Acta Univ. Sapientiae, Mathematica, **15**, 1 (2023) 91–108

DOI: 10.2478/ausm-2023-0006

On a new $p(x)$-Kirchhoff type problems with $p(x)$-Laplacian-like operators and Neumann boundary conditions

Mohamed El Ouaarabi
Laboratory LMACS,
Faculty of Science and Technology of Beni Mellal,
Sultan Moulay Slimane University,
Beni Mellal, BP 523, 23000, Morocco
email: mohamed.elouaarabi@usms.ma

Chakir Allalou
Laboratory LMACS, Faculty of Science and Technology of Beni Mellal,
Sultan Moulay Slimane University,
Beni Mellal, BP 523, 23000, Morocco
email: c.allalou@usms.ma

Said Melliani
Laboratory LMACS, Faculty of Science and Technology of Beni Mellal,
Sultan Moulay Slimane University,
Beni Mellal, BP 523, 23000, Morocco
email: s.melliani@usms.ma

Abstract. In this paper we study a Neumann boundary value problem of a new $p(x)$-Kirchhoff type problems driven by $p(x)$-Laplacian-like operators. Using the theory of variable exponent Sobolev spaces and the method of the topological degree for a class of demicontinuous operators of generalized (S_+) type, we prove the existence of a weak solutions of this problem. Our results are a natural generalisation of some existing ones in the context of $p(x)$-Kirchhoff type problems.

2010 Mathematics Subject Classification: 35J60, 35J70, 35D30, 47H11
Key words and phrases: $p(x)$-Kirchhoff type problems, $p(x)$-Laplacian-like operators, Neumann boundary conditions, topological degree methods, weak solutions, variable exponent Sobolev spaces

1 Introduction and motivation

The study of differential equations and variational problems with nonlinearities and nonstandard $p(x)$-growth conditions or nonstandard $(p(x), q(x))-$ growth conditions have received a lot of attention. Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents (see [5, 9, 11, 21]). The motivation for this research comes from the application of similar problems in physics to model the behavior of elasticity [26] and electrorheological fluids [24], which have the ability to modify their mechanical properties when exposed to an electric field (see [19, 20, 22, 23]), specifically the phenomenon of capillarity, which depends solid-liquid interfacial characteristics as surface tension, contact angle, and solid surface geometry.

Let Ω be a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary denoted by $\partial\Omega$, and let λ be a real parameters, $\delta \in L^\infty(\Omega)$ and $p(\cdot), \alpha(\cdot) \in C_+(\overline{\Omega})$ such that the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $a > 0$ such that for every $x, y \in \Omega, x \neq y$ with $|x - y| \leq \dfrac{1}{2}$ one has

$$|p(x) - p(y)| \leq \frac{a}{-\log|x-y|}. \tag{1}$$

In this paper, we consider a certain class of $p(x)$-Kirchhoff type problems involving the $p(x)$-Laplacian-like operators under Neumann boundary conditions of the following form:

$$\begin{cases} -M\Big(\Theta(u)\Big)\Big(\Delta^l_{p(x)}u - |u|^{p(x)-2}u\Big) + \delta(x)|u|^{\alpha(x)-2}u = \lambda f(x, u, \nabla u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \eta}(\Delta^l_{p(x)}u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where

$$\Theta(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} + |u|^{p(x)}\Big)dx,$$

and

$$\Delta^l_{p(x)}u := \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)$$

is the $p(x)$-Laplacian-like operators, η is the outer unit normal to $\partial\Omega$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that satisfies the assumption of growth and $M(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

Studying this type of problems is both significant and relevant. In the one hand, we have the physical motivation; since the $p(x)$-Laplacian-like operators and $p(x)$-Laplacian operators has been used to model the steady-state solutions of reactiondiffusion problems, that arise in biophysic, plasma-physic and in the study of chemical reactions. In the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point (see [1, 6, 10, 21, 24] and the references given there).

Problems related to (2) have been studied by many scholars, for example, Ni and Serrin [18] study the following equation

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(u) \quad \text{in } \mathbb{R}^N.$$

The operator $-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)$ is most often denoted by the specified mean curvature operator and $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ is the Kirchhoff stress term.

Elliptic boundary value problems involving the mean curvature operator play apivotal role in the mathematical analysis of several physical or geometrical issues,such as capillarity phenomena for incompressible or compressible fluids, mathematicalmodels in physiology or in electrostatics, flux-limited diffusion phenomena, prescribedmean curvature problems for Cartesian surfaces in the Euclidean space: relevantreferences on these topics include [3, 7, 8, 13].

