatrix}$ and $\theta'\theta(k) = \begin{pmatrix} \alpha'\beta + \beta'\delta \\ \delta'\delta \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix}$. Thus, $\theta'\theta(hk) = \begin{pmatrix} \alpha'\alpha & \alpha'\beta + \beta'\delta \\ 0 & \delta'\delta \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$. Therefore, $T(\theta'\theta) = \begin{pmatrix} \alpha'\alpha & \alpha'\beta + \beta'\delta \\ 0 & \delta'\delta \end{pmatrix} = T(\theta)T(\theta')$. Hence, T is an isomorphism of groups. □

From here on, we will identify the automorphisms of G fixing the subgroup H with the matrices in $\mathcal{A}_H$. Now, we have the following remarks,

(i) $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{A}_H$ if and only if $\alpha \in \mathrm{Aut}(H)$, $k \cdot \alpha(h) = \alpha(k \cdot h)$ and $k^h = k^{\alpha(h)}$ for all $h \in H$ and $k \in K$.

(ii) $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \mathcal{A}_H$ if and only if $\beta(kk') = \beta(k)(k \cdot \beta(k'))$, $k = k^{\beta(k')}$, $\beta(k) = \beta(k^h)$ for all $h \in H$ and $k \in K$.

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \in \mathcal{A}_H$ if and only if $\delta \in \mathrm{Aut}(K)$, $\delta(k) \cdot h = k \cdot h$, $\delta(k)^h = \delta(k^h)$ for all $h \in H$ and $k \in K$.

(iv) $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in \mathcal{A}_H$ if and only if $\alpha \in \mathrm{Aut}(H)$, $\delta \in \mathrm{Aut}(K)$, $\delta(k) \cdot \alpha(h) = \alpha(k \cdot h)$, and $\delta(k)^{\alpha(h)} = \delta(k^h)$ for all $h \in H$ and $k \in K$.

Let

$$\begin{aligned}
P =& \{\alpha \in \mathrm{Aut}(H) \mid k \cdot \alpha(h) = \alpha(k \cdot h) \text{ and } k^{\alpha(h)} = k^h\}, \\
Q =& \{\beta \in \mathrm{Map}(K, H) \mid \beta(kk') = \beta(k)(k \cdot \beta(k')), k = k^{\beta(k')}, \beta(k) = \beta(k^h)\}, \\
S =& \{\delta \in \mathrm{Aut}(K) \mid \delta(k) \cdot h = k \cdot h, \delta(k)^h = \delta(k^h)\}, \\
X =& \{(\alpha, \delta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K) \mid \delta(k) \cdot \alpha(h) = \alpha(k \cdot h), \delta(k)^{\alpha(h)} = \delta(k^h)\}, \\
Y =& \{(\beta, \delta) \in \mathrm{Map}(K, H) \times \mathrm{Map}(K, K) \mid \beta(kk') = \beta(k)(\delta(k) \cdot \beta(k')),
\end{aligned}$$

$$\delta(kk') = \delta(k)^{\beta(k')}\delta(k'), \beta(k)(\delta(k)\cdot h) = (k\cdot h)\beta(k^h), \delta(k)^h = \delta(k^h)\}.$$

Then one can easily check that P, S, X and Y are all subgroups of the group $\mathrm{Aut}_H(G)$. But Q need not be subgroup of the group $\mathrm{Aut}_H(G)$. However, if H is abelian, then Q is subgroups of $\mathrm{Aut}_H(G)$. Also, note that $P \times S \leq X$. Let

$$A = \left\{\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} \mid \alpha \in P\right\}, \qquad B = \left\{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \mid \beta \in Q\right\},$$
$$D = \left\{\begin{pmatrix}1 & 0\\ 0 & \delta\end{pmatrix} \mid \delta \in S\right\}, \qquad E = \left\{\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix} \mid (\alpha,\delta) \in X\right\},$$
$$F = \left\{\begin{pmatrix}1 & \beta\\ 0 & \delta\end{pmatrix} \mid (\beta,\delta) \in Y\right\}.$$

be the corresponding subsets of $\mathcal{A}_H$. Then one can easily check that A, D, E and F are subgroups of $\mathcal{A}_H$, and if H is abelian group, then B is also a subgroup of $\mathcal{A}_H$.

Theorem 2 *If either* $P = \mathrm{Aut}(H)$ *or* $S = \mathrm{Aut}(K)$, *then* $X = P \times S$. *Equivalently,* $E = A \times D$.

Proof. Let $(\alpha,\delta) \in X$. Then $\delta(k)\cdot\alpha(h) = \alpha(k\cdot h)$ and $\delta(k)^{\alpha(h)} = \delta(k^h)$. Now, if $P = \mathrm{Aut}(H)$, then $k\cdot\alpha(h) = \alpha(k\cdot h)$. Therefore, using $\delta(k)\cdot\alpha(h) = \alpha(k\cdot h)$, we get $\delta(k)\cdot\alpha(h) = k\cdot\alpha(h)$. Also, since $P = \mathrm{Aut}(H)$, $k^{\alpha(h)} = k^h$. So, using $\delta(k)^{\alpha(h)} = \delta(k^h)$, we get $\delta(k^h) = \delta(k)^{\alpha(h)} = \delta(k)^h$. Thus, $\delta \in S$ and so, $(\alpha,\delta) \in P \times S$. Hence, $X = P \times S$. By the similar argument, if $S = \mathrm{Aut}(K)$, then $X = P \times S$. □

Theorem 3 *Let* $\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \in \mathcal{A}_H$. *If* $\beta \in Q$, *then* $\mathrm{Aut}_H(G) \simeq B \rtimes E$.