Note that, in the case when $\Theta(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)})dx$, $\delta \equiv 0$, $\lambda = 1$, f independent of ∇u and without the term $|u|^{p(x)-2}u$ with Dirichlet boundary condition, then we obtain the following problem

$$\begin{cases} -M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)})dx\Big)\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

which is called the $p(x)$-Kirchhoff type problem.

In this case, Dai et al. [4], by a direct variational approach, established conditions ensuring the existence and multiplicity of solutions to (3). Furthermore, the problem (3) is a generalization of the stationary problem of a model introduced by Kirchhoff [15] of the following form:

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{4}$$

where ρ, ρ_0, h, E, L are all constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration.

In the present paper, we will generalized these works, by proving, under some conditions on M and f, the existence of weak solutions for the problem (2). Note that the problem (2) does not have a variational structure, so the most usual variational methods can not used to study it. To attack it we will employ a topological degree for a class of demicontinuous operators of generalized (S_+) type of [2].

The remainder of the paper is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problem. In Section 3, we introduce some classes of operators of generalized (S_+) type, as well as the Berkovits topological degrees. Finaly, in the Section 4, we give our basic assumptions, some technical lemmas, and we will state and prove the main result of the paper.

2 Preliminaries

In order to deal with the Problem (2), we need some theory of the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [12, 16] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. Denote

$$C_+(\overline{\Omega}) = \Big\{ h : h \in C(\overline{\Omega}) \text{ such that } h(x) > 1 \text{ for any } x \in \overline{\Omega} \Big\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ := \max \Big\{ h(x),\ x \in \overline{\Omega} \Big\} \quad \text{and} \quad h^- := \min \Big\{ h(x),\ x \in \overline{\Omega} \Big\}.$$

For any $p \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \ \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \Big\},$$

with the norm

$$|u|_{p(x)} = \inf \Big\{ \lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leq 1 \Big\}.$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx,\ \forall\, u \in L^{p(x)}(\Omega).$$

Proposition 1 *[12] Let (u_n) and $u \in L^{p(x)}(\Omega)$, then*

$$|u|_{p(x)} < 1(\text{resp.} = 1; > 1) \;\Leftrightarrow\; \rho_{p(x)}(u) < 1(\text{resp.} = 1; > 1), \tag{5}$$

$$|u|_{p(x)} > 1 \;\Rightarrow\; |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{6}$$

$$|u|_{p(x)} < 1 \;\Rightarrow\; |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{7}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{8}$$

Remark 1 *Notice that, from (6) and (7), we can deduce the inequalities*

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{9}$$

$$\rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} + |u|_{p(x)}^{p^+}. \tag{10}$$

Proposition 2 *[16] The space $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ is a separable and reflexive Banach space.*

Proposition 3 *[16] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\left|\int_\Omega uv \;dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{11}$$

Remark 2 *If $p_1, p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*

Now, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$|u|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Furthermore, we have the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$ (see [16]).

Remark 3 *Note that for all $u \in W^{1,p(x)}(\Omega)$, we have*

$$|u|_{p(x)} \leq |u|_{1,p(x)} \quad \text{and} \quad |\nabla u|_{p(x)} \leq |u|_{1,p(x)}.$$

Next, for all $u \in W^{1,p(x)}(\Omega)$, we introduce the following notation

$$\rho_{1,p(x)}(u) = \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u).$$

Then, from [12, Theorem 1.3], we have the following result.

Proposition 4 *If* $u \in W^{1,p(x)}(\Omega)$, *then the following properties hold true*

$$|u|_{1,p(x)} < 1(\text{resp.} = 1; > 1) \;\Leftrightarrow\; \rho_{1,p(x)}(u) < 1(\text{resp.} = 1; > 1), \tag{12}$$

$$|u|_{1,p(x)} > 1 \;\Rightarrow\; |u|_{1,p(x)}^{p^-} \leq \rho_{1,p(x)}(u) \leq |u|_{1,p(x)}^{p^+}, \tag{13}$$

$$|u|_{1,p(x)} < 1 \;\Rightarrow\; |u|_{1,p(x)}^{p^+} \leq \rho_{1,p(x)}(u) \leq |u|_{1,p(x)}^{p^-}. \tag{14}$$

Proposition 5 *[12, 16] The space* $\left(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ *is a separable and reflexive Banach space.*

Remark 4 *The dual space of* $W^{1,p(x)}(\Omega)$ *denoted* $W^{-1,p'(x)}(\Omega)$, *is equipped with the norm*

$$|u|_{-1,p'(x)} = \inf\left\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\right\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div}F$ *with* $u_0 \in L^{p'(x)}(\Omega)$ *and* $F = (u_1, \ldots, u_N) \in (L^{p'(x)}(\Omega))^N$.