Proof. Let $\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \in B$ and $\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix} \in E$. Then

$$\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}^{-1} = \begin{pmatrix}1 & \alpha\beta\delta^{-1}\\ 0 & 1\end{pmatrix}. \tag{1}$$

Now, for all $h \in H$ and $k, k' \in K$,

$$\begin{aligned}\alpha\beta\delta^{-1}(kk') &= \alpha\beta(\delta^{-1}(k)\delta^{-1}(k')) = \alpha(\beta(\delta^{-1}(k))(\delta^{-1}(k)\cdot\beta(\delta^{-1}(k'))))\\ &= \alpha(\beta(\delta^{-1}(k))\alpha(\delta^{-1}(k)\cdot\beta\delta^{-1}(k'))\\ &= \alpha\beta\delta^{-1}(k)(\delta(\delta^{-1}(k))\cdot\alpha(\beta\delta^{-1}(k')))\end{aligned}$$

$$= \alpha\beta\delta^{-1}(k)(k \cdot \alpha\beta\delta^{-1}(k')).$$

Also, one can easily observe that $k^{\alpha\beta\delta^{-1}(k')} = k$ and $\alpha\beta\delta^{-1}(k^h) = \alpha\beta\delta^{-1}(k)$. Thus, $\alpha\beta\delta^{-1} \in Q$. Therefore, by the Equation (1), we get

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \alpha\beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \in B.$$

Thus, $B \lhd \mathcal{A}_H$. Clearly, $B \cap E = \{1\}$. If $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathcal{A}_H$, then

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{pmatrix} \in EB.$$

Hence, $\mathcal{A}_H = B \rtimes E$ and so, $\mathrm{Aut}_H(G) \simeq B \rtimes E$. □

3 $\mathrm{Aut}_H(G)$ of Zappa-Szép product of groups $\mathbb{Z}_4$ and $\mathbb{Z}_m$

In [8], Yacoub classified the groups which are Zappa-Szép product of cyclic groups of order 4 and order m. He found that these are of the following type (see [8, Conclusion, p. 126])

$$\begin{aligned} L_1 =& \langle a, b \mid a^m = 1 = b^4, ab = ba^r, r^4 \equiv 1 \pmod{m} \rangle, \\ L_2 =& \langle a, b \mid a^m = 1 = b^4, ab = b^3a^{2t+1}, a^2b = ba^{2s} \rangle, \end{aligned}$$

where in L_2, m is even. These are not non-isomorphic classes. The group L_1 may be isomorphic to the group L_2 depending on the values of m, r and t (see [8, Theorem 5, p. 126]). Clearly, L_1 is a semidirect product. Throughout this section G will denote the group L_2 and we will be only concerned about groups L_2 which are Zappa-Szép product but not the semidirect product. Note that $G = H \bowtie K$, where $H = \langle b \rangle$ and $K = \langle a \rangle$. For the group G, the mutual actions of H and K are defined by $a \cdot b = b^3, a^b = a^{2t+1}$ along with $a^2 \cdot b = b$ and $(a^2)^b = a^{2s}$, where t and s are the integers satisfying the conditions

(G1) $2s^2 \equiv 2 \pmod{m}$,

(G2) $4t(s+1) \equiv 0 \pmod{m}$,

(G3) $2(t+1)(s-1) \equiv 0 \pmod{m}$,

(G4) $\gcd(s, \frac{m}{2}) = 1$.

Now, one can easily observe that for the given group G, $k \cdot \alpha(h) = \alpha(k \cdot h)$, $\beta(k) = \beta(k^h)$, $\delta(k) \cdot h = k \cdot h$, $\delta(k) \cdot \alpha(h) = \alpha(k \cdot h)$ and $\beta(k)(\delta(k) \cdot \alpha(h)) = \alpha(k \cdot h)\beta(k^h)$ always holds for all $\alpha \in P$, $\beta \in Q$, $\delta \in S$, $(\alpha, \delta) \in X$, and $(\beta, \delta) \in Y$ respectively. Thus the subgroups P, Q, S, X, and Y reduces to the following,

$$\begin{aligned}
P =& \{\alpha \in \mathrm{Aut}(H) \mid k^{\alpha(h)} = k^h\},\\
Q =& \{\beta \in \mathrm{Hom}(K, H) \mid k = k^{\beta(k')}\} = \mathrm{Hom}(K, \mathrm{Stab}_H(K)),\\
S =& \{\delta \in \mathrm{Aut}(K) \mid \delta(k)^h = \delta(k^h)\},\\
X =& \{(\alpha, \delta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K) \mid \delta(k)^{\alpha(h)} = \delta(k^h)\},\\
Y =& \{(\beta, \delta) \in \mathrm{Map}(K, H) \times \mathrm{Map}(K, K) \mid \beta(kk') = \beta(k)(\delta(k) \cdot \beta(k')),\\
& \delta(kk') = \delta(k)^{\beta(k')}\delta(k'), \delta(k)^{\alpha(h)} = \delta(k^h)\}.
\end{aligned}$$

Now, we will find the structure of the automorphism group $\mathrm{Aut}_H(G)$. For this, we will proceed by first taking t to be such that $\gcd(t, m) = 1$ and then by taking t such that $\gcd(t, m) = d$, where $d > 1$.