3 A review on the topological degree theory

We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot \rangle$ and given a nonempty subset $\mathcal{D}$ of X. Strong (weak) convergence is represented by the symbol $\rightarrow$ ($\rightharpoonup$).

Definition 1 *Let* Y *be another real Banach space. An operator* $F : \mathcal{D} \subset X \rightarrow Y$ *is said to be*

1. *bounded, if it maps any bounded set to a bounded set.*

2. *demicontinuous, if* $(u_n) \subset \mathcal{D}$, *and* $u_n \rightarrow u$ *in* X *as* $n \rightarrow \infty$, *then* $F(u_n) \rightharpoonup F(u)$ *in* Y.

3. *compact, if it is continuous and the image of any bounded set in* X *is relatively compact in* Y.

Definition 2 *A mapping* $F : \mathcal{D} \subset X \to X^*$ *is said to be*

1. *of class* (S_+), *if for any sequence* $(u_n) \subset \mathcal{D}$ *with* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty} \langle Fu_n, u_n - u\rangle \leq 0$, *we have* $u_n \to u$ *in* X.

2. *quasimonotone, if for any sequence* $(u_n) \subset \mathcal{D}$ *with* $u_n \rightharpoonup u$ *in* X, *we have* $\limsup_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 3 *Let* $T : \mathcal{D}_1 \subset X \to X^*$ *be a bounded operator such that* $\mathcal{D} \subset \mathcal{D}_1$. *For any operator* $F : \mathcal{D} \subset X \to X$ *we say that*

1. F *of class* $(S_+)_T$, *if for any sequence* $(u_n) \subset \mathcal{D}$ *with* $u_n \rightharpoonup u$ *in* X, $y_n := Tu_n \rightharpoonup y$ *in* X^* *and* $\limsup_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \leq 0$, *we have* $u_n \to u$ *in* X.

2. F *has the property* $(QM)_T$, *if for any sequence* $(u_n) \subset \mathcal{D}$ *with* $u_n \rightharpoonup u$ *in* X, $y_n := Tu_n \rightharpoonup y$ *in* X^*, *we have* $\limsup_{n\to\infty} \langle Fu_n, y - y_n\rangle \geq 0$.

In the sequel, we consider the following classes of operators:

$$\mathcal{F}_1(\mathcal{D}) := \Big\{F : \mathcal{D} \to X^* : F \text{ is bounded, demicontinuous and of class } (S_+)\Big\},$$
$$\mathcal{F}_T(\mathcal{D}) := \Big\{F : \mathcal{D} \to X : F \text{ is demicontinuous and of class } (S_+)_T\Big\},$$
$$\mathcal{F}_{T,B}(\mathcal{D}) := \Big\{F \in \mathcal{F}_T(\mathcal{D}) : F \text{ is bounded }\Big\},$$

for any $\mathcal{D} \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\mathcal{D})$. Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) : E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 1 *[14, Lemma 2.3] Let* E *be a bounded open set in a real reflexive Banach space* X, *and let* $T \in \mathcal{F}_1(\overline{E})$ *be a continuous operator. Let* $S : D(S) \subset X^* \to X$ *be a demicontinuous operator, such that* $T(\overline{E}) \subset D(S)$. *Then, the following statements hold.*

1. *If* S *is quasimonotone, then* $\mathcal{I} + S \circ T \in \mathcal{F}_T(\overline{E})$*, where* $\mathcal{I}$ *denotes the identity operator.*

2. *If* S *is of class* (S_+)*, then* $S \circ T \in \mathcal{F}_T(\overline{E})$*.*

Definition 4 *Suppose that* E *is bounded open subset of a real reflexive Banach space* X*,* $T \in \mathcal{F}_1(\overline{E})$ *is continuous and* $F, S \in \mathcal{F}_T(\overline{E})$*. Then the affine homotopy* $\mathcal{H} : [0,1] \times \overline{E} \to X$ *defined by*

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \textit{for all} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T*.*

Remark 5 *[14, Lemma 2.5]* The above affine homotopy is of class $(S_+)_T$.

As in [14] we give the topological degree for the class $\mathcal{F}(X)$.