Theorem 4 *Let* 4 *divides* m *and* t *be odd such that* $\gcd(t, m) = 1$*. Then*

$$\mathrm{Aut}_H(G) \simeq \begin{cases} \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m)), & \textit{if } s \in \{\frac{m}{2} - 1, m - 1\} \\ \mathbb{Z}_2 \times U(m), & \textit{if } s \in \{\frac{m}{4} - 1, \frac{3m}{4} - 1\} \end{cases}.$$

Proof. Let $\gcd(t, m) = 1$. Then, using (G2), we get, $s \equiv -1 \pmod{\frac{m}{4}}$ which implies that $s \in \{\frac{m}{4} - 1, \frac{m}{2} - 1, \frac{3m}{4} - 1, m - 1\}$. Now, using (G3), we get $t \equiv -1 \pmod{\frac{m}{4}}$. Then $t \in \{\frac{m}{4} - 1, \frac{m}{2} - 1, \frac{3m}{4} - 1, m - 1\}$.

Let $(\alpha, \delta) \in X$ be such that $\alpha(b) = b^i$, and $\delta(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. Then, using $\delta(a)^{\alpha(b)} = \delta(a^b)$, $a^{(2t+1)r} = \delta(a^{2t+1}) = \delta(a^b) = \delta(a)^{\alpha(b)} = (a^r)^{b^i} = a^{2t+1+(r-1)s+\frac{i-1}{2}2t(s+1)}$. Thus

$$(r - 1)(2t + 1 - s) \equiv \frac{i - 1}{2}2t(s + 1) \pmod{m}. \tag{2}$$

If $s \in \{\frac{m}{2} - 1, m - 1\}$, then the Equation (2) holds for all values of t and r. If $s \in \{\frac{m}{4} - 1, \frac{3m}{4} - 1\}$, then the Equation (2) holds for all t and $r \equiv i \pmod 4$. Thus, the choices for the maps α and δ are, $\alpha_i(b) = b^i$ and $\delta_r(a) = a^r$, for all $i \in \{1, 3\}$ and $r \in U(m)$. So, $X \simeq A \times D \simeq \mathbb{Z}_2 \times U(m)$. Now, if $s \in \{\frac{m}{2} - 1, m - 1\}$, then $2t(s+1) \equiv 0 \pmod m$. Therefore, using [3, Lemma 3.3, p. 9], $\mathrm{Im}(\beta) = \{b^2\}$ and so, $B \simeq \mathbb{Z}_2$. If $s \in \{\frac{m}{4} - 1, \frac{3m}{4} - 1\}$, then $2t(s + 1) \not\equiv 0 \pmod m$. Therefore, using [3, Lemma 3.3, p. 9], $\mathrm{Im}(\beta) = \{1\}$ and so, B is a trivial group. Hence, by the Theorem 3,

$$\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \begin{cases} \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m)), & \text{if } s \in \{\frac{m}{2}-1, m-1\} \\ \mathbb{Z}_2 \times U(m), & \text{if } s \in \{\frac{m}{4}-1, \frac{3m}{4}-1\} \end{cases}.$$

□

Theorem 5 *Let* $m = 2q$*, where* $q > 1$ *is odd and* $\gcd(t, m) = 1$*. Then,* $\mathrm{Aut}_H(G) \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m))$.

Proof. Using $(G1), (G2)$, and $(G3)$, we get $s, t \in \{\frac{m}{2}-1, m-1\}$. Then, the result follows on the lines of the proof of the Theorem 4. □

Theorem 6 *Let* $m = 2^n$*,* $n \geq 3$ *and* t *be even. Then*

$$\mathrm{Aut}_H(G) \simeq \begin{cases} (\mathbb{Z}_4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})) \rtimes \mathbb{Z}_2, & \textit{if } 2t(s+1) \equiv 0 \pmod{2^n} \\ \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})), & \textit{if } 2t(s+1) \not\equiv 0 \pmod{2^n} \end{cases}.$$

Proof. Let t be even. Then $2(t+1)(s-1) \equiv 0 \pmod{2^n}$ implies that $s \equiv 1 \pmod{2^{n-1}}$. Therefore, $s = 1, 2^{n-1}+1$. Now, $4t(s+1) \equiv 0 \pmod{2^n}$ implies that $t \equiv 0 \pmod{2^{n-3}}$. Therefore, by the defining relations of the group G, $t \in \{2^{n-3}, 2^{n-2}, 3\cdot 2^{n-3}, 2^{n-1}, 5\cdot 2^{n-3}, 3\cdot 2^{n-2}, 7\cdot 2^{n-3}, 2^n\}$. Note that, for $t = 2^{n-1}$ or $t = 2^n$, G is the semidirect product of H and K. So, we consider the other values of t.