Theorem 1 *Let*

$$M = \Big\{(F, E, h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ *that satisfies the following properties:*

1. *(Normalization) For any* $h \in F(E)$*, we have*
$$d(\mathcal{I}, E, h) = 1.$$

2. *(Additivity) Let* $F \in \mathcal{F}_{T,B}(\overline{E})$*. If* E_1 *and* E_2 *are two disjoint open subsets of* E *such that* $h \notin F(\overline{E} \backslash (E_1 \cup E_2))$*, then we have*
$$d(F, E, h) = d(F, E_1, h) + d(F, E_2, h).$$

3. *(Homotopy invariance) If* $\mathcal{H} : [0,1] \times \overline{E} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* h*:* $[0,1] \to X$ *is a continuous path in* X *such that* $h(t) \notin \mathcal{H}(t, \partial E)$ *for all* $t \in [0,1]$*, then*
$$d(\mathcal{H}(t,\cdot), E, h(t)) = C \textit{ for all } t \in [0,1].$$

4. *(Existence) If* $d(F, E, h) \neq 0$*, then the equation* $Fu = h$ *has a solution in* E*.*

Definition 5 *[14, Definition 3.3] The above degree is defined as follows:*

$$d(F,E,h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B *is the Berkovits degree and* E_0 *is any open subset of* E *with* $F^{-1}(h) \subset E_0$ *and* F *is bounded on* $\overline{E}_0$*.*

4 Weak solutions

In this section, we will discuss the existence of weak solutions of (2). We assume that $\Omega \subset \mathbb{R}^N (N > 1)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1), $\alpha \in C_+(\overline{\Omega})$ with $2 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < p^-$, $\delta \in L^\infty(\Omega)$, $M(t) : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(M_0) $M(t) : [0,+\infty) \to (m_0,+\infty)$ is a continuous-increasing function with $m_0 > 0$.

(A_1) f is a Carathéodory function.

(A_2) There exists $C_1 > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,\zeta,\xi)| \leq C_1(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}),$$

for all $(\zeta,\xi) \in \mathbb{R}\times\mathbb{R}^N$, where $q \in C_+(\overline{\Omega})$ with $2 \leq q^- \leq q(x) \leq q^+ < p^-$.

Remark 6

- *Note that, for all* $u, v \in W^{1,p(x)}(\Omega)$

$$M\Big(\Theta(u)\Big)\int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big)dx$$

is well defined (see [17]).

- *We have* $\delta(x)|u|^{\alpha(x)-2}u \in L^{p'(x)}(\Omega)$ *and* $\lambda f(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ *under* $u \in W^{1,p(x)}(\Omega)$*, the assumptions* (A_2) *and the given hypotheses about the exponents* p, α *and* q *because:* $r(x) = (q(x)-1)p'(x) \in C_+(\overline{\Omega})$ *with* $r(x) < p(x)$*,* $\beta(x) = (\alpha(x)-1)p'(x) \in C_+(\overline{\Omega})$ *with* $\beta(x) < p(x)$*.*
Then, by Remark 2 we can conclude that $L^{p(x)} \hookrightarrow L^{r(x)}$ *and* $L^{p(x)} \hookrightarrow L^{\beta(x)}$*.*
Hence, since $v \in L^{p(x)}(\Omega)$*, we have*

$$\Big(-\delta(x)|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big)v \in L^1(\Omega).$$

This implies that, the integral

$$\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big) v dx$$

exist.

Then, we shall use the definition of weak solution for problem (2) in the following sense:

Definition 6 *We say that a function* $u \in W^{1,p(x)}(\Omega)$ *is a weak solution of* (2)*, if for any* $v \in W^{1,p(x)}(\Omega)$*, it satisfies the following:*

$$M\Big(\Theta(u)\Big)\int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big)dx$$
$$= \int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big) v dx.$$

Before giving the existence result for problem 2, we first give two lemmas that will be used in the proof of this result.
First, let us consider the following functional:

$$\mathcal{J}(u) := \widehat{M}\Big(\Theta(u)\Big), \quad \text{where} \quad \widehat{M}(t) = \int_0^t M(s)ds,$$

such that $M(s)$ satisfies the assumption (M_0).
From [17], it is obvious that the derivative operator of the functional $\mathcal{J}$ in the weak sense at the point $u \in W^{1,p(x)}(\Omega)$ is the functional $\mathcal{M} := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$ given by

$$\langle \mathcal{M}u, v\rangle = M\Big(\Theta(u)\Big)\int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big)dx,$$

for all $u, v \in W^{1,p(x)}(\Omega)$, where $\langle \cdot,\cdot\rangle$ means the duality pairing between $W^{-1,p'(x)}$ (Ω) and $W^{1,p(x)}(\Omega)$. In addition, the following lemma summarizes the properties of the operator $\mathcal{M}$ (see [17]).