Case(i). Let $t \in \{2^{n-2}, 3\cdot 2^{n-2}\}$. Then, one can easily observe that $2t(s+1) \equiv 0 \pmod{2^n}$. Therefore, for all $\alpha \in P$, $(a^l)^{\alpha(b)} = (a^l)^{b^i} = a^{2it+l} = a^{2t+l} = (a^l)^b$. Thus, $P \simeq A \simeq \mathbb{Z}_2$. Now, let $(\beta, \delta) \in Y$ be such that $\beta(a) = b^j$, and $\delta(a) = a^r$, where $0 \leq j \leq 3$ and $0 \leq r \leq 2^n - 1$ and r is odd. Using [3, Lemma 3.2 (ii), p. 7], $\beta(kk') = \beta(k)(\delta(k)\cdot\beta(k'))$ holds, for all $k, k' \in K$. Now, using $\delta(kk') = \delta(k)^{\beta(k')}\delta(k')$, we get

$$\delta(a^l) = \begin{cases} a^{(l-1)(jt+r)+r}, & \text{if } l \text{ is odd} \\ a^{l(jt+r)}, & \text{if } l \text{ is even} \end{cases}. \tag{3}$$

Finally, using $\delta(k^h) = \delta(k)^h$, $a^{2t+r} = (a^r)^b = \delta(a)^b = \delta(a^b) = \delta(a^{2t+1}) = a^{2t(jt+r)+r} = a^{2tr+r}$. Thus, $2t(r-1) \equiv 0 \pmod{2^n}$ which is true for all $r \in U(2^n)$. So, $Y \simeq B \rtimes D \simeq \mathbb{Z}_4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})$. Now, let $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in A$ and $\begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix} \in F$. Then

$$\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & \delta\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \alpha\beta\\ 0 & \delta\end{pmatrix} \in F.$$

Thus $F \lhd \mathcal{A}_H$. Clearly, $A \cap F = \{1\}$. Also, if $\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \in \mathcal{A}_H$, then

$$\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} = \begin{pmatrix}1 & \beta\\ 0 & \delta\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} \in FA.$$

Hence, $\mathcal{A}_H = F \rtimes A$ and so, $\mathrm{Aut}_H(G) \simeq F \rtimes A \simeq (\mathbb{Z}_4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})) \rtimes \mathbb{Z}_2$.

$\mathtt{Case}(\mathrm{ii})$. Let $t \in \{2^{n-3}, 3\cdot 2^{n-3}, 5\cdot 2^{n-3}, 7\cdot 2^{n-3}\}$. Then, one can easily observe that $2t(s+1) \not\equiv 0 \pmod{2^n}$. Let $(\alpha,\beta,\delta) \in \mathrm{Aut}_H(G)$ be such that $\alpha(b) = b^i, \beta(a) = b^j$, and $\delta(a) = a^r$, where $i \in \{1,3\}$, $0 \le j \le 3$, $0 \le r \le 2^n - 1$ and r is odd. Using [3, Lemma 3.2 (ii), p. 7], $\beta(kk') = \beta(k)(\delta(k)\cdot\beta(k'))$ holds, for all $k, k' \in K$. Now, using (A5), for any $b^j a^l \in G$ there is unique $b^j \in H$ such that $b^j = \alpha(b^j)\beta(a^l)$. Note that, if $\alpha(b^j) = b^j$, then $\beta(a^l) = 1$ and if $\alpha(b^j) = b^{-j}$, then $\beta(a^l) = b^2$. Thus, $\mathrm{Im}(\beta) = \langle b^2\rangle$.

Finally, using the definition of the map δ in the Equation (3) and $\delta(k^h) = \delta(k)^{\alpha(h)}$, we get $a^{2it+r} = (a^r)^{b^i} = \delta(a)^{\alpha(b)} = \delta(a^b) = \delta(a^{2t+1}) = a^{2t(jt+r)+r}$. Thus, $2t(jt + r - i) \equiv 0 \pmod{2^n}$ which implies that

$$\begin{cases} r \equiv i \pmod 4, & \text{if } t \in \{2^{n-3}, 3\cdot 2^{n-3}, 5\cdot 2^{n-3}, 7\cdot 2^{n-3}\} \text{ and } n \ge 5\\ r \equiv i + 2j \pmod 4, & \text{if } t \in \{2^{n-3}, 3\cdot 2^{n-3}, 5\cdot 2^{n-3}, 7\cdot 2^{n-3}\} \text{ and } n = 4\end{cases}.$$

Thus $r \equiv i \pmod 4$ and so, the choices for the maps α and δ are, $\alpha_i(b) = b^i$ and $\delta_r(a) = a^r$, where $i \in \{1,3\}$ and $r \in U(m)$. Note that, if $\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \in \mathcal{A}_H$, then

$$\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & \alpha^{-1}\beta\\ 0 & 1\end{pmatrix} \in EB.$$

Clearly, $E \cap B = \{1\}$ and E normalizes B. So, $B \lhd \mathcal{A}_H$. Hence, $\mathcal{A}_H = B \rtimes E$ and so, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}))$. □

Now, we will discuss the structure of the automorphism group $\mathrm{Aut}_H(G)$ in the case when $\gcd(t, m) > 1$.

Theorem 7 *Let* $m = 4q$, *where* $q > 1$ *is odd and* $\gcd(t, m) = 2^i d$, *where* $i \in \{0, 1, 2\}$, *and* d *divides* q. *Then* $\mathrm{Aut}_H(G) \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m))$.

Proof. Let $q = du$, for some integer u. Then, using (G2), $s \equiv -1 \pmod{u}$ which implies that $s = lu - 1$, where $1 \leq l \leq 4d$. Since, $\gcd(s, \frac{m}{2}) = 1$, s is odd and so, l is even. Using (G1) and (G3), we get $l(u\frac{l}{2} - 1) \equiv 0 \pmod{d}$ and $t+1 \equiv u\frac{l}{2} \pmod{q}$. Now, one can easily observe that $\gcd(l, d) = 1$ which implies that $u\frac{l}{2} - 1 \equiv 0 \pmod{d}$. Thus, $2t(s+1) \equiv 2ltu \equiv 0 \pmod{m}$ and $\gcd(s+1, \frac{m}{2}) \neq 1$. Therefore, using [3, Lemma 3.3, p. 9], $B \simeq \mathbb{Z}_2$.

Let $(\alpha, \delta) \in X$ be such that $\alpha(b) = b^i$ and $\delta(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. Then, using $\delta(a)^{\alpha(b)} = \delta(a^b)$ and the fact that $2t(s+1) \equiv 0 \pmod{m}$, we get $a^{(2t+1)r} = \delta(a^{2t+1}) = \delta(a^b) = \delta(a)^{\alpha(b)} = (a^r)^{b^i} = a^{2t+1+(r-1)s+\frac{i-1}{2}2t(s+1)} = a^{2t+1+(r-1)s}$. Thus

$$(r-1)(s-2t-1) \equiv 0 \pmod{m}. \tag{4}$$

Since $2t(s+1) \equiv 0 \pmod{m}$, using (G3), we get $2(s-2t-1) \equiv 0 \pmod{m}$. Therefore, the Equation (4) holds for all $r \in U(m)$. Thus, using the Theorem 2, $X \simeq A \times D \simeq \mathbb{Z}_2 \times U(m)$. Hence, using the Theorem 3, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m))$. □

Theorem 8 *Let* $m = 2q$, *where* $q > 1$ *is odd and* $\gcd(t, m) = 2^i d$, *where* $i \in \{0, 1\}$, *and* d *divides* q. *Then* $\mathrm{Aut}_H(G) \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m))$.

Proof. Follows on the lines of the proof of the Theorem 7. □

Theorem 9 *Let* $m = 2^n q$, t *be even and* $\gcd(m, t) = 2^i d$, *where* $1 \leq i \leq n$, $n \geq 3$, $q > 1$ *and* d *divides* q. *Then*

$$\mathrm{Aut}_H(G) \simeq \begin{cases} (\mathbb{Z}_4 \rtimes U(m)) \rtimes \mathbb{Z}_2, & \text{if } d = q \text{ and } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m)), & \text{if } d = q \text{ and } 2t(s+1) \not\equiv 0 \pmod{m} \\ \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m)), & \text{if } d \neq q \text{ and } n-2 \leq i \leq n \\ \mathbb{Z}_2 \times U(m), & \text{if } d \neq q \text{ and } i = n-3 \end{cases}.$$

Proof. We consider the following four cases to find the structure of $\mathrm{Aut}_H(G)$.

Case(i): Let $d = q$ and $\gcd(t+1, m) = u$. Since, $t+1$ is odd, u is odd and u divides q. Thus, u divides t and so, $u = 1$. Therefore, using (G2) and (G3), $s \equiv 1 \pmod{\frac{m}{2}}$ and $t \equiv 0 \pmod{\frac{m}{8}}$. By the similar argument used in the proof of the Theorem 6 (i), we get,

$$\mathrm{Aut}_H(G) \simeq \begin{cases} (\mathbb{Z}_4 \rtimes U(m)) \rtimes \mathbb{Z}_2, & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m)), & \text{if } 2t(s+1) \not\equiv 0 \pmod{m} \end{cases}.$$

Case(ii): Let $n-2 \leq i \leq n$ $d \neq q$ and $q = du$, for some odd integer u. Then using (G2), $s \equiv -1 \pmod{u}$ and so, $s = lu - 1$, where $0 \leq l \leq 2^n d$. Since, $\gcd(s, \frac{m}{2}) = 1$, s is odd and so, l is even. Now, using (G1), $\frac{l}{2}(\frac{l}{2}u - 1) \equiv 0 \pmod{2^{n-3}d}$ and by (G3), $t \equiv \frac{l}{2}u - 1 \pmod{2^{n-2}q}$. Since, t is even, $\frac{l}{2}$ is odd and $\gcd(\frac{l}{2}, d) = 1$. Thus, $\frac{l}{2}u \equiv 1 \pmod{2^{n-3}d}$ and $t \equiv 2^i d \pmod{2^{n-2}q}$. One can easily observe that $2t(s+1) \equiv 0 \pmod{m}$. Therefore, using the similar argument as in the proof of the Theorem 4, we get, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m))$.