Lemma 2 *If* (M_0) *holds, then* $\mathcal{M} : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator, and is a mapping of class* (S_+)*.*

Lemma 3 *Assume that the assumptions* $(A_1)-(A_2)$ *hold, then the operator*

$$\begin{aligned} &\mathcal{N}: W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{N}u, v\rangle = -\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big) v dx, \end{aligned}$$

for all $u, v \in W^{1,p(x)}(\Omega)$, *is compact.*

Proof. In order to prove this lemma, we proceed in three steps.
Step 1 : We define the operator $\Psi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi u(x) := \delta(x)|u(x)|^{\alpha(x)-2}u(x).$$

We will prove that Ψ is bounded and continuous.
It is clear that Ψ is continuous. Next we show that Ψ is bounded.
Let $u \in W^{1,p(x)}(\Omega)$ and using (9) and (10), we obtain

$$\begin{aligned} |\Psi u|_{p'(x)} &\le \rho_{p'(x)}(\Psi u) + 1 \\ &= \int_\Omega |\delta(x)|u|^{\alpha(x)-2}u|^{p'(x)} dx + 1 \\ &= \int_\Omega |\delta(x)|^{p'(x)}|u|^{(\alpha(x)-1)p'(x)} dx + 1 \\ &\le ||\delta||^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^{\beta(x)} dx + 1 \\ &= ||\delta||^{p'}_{L^\infty(\Omega)} \rho_{\beta(x)}(u) + 1 \\ &\le ||\delta||^{p'}_{L^\infty(\Omega)} \Big(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)} \Big) + 1. \end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and Remark 3 that

$$|\Psi u|_{p'(x)} \le C\Big(|u|^{\beta^-}_{1,p(x)} + |u|^{\beta^+}_{1,p(x)} \Big) + 1,$$

and consequently, Ψ is bounded on $W^{1,p(x)}(\Omega)$.
Step 2 : Let us define the operator $\Phi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\lambda f(x, u(x), \nabla u(x)).$$

We will show that Φ is bounded and continuous.
Let $u \in W^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (9) and (10), we

obtain

$$\begin{aligned}|\Phi u|_{p'(x)} &\le \rho_{p'(x)}(\Phi u)+1\\ &= \int_\Omega |\lambda f(x,u(x),\nabla u(x))|^{p'(x)}dx+1\\ &= \int_\Omega |\lambda|^{p'(x)}|f(x,u(x),\nabla u(x))|^{p'(x)}dx+1\\ &\le \Big(|\lambda|^{p'^-}+|\lambda|^{p'^+}\Big)\int_\Omega |C_1\Big(\gamma(x)+|u|^{q(x)-1}+|\nabla u|^{q(x)-1}\Big)|^{p'(x)}dx+1\\ &\le C\Big(|\lambda|^{p'^-}+|\lambda|^{p'^+}\Big)\int_\Omega \Big(|\gamma(x)|^{p'(x)}+|u|^{r(x)}+|\nabla u|^{r(x)}\Big)dx+1\\ &\le C\Big(|\lambda|^{p'^-}+|\lambda|^{p'^+}\Big)\Big(\rho_{p'(x)}(\gamma)+\rho_{r(x)}(u)+\rho_{r(x)}(\nabla u)\Big)+1\\ &\le C\Big(|\gamma|^{p'^+}_{p(x)}+|u|^{r^+}_{r(x)}+|u|^{r^-}_{r(x)}+|\nabla u|^{r^+}_{r(x)}+|\nabla u|^{r^-}_{r(x)}\Big)+1.\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and Remark 3, we have then

$$|\Phi u|_{p'(x)} \le C\Big(|\gamma|^{p'^+}_{p(x)}+|u|^{r^+}_{1,p(x)}+|u|^{r^-}_{1,p(x)}\Big)+1,$$

and consequently Φ is bounded on $W^{1,p(x)}(\Omega)$.
It remains to show that Φ is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$, we need to show that $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem. Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad \nabla u_k(x) \to \nabla u(x), \tag{15}$$

$$|u_k(x)| \le \phi(x) \quad \text{and} \quad |\nabla u_k(x)| \le |\psi(x)|, \tag{16}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1) and (15), we get, as $k \longrightarrow \infty$

$$f(x,u_k(x),\nabla u_k(x)) \to f(x,u(x),\nabla u(x)) \quad \text{a.e.} \quad x \in \Omega.$$

On the other hand, from (A_2) and (16), we can deduce the estimate

$$|f(x,u_k(x),\nabla u_k(x))| \le C_1(\gamma(x)+|\phi(x)|^{q(x)-1}+|\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Seeing that

$$\gamma+|\phi|^{q(x)-1}+|\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (8) that

$$\Phi u_k \to \Phi u \quad \text{in} \quad L^{p'(x)}(\Omega)$$

and consequently

$$\Phi u_n \to \Phi u \quad \text{in} \quad L^{p'(x)}(\Omega),$$

and then Φ is continuous.