Case(iii): Let $i = n-3$, $d \neq q$ and $q = du$, for some odd integer u. Then using (G2), $s \equiv -1 \pmod{2u}$, that is, $s = 2lu - 1$, where $1 \leq l \leq 2^{n-1}d$. Now, using (G1) and (G3), $l(lu-1) \equiv 0 \pmod{2^{n-3}d}$ and $(t+1)(lu-1) \equiv 0 \pmod{2^{n-2}q}$. If l is even, then $t \equiv lu - 1 \pmod{2^{n-2}q}$ gives that t is odd, which is a contradiction. Therefore, l is odd. Using $(t+1)(lu-1) \equiv 0 \pmod{2^{n-2}q}$, one can easily observe that $\gcd(l, d) = 1$. Then, $lu - 1 = 2^{n-3}dl'$ and $s = 2^{n-2}dl' + 1$, where $1 \leq l' \leq 8u$. Clearly, $\gcd(l', u) = 1$. Thus, $(t+1)l' \equiv 0 \pmod{2u}$. If l' is odd, then $(t+1) \equiv 0 \pmod{2u}$ which implies that t is odd. So, l' is even and so, $t = uq' - 1$, $1 \leq q' < 2^{n-1}d$, q' is odd as t is even. Note that $s - 2t - 1 = 2^{n-2}dl' - 2t = 2^{n-2}d(l' - \frac{t}{2^{n-3}d}) = 2^{n-2}d\left(\frac{lu-1}{2^{n-3}d} - \frac{uq'-1}{2^{n-3}d}\right) = 2^{n-2}du\left(\frac{l-q'}{2^{n-3}d}\right)$.

Let $(\alpha, \delta) \in X$ be such that $\alpha(b) = b^i$, and $\delta(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. Then, using $\delta(a)^{\alpha(b)} = \delta(a^b)$, $a^{(2t+1)r} = \delta(a^{2t+1}) = \delta(a^b) = \delta(a)^{\alpha(b)} = (a^r)^{b^i} = a^{2t+1+(r-1)s+\frac{i-1}{2}2t(s+1)}$. Thus

$$(r-1)(2t+1-s) \equiv \frac{i-1}{2} 2t(s+1) \pmod{m}.$$

Therefore, $-2^{n-2}du(r-1)\left(\frac{l-q'}{2^{n-3}d}\right) \equiv \frac{i-1}{2}(4tul) \pmod{2^n q}$ which implies that $-(r-1)\left(\frac{l-q'}{2^{n-3}d}\right) \equiv (i-1)l \pmod{4}$. Since, $\frac{l-q'}{2^{n-3}d}$ and l is odd, $r \equiv i \pmod{4}$. Thus, the choices for the maps α and δ are, $\alpha_i(b) = b^i$ and $\delta_r(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. So, $X \simeq A \times D \simeq \mathbb{Z}_2 \times U(m)$. At last, since, l is odd, $2t(s+1) \equiv 4tlu \not\equiv 0 \pmod{m}$. Therefore, using [3, Lemma 3.3, p. 9], $\mathrm{Im}(\beta) = \{1\}$. Thus, B is a trivial group. Hence, using the Theorem 3, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \times U(m)$.

Case(iv): Let $1 \leq i \leq n-4$. and $q = du$, for some odd integer u. Then using (G2), $s \equiv -1 \pmod{2^{n-i-2}u}$, that is, $s = 2^{n-i-2}lu - 1$, where $1 \leq l \leq 2^{i+2}d$. Now, using (G1) and (G3), $l(2^{n-i-3}lu - 1) \equiv 0 \pmod{2^i d}$ and $(t+1)(lu2^{n-i-3} - 1) \equiv 0 \pmod{2^{n-2}q}$. Since, $n - i - 3 > 0$, $lu2^{n-i-3} - 1$ is odd. If l is even, then $t \equiv lu2^{n-i-3} - 1 \pmod{2^{n-2}q}$ gives that t is odd, which is a contradiction. Now, if l is odd, then Using $(t+1)(lu-1) \equiv 0 \pmod{2^{n-2}q}$, one can easily observe that $\gcd(l, d) = 1$. Thus, $2^{n-i-3}lu - 1 \equiv 0 \pmod{2^i d}$, which is impossible. Hence, there is no such l exist and so, no such t and s exist and hence no group G exists as the Zappa-Szép product of H and K in this case. □

Theorem 10 *Let* $m = 2^n q$, t *be odd and* $\gcd(t, m) = d$, *where* $n \geq 4$ *and* q *is odd. Then*

$$\mathrm{Aut}_H(G) \simeq \begin{cases} \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m)), & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \times U(m), & \text{if } 2t(s+1) \not\equiv 0 \pmod{m} \end{cases}.$$

Proof. Let $q = du$, for some odd integer u. Then using (G2), we have $s \equiv -1 \pmod{2^{n-2}u}$ which implies that $s = 2^{n-2}lu - 1$, where $1 \leq l \leq 4d$. Now, using (G1), $l(2^{n-3}ul - 1) \equiv 0 \pmod{d}$. Using (G3), we get

$$(t+1)(lu2^{n-3} - 1) \equiv 0 \pmod{2^{n-2}q}. \tag{5}$$

Case(i): If l is even, then by the Equation (5), $t \equiv lu2^{n-3} - 1 \pmod{2^{n-2}q}$. Note that, $2t(s+1) \equiv 2t(2^{n-2}lu) \equiv 0 \pmod{m}$. Using the similar argument as in the proof of the Theorem 4, we get $X \simeq A \times D \simeq \mathbb{Z}_2 \times U(m)$ and $B \simeq \mathbb{Z}_2$. Hence, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m))$.

Case(ii): If l is odd, then using the Equation (5), one can easily observe that $\gcd(l, d) = 1$ which means that $2^{n-3}lu - 1 = dl'$, where l' is odd, $\gcd(l', u) = 1$ and $1 \leq l' \leq 2^n u$. Thus, using the Equation (5), $(t+1)dl' \equiv 0 \pmod{2^{n-2}q}$. Since, $\gcd(l', u) = 1$, $t = 2^{n-2}uq' - 1$, where $1 \leq q' \leq 4d$. Now, $s - 2t - 1 = 2dl' - 2t = 2d(l' - \frac{t}{d}) = 2d(\frac{2^{n-3}ul - 2^{n-2}uq'}{d}) = 2^{n-2}du\frac{l-2q'}{d}$.

Let $(\alpha, \delta) \in X$ be such that $\alpha(b) = b^i$, and $\delta(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. Then, using $\delta(a)^{\alpha(b)} = \delta(a^b)$, $a^{(2t+1)r} = \delta(a^{2t+1}) = \delta(a^b) = \delta(a)^{\alpha(b)} = (a^r)^{b^i} = a^{2t+1+(r-1)s+\frac{i-1}{2}2t(s+1)}$. Thus

$$(r-1)(2t+1-s) \equiv \frac{i-1}{2} 2t(s+1) \pmod{m}.$$

Therefore, $-2^{n-2}du(r-1)\left(\frac{l-2q'}{d}\right) \equiv (i-1)2^{n-2}tul \pmod{2^n q}$ which implies that $-(r-1)\left(\frac{l-2q'}{d}\right) \equiv (i-1)l \pmod{4}$. Since, $\frac{l-2q'}{d}$ and l is odd, $r \equiv i \pmod{4}$. Thus, the choices for the maps α and δ are, $\alpha_i(b) = b^i$ and $\delta_r(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. So, $X \simeq A \times D \simeq \mathbb{Z}_2 \times U(m)$. At last, since, l is odd, $2t(s+1) \equiv 2^{n-1}tlu \not\equiv 0 \pmod{m}$. Therefore, using [3, Lemma 3.3,

p. 9], $\mathrm{Im}(\beta) = \{1\}$. Thus, B is a trivial group. Hence, using the Theorem 3, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \times U(m)$. □

Theorem 11 *Let* $m = 8q$, t *is odd, and* $\gcd(t, m) = d$, *where* $q > 1$ *is odd. Then*

$$\mathrm{Aut}_H(G) \simeq \begin{cases} \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m)), & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \times U(m), & \text{if } 2t(s+1) \not\equiv 0 \pmod{m} \end{cases}.$$

Proof. Let $q = du$, for some odd integer u. Then using (G2), $s \equiv -1 \pmod{2u}$ which implies that $s = 2lu - 1$, where $1 \le l \le 4d$. Now, using (G1), $l(lu-1) \equiv 0 \pmod{d}$. Using (G3), we get

$$(t+1)(lu-1) \equiv 0 \pmod{2q}. \tag{6}$$

Case(i)*:* If l is even, then by the Equation (6), $t \equiv lu - 1 \pmod{2q}$. Note that, $2t(s+1) \equiv 2t(2lu) \equiv 0 \pmod{m}$. Using the similar argument as in the proof of the Theorem 4, we get $X \simeq A \times D \simeq \mathbb{Z}_2 \times U(m)$ and $B \simeq \mathbb{Z}_2$. Hence, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times U(m))$.

Case(ii)*:* If l is odd, then using the Equation (6), one can easily observe that $\gcd(l, d) = 1$ which means that $lu - 1 = dl'$, where $1 \le l' \le 8u$ and $\gcd(l', u) = 1$. Since $lu - 1$ is even, l' is even. Thus using the Equation (6), $(t+1)dl' \equiv 0 \pmod{2q}$. Since, $\gcd(l', u) = 1$, $t = uq' - 1$, where $1 \le q' \le 8d$ and q' is even, as t is odd. Now, $s - 2t - 1 = 2dl' - 2t = 2d(l' - \frac{t}{d}) = 2d(\frac{ul - uq'}{d}) = 2du\frac{l-q'}{d}$.