Step 3: Let $\mathcal{I}^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the natural embedding mapping $\mathcal{I} : W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. Then, we define

$$\mathcal{I}^* \circ \Psi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$\mathcal{I}^* \circ \Phi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, due to the compactness of $\mathcal{I}$, $\mathcal{I}^*$ also becomes compact. Thus, the compositions $\mathcal{I}^* \circ \Psi$ and $\mathcal{I}^* \circ \Phi$ are compact, that means $\mathcal{N} = \mathcal{I}^* \circ \Psi + \mathcal{I}^* \circ \Phi$ is compact. With this last step the proof of Lemma 3 is completed. □

We are now in the position to get the existence result of weak solution for (2).

Theorem 2 *If* $(A_1) - (A_2)$ *and* (M_0) *hold, then for every* $\delta \in L^\infty(\Omega)$ *and* $\lambda \in \mathbb{R}$ *the problem* (2) *admits at least one weak solution* u *in* $W^{1,p(x)}(\Omega)$.

Proof. The basic idea of our proof is to reformulate the problem (2) to an abstract formula governed by a Hammerstein equation, and apply the theory of topological degree introduced in Section 3 to show the existence of a weak solutions to the state problem.

First, for all $u, v \in W^{1,p(x)}(\Omega)$, we define the operators $\mathcal{M}$ and $\mathcal{N}$, as defined in Lemmas 2 and 3 respectively,

$$\begin{aligned} &\mathcal{M} : W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{M}u, v\rangle = M\Big(\Theta(u)\Big)\int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big) dx, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{N} : W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{N}u, v\rangle = -\int_\Omega \Big(- \delta|u|^{\alpha(x)-2}u + \lambda f(x, u, \nabla u)\Big) v dx. \end{aligned}$$

Consequently, the problem (2) is equivalent to the equation

$$\mathcal{M}u + \mathcal{N}u = 0, \quad u \in W^{1,p(x)}(\Omega). \tag{17}$$

Taking into account that, by Lemma 2, the operator $\mathcal{M}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [25, Theorem 26 A], the inverse operator

$$\mathcal{P} := \mathcal{M}^{-1} : W^{-1,p'(x)}(\Omega) \to W^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).
On another side, according to Lemma 3, we have that the operator $\mathcal{N}$ is bounded, continuous and quasimonotone.
Consequently, following Zeidler's terminology [25], the equation (17) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{P}v \ \text{ and } \ v + \mathcal{N} \circ \mathcal{P}v = 0, \quad u \in W^{1,p(x)}(\Omega) \text{ and } v \in W^{-1,p'(x)}(\Omega). \tag{18}$$

Due to the equivalence of (17) and (18), it will be sufficient to solve (18). In order to solve (18), we will apply the Berkovits topological degree introduced in Section 3. First, let us set

$$\mathcal{E} := \Big\{ v \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ v + t\mathcal{N} \circ \mathcal{P}v = 0 \Big\}.$$

Next, we show that $\mathcal{E}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.
Let $v \in \mathcal{E}$ and set $u := \mathcal{P}v$ for all $v \in \mathcal{E}$. Since $|\mathcal{P}v|_{1,p(x)} = |u|_{1,p(x)}$, then we have two cases:
Case 1 : If $|u|_{1,p(x)} \leq 1$, then $|\mathcal{P}v|_{1,p(x)} \leq 1$, that means $\Big\{ \mathcal{P}v : v \in \mathcal{E} \Big\}$ is bounded.
Case 2 : If $|u|_{1,p(x)} > 1$, then, we deduce from (13), the growth condition (A_2),

the inequalities (10) and (11) and the Young inequality that

$$\begin{aligned}
|\mathcal{P}v|_{1,p(x)}^{p^-} &= |u|_{1,p(x)}^{p^-} \\
&\le \rho_{1,p(x)}(u) \\
&= \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u) \\
&\le \langle v,\ \mathcal{P}v\rangle \\
&= -t\langle \mathcal{N}\circ\mathcal{P}v,\ \mathcal{P}v\rangle \\
&= t\int_\Omega \Big(-\delta(x)|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big)u\,dx \\
&\le t\max(\|\delta\|_{L^\infty(\Omega)}, C_1|\lambda|)\Big(\rho_{\alpha(x)}(u) + \int_\Omega |\gamma(x)u(x)|dx + \rho_{q(x)}(u) \\
&\qquad\qquad + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&\le C\Big(|u|_{\alpha(x)}^{\alpha^-} + |u|_{\alpha(x)}^{\alpha^+} + |\gamma|_{p'(x)}|u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} \\
&\qquad\qquad + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big) \\
&\le C\Big(|u|_{\alpha(x)}^{\alpha^-} + |u|_{\alpha(x)}^{\alpha^+} + |u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+}\Big),
\end{aligned}$$

then, according to Remark 3 and the continuous embeddings $L^{p(x)} \hookrightarrow L^{\alpha(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{P}v|_{1,p(x)}^{p^-} \le C\Big(|\mathcal{P}v|_{1,p(x)}^{\alpha^+} + |\mathcal{P}v|_{1,p(x)} + |\mathcal{P}v|_{1,p(x)}^{q^+}\Big),$$

what implies that $\Big\{\mathcal{P}v : v \in \mathcal{E}\Big\}$ is bounded. On the other hand, we have that the operator is $\mathcal{N}$ is bounded, then $\mathcal{N}\circ\mathcal{P}v$ is bounded. Thus, thanks to (18), we have that $\mathcal{E}$ is bounded in $W^{-1,p'(x)}(\Omega)$. However, there exists a constant $a > 0$ such that

$$|v|_{-1,p'(x)} < a \quad \text{for all} \quad v \in \mathcal{E},$$

which leads to

$$v + t\mathcal{N}\circ\mathcal{P}v \neq 0, \quad v \in \partial\mathcal{E}_a(0) \quad \text{and} \quad t \in [0,1],$$

where $\mathcal{E}_a(0)$ is the ball of center 0 and radius a in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 1, we conclude that

$$\mathcal{I} + \mathcal{N}\circ\mathcal{P} \in \mathcal{F}_{\mathcal{P}}(\overline{\mathcal{E}_a(0)}) \quad \text{and} \quad \mathcal{I} = \mathcal{M}\circ\mathcal{P} \in \mathcal{F}_{\mathcal{P}}(\overline{\mathcal{E}_a(0)}).$$

On the other hand, taking into account the fact that $\mathcal{I}$, $\mathcal{N}$ and $\mathcal{P}$ are bounded, then $\mathcal{I}+\mathcal{N}\circ\mathcal{P}$ is bounded. Therefore, we deduce that

$$\mathcal{I}+\mathcal{N}\circ\mathcal{P}\in\mathcal{F}_{\mathcal{P},\mathrm{B}}(\overline{\mathcal{E}_a(0)}) \quad\text{and}\quad \mathcal{I}=\mathcal{M}\circ\mathcal{P}\in\mathcal{F}_{\mathcal{P},\mathrm{B}}(\overline{\mathcal{E}_a(0)}).$$

Next, we define the homotopy $\mathcal{H}:[0,1]\times\overline{\mathcal{E}_a(0)}\to W^{-1,p'(x)}(\Omega)$ by

$$(t,v)\mapsto\mathcal{H}(t,v):=v+t\mathcal{N}\circ\mathcal{P}v.$$

Applying the homotopy invariance and normalization properties of the degree d as in Theorem 1, we obtain

$$d(\mathcal{I}+\mathcal{N}\circ\mathcal{P},\mathcal{E}_a(0),0)=d(\mathcal{I},\mathcal{E}_a(0),\ 0)=1\neq 0.$$

Since $d(\mathcal{I}+\mathcal{N}\circ\mathcal{P},\mathcal{E}_a(0),0)\neq 0$, then by the existence property of the degree d stated in Theorem 1, we conclude that there exists $v\in\mathcal{B}_R(0)$ which verifies

$$\big(\mathcal{I}+\mathcal{N}\circ\mathcal{P}\big)(v)=0\Leftrightarrow v+\mathcal{N}\circ\mathcal{P}v=0\Leftrightarrow\mathcal{M}\circ\mathcal{P}v+\mathcal{N}\circ\mathcal{P}v=0.$$

At last, we deduce that $u=\mathcal{P}v$ is a weak solution of (2). The proof of Theorem 2 is completed. □

Acknowledgment

The authors would like to thank the referees and the editor for their careful reading and useful comments and suggestions that substantially helped improving the quality of the paper.

References

[1] C. Allalou, M. El Ouaarabi and S. Melliani, Existence and uniqueness results for a class of $p(x)$-Kirchhoff-type problems with convection term and Neumann boundary data, *J. Elliptic Parabol. Equ.*, **8** (2022), no. 1, 617–633.

[2] J. Berkovits, Extension of the Leray-Schauder degree for abstract Hammerstein type mappings, *J. Differ. Equ.*, **234** (2007), 289–310.