Let $(\alpha, \delta) \in X$ be such that $\alpha(b) = b^i$, and $\delta(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. Then, using $\delta(a)^{\alpha(b)} = \delta(a^b)$, $a^{(2t+1)r} = \delta(a^{2t+1}) = \delta(a^b) = \delta(a)^{\alpha(b)} = (a^r)^{b^i} = a^{2t+1+(r-1)s+\frac{i-1}{2}2t(s+1)}$. Thus

$$(r-1)(2t+1-s) \equiv \frac{i-1}{2}2t(s+1) \pmod{m}.$$

Therefore, $-2du(r-1)\left(\frac{l-q'}{d}\right) \equiv (i-1)2tul \pmod{8q}$ which implies that $-(r-1)\left(\frac{l-q'}{d}\right) \equiv (i-1)l \pmod{4}$. Since, $\frac{l-q'}{d}$ and l is odd, $r \equiv i \pmod{4}$. Thus, the choices for the maps α and δ are, $\alpha_i(b) = b^i$ and $\delta_r(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. So, $X \simeq A \times D \simeq \mathbb{Z}_2 \times U(m)$. At last, since, l is odd, $2t(s+1) \equiv 4tlu \not\equiv 0 \pmod{8q}$. Therefore, using [3, Lemma 3.3, p. 9], $\mathrm{Im}(\beta) = \{1\}$. Thus, B is a trivial group. Hence, using the Theorem 3, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_2 \times U(m)$. □

4 $\mathrm{Aut}_H(G)$ of Zappa-Szép product of groups $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_m$, p is odd prime

In [9], Yacoub classified the groups which are Zappa-Szép product of cyclic groups of order p^2 and order m. He found that these are of the following type (see [9, Conclusion, p. 38])

$$\begin{aligned}
M_1 =& \langle a, b \mid a^m = 1 = b^{p^2}, ab = ba^u, u^{p^2} \equiv 1 \pmod m\rangle,\\
M_2 =& \langle a, b \mid a^m = 1 = b^{p^2}, ab = b^t a, t^m \equiv 1 \pmod{p^2}\rangle,\\
M_3 =& \langle a, b \mid a^m = 1 = b^{p^2}, ab = b^t a^{pr+1}, a^p b = ba^{p(pr+1)}\rangle,
\end{aligned}$$

where p is an odd prime and in M_3, p divides m. These are not non isomorphic classes. The groups M_1 and M_2 may be isomorphic to the group M_3 depending on the values of m, r and t. Clearly, M_1 and M_2 are semidirect products. Throughout this section G will denote the group M_3 and we will be only concerned about groups M_3 which are the Zappa-Szép product but not the semidirect product. Note that $G = H \bowtie K$, where $H = \langle b\rangle$ and $K = \langle a\rangle$. For the group G, the mutual actions of H and K are defined by $a \cdot b = b^t, a^b = a^{pr+1}$ along with $a^p \cdot b = b$ and $(a^p)^b = a^{p(pr+1)}$, where t and r are integers satisfying the conditions

(G1) $\gcd(t-1, p^2) = p$, that is, $t = 1 + \lambda p$, where $\gcd(\lambda, p) = 1$,

(G2) $\gcd(r, p) = 1$,

(G3) $p(pr+1)^p \equiv p \pmod m$.

Theorem 12 *Let* G *be as above. Then* $\mathrm{Aut}_H(G) \simeq \mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \tilde{D})$, *where* $\tilde{D}$ *is a subgroup of* $U(m)$ *of order* $\frac{\phi(m)}{p-1}$.

Proof. Let $\beta \in Q$. Then using [3, Lemma 4.4 (i), p. 22], we have that $\beta(a^l) = b^{jl}$, where $j \equiv 0 \pmod p$. Thus, $B \simeq \mathbb{Z}_p$. Now, let $(\alpha, \delta) \in X$ be such that $\alpha(b) = b^i$ and $\delta(a) = a^s$, where $i \in U(\mathbb{Z}_{p^2})$ and $s \in U(m)$.

Now, $\delta(k) \cdot \alpha(h) = \alpha(k \cdot h)$, $b^{it} = \alpha(b^t) = \alpha(a \cdot b) = \delta(a) \cdot \alpha(b) = a^s \cdot b^i = b^{it^s}$. Thus, $it^s \equiv it \pmod{p^2}$ which implies that $(1 + p\lambda)^{s-1} \equiv 1 \pmod{p^2}$. Therefore, $s \equiv 1 \pmod p$. Using $\delta(k)^{\alpha(h)} = \delta(k^h)$, (G3) and the fact that $s \equiv 1 \pmod p$, we get, $a^{(pr+1)s} = \delta(a^{pr+1}) = \delta(a^b) = \delta(a)^{\alpha(b)} = (a^s)^{b^i} =$

$a^{\frac{is(s-1)}{2}((pr+1)^{\lambda p}-1)+s(pr+1)^i} = a^{s(pr+1)^i}$. Thus $(pr+1)s \equiv s(pr+1)^i \pmod{m}$. Therefore, $i \equiv 1 \pmod{p}$.

Thus, the choices for the maps α and δ are, $\alpha_i(b) = b^i$ and $\delta_s(a) = a^s$, where $i \in U(p^2)$, $i \equiv 1 \pmod{p}$, $s \in U(m)$, and $s \equiv 1 \pmod{p}$. So, $X \simeq A \times D \simeq \mathbb{Z}_p \times \tilde{D}$, where $\tilde{D}$ is a subgroup of $U(m)$ of order $\frac{\phi(m)}{p-1}$. Hence, using the Theorem 3, $\mathrm{Aut}_H(G) \simeq B \rtimes E \simeq \mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \tilde{D}))$. □

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