[3] C. Corsato, C. De Coster, F. Obersnel and P. Omari, Qualitative analysis of a curvature equation modeling MEMS with vertical loads, *Nonlinear Anal. Real World Appl.*, **55** (2020), 103–123.

[4] G. Dai and R. Hao, Existence of solutions for a p(x)-Kirchhoff-type equation, *J. Math. Anal. Appl.*, **359** (2009), 275–284.

[5] M. El Ouaarabi, C. Allalou and S. Melliani, Existence result for Neumann problems with $p(x)$-Laplacian-like operators in generalized Sobolev spaces, *Rend. Circ. Mat. Palermo, II. Ser*, (2022). https://doi.org/10.1007/s12215-022-00733-y.

[6] M. El Ouaarabi, C. Allalou and S. Melliani, On a class of $p(x)$-Laplacian-like Dirichlet problem depending on three real parameters, *Arab. J. Math.*, **11** (2022), no. 4, 227–239.

[7] M. El Ouaarabi, C. Allalou and S. Melliani, Weak solution of a Neumann boundary value problem with $p(x)$-Laplacian-like operator, *Analysis*, **42** (2022), no. 4, 271–280.

[8] M. El Ouaarabi, C. Allalou and S. Melliani, Existence of weak solutions for $p(x)$-Laplacian-like problem with $p(x)$-Laplacian operator under Neumann boundary condition, *São Paulo J. Math. Sci.*, (2022). https://doi.org/10.1007/s40863-022-00321-z.

[9] M. El Ouaarabi, C. Allalou and S. Melliani, Existence of weak solution for a class of $p(x)$-Laplacian problems depending on three real parameters with Dirichlet condition, *Bol. Soc. Mat. Mex.*, **28** (2022), no. 2, 31.

[10] M. El Ouaarabi, C. Allalou and S. Melliani, Existence result for a Neumann boundary value problem governed by a class of $p(x)$-Laplacian-like equation, *Asymptot. Anal.*, (2022). https://doi.org/10.3233/ASY-221791.

[11] M. El Ouaarabi, C. Allalou and S. Melliani, On a class of nonlinear degenerate elliptic equations in weighted Sobolev spaces, *Georgian Mathematical Journal*, **30** (2023), no. 1, 81–94.

[12] X.L. Fan and D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446.

[13] R. Finn, *Equilibrium Capillary Surfaces*, vol. 284, Springer-Verlag, New York, 2012.

[14] I.S. Kim and S.J. Hong, A topological degree for operators of generalized (S_+) type, *Fixed Point Theory and Appl.*, **1** (2015), 1–16.

[15] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[16] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.*, **41** (1991), no. 4, 592–618.

[17] E. C. Lapa, V. P. Rivera and J. Q. Broncano, No-flux boundary problems involving $p(x)$-Laplacian-like operators, *Electron. J. Diff. Equ.*, **2015** (2015), no. 219, 1–10.

[18] W. M. Ni and J. Serrin, Existence and non-existence theorems for ground states for quasilinear partial differential equations, *Att. Conveg. Lincei.*, **77** (1986), 231–257.

[19] M. E. Ouaarabi, A. Abbassi and C. Allalou, Existence result for a Dirichlet problem governed by nonlinear degenerate elliptic equation in weighted Sobolev spaces, *J. Elliptic Parabol Equ.*, **7** (2021), no. 1, 221–242.

[20] M. E. Ouaarabi, A. Abbassi and C. Allalou, Existence and uniqueness of weak solution in weighted Sobolev spaces for a class of nonlinear degenerate elliptic problems with measure data, *Int. J. Nonlinear Anal. Appl.*, **13** (2021), no. 1, 2635–2653.

[21] V. D. Rădulescu and D. D. Repoveš, *Partial Differential Equations with Variable Exponents, Variational Methods and Qualitative Analysis*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, 2015.

[22] M. A. Ragusa and A. Tachikawa, On continuity of minimizers for certain quadratic growth functionals, *Journal of the Mathematical Society of Japan*, **57** (2005), no. 3, 691–700.

[23] M. A. Ragusa and A. Tachikawa, Regularity of Minimizers of some Variational Integrals with Discontinuity, *Zeitschrift für Analysis und ihre Anwendungen*, **27** (2008), no. 4, 469–482.

[24] K. R. Rajagopal and M. Růžička, Mathematical modeling of electrorheological materials, *Continuum mechanics and thermodynamics*, **13** (2001), no. 1, 59–78.

[25] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B*, Springer-Verlag, New York, 1990.

[26] V. V. E. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, **50** (1986), no. 4, 675–710.

Received: January 31, 2022