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On Hankel determinant problems of functions associated to the lemniscate of Bernoulli and involving conjugate points

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Abstract. The notion of Hankel determinant H_q in univalent functions theory is initiated by Noonan and Thomas while studying it for areally mean multivalent mappings. This determinant has significant role while dealing with singularities and particularly it's important for analyzing integral coefficient. Fekete-Szegö functional used in the study of the area theorem is a particular case of this determinant. We explore a known class of holomorphic mappings which is related with the various classes of functions with conjugate symmetric points We also study upper bounds in different settings of the coefficients of these mappings. We also relate our exploration with the existing literature of the subject.

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1 Introduction

Suppose that an analytic function f is expressed in the following series form:

$$f(z) = z + \sum_{j=2}^{\infty} \eta_j z^j, z \in \mathbb{E}_1^0 \quad (1)$$

where $\mathbb{E}_1^0 \subset \mathbb{E}_r^{z_0} = \{|z \in \mathbb{C} : z - z_0| < r\}$. We use $\mathcal{A}$ to represent the family of these functions. Also $\mathcal{S} \subset \mathcal{A}$ deputize for the family of one-to-one or univalent functions defined in $\mathbb{E}_1^0$. Let $\mathcal{Q}$ stand for the collection of functions $\hbar$ such that

$$\hbar(z) = 1 + \sum_{j=1}^{\infty} \vartheta_j z^j \ : \operatorname{Re} \hbar(z) > 0, z \in \mathbb{E}_1^0. \quad (2)$$

If for a Schwarz mapping w, we write $f(z) = g(w(z))$, where f and g are analytic in $\mathbb{E}_1^0$, then it is said that f is subordinate g, and mathematically, we write $f \prec g$.

A large number of subfamilies are related with the class $\mathcal{P}$ and some of its generalizations. These may include the family $\mathcal{S}^*$ of starlike and a related family $\mathcal{C}$ of convex mappings. These families are further studied with the order and arguments or in such a way that the function f maps on to the right half plane as well as some specific plane region. Ma and Minda as seen in [8] introduced two classes of analytic functions namely;

$$\mathcal{S}^*(\psi) = \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec \psi(z) \quad (z \in \mathbb{E}_1^0) \right\}$$

and

$$\mathcal{C}(\psi) = \left\{ g \in \mathcal{A} : \phi(z) = \frac{zg'(z)}{g(z)} \prec \psi(z) \quad (z \in \mathbb{E}_1^0 \right\},$$

where the function ψ is an analytic univalent function such that $\Re(\psi) > 0$ in $\mathbb{U}$ with $\psi(0) = 1$, $\psi'(0) > 0$ and g maps $z \in \mathbb{E}_1^0$ onto a region starlike with respect to 1 and the symbol $\prec$ denotes the subordination between two analytic functions ϕ and ψ. By varying the function ψ, several familiar families will be deduced as seen below:

(i) For $\psi = \frac{1+Az}{1+Bz}$ $(-1 \leq B \leq A \leq 1)$, we get the family $\mathcal{S}^*(A, B)$, see [5].

(ii) For $A = 1 - 2\alpha$ and $B = -1$, the family $\mathcal{S}^*(\alpha)$ is studied at large as seen in [11].

(iii) In case $\psi = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, the desired family is studied in [12].

Recently as seen in [7] and by choosing a particular function for ψ as above, inequalities related with coefficient bounds of some subfamiles of univalent functions have been discussed extensively.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^{\ell B}$, if and only if

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, z \in \mathbb{E}_1^0. \tag{3}$$

For $f \in \mathcal{S}^{\ell B}$, $\frac{zf'(z)}{f(z)}$ is bounded by the lemniscate of Bernoulli

$$\left\{\psi \in \mathbb{C} \text{ with } \operatorname{Re}(\psi) > 0 : \left|\psi^2 - 1\right| < 1\right\} \tag{4}$$

in the right half of the $w-$plane. In term of subordination, we say that $f \in \mathcal{S}^{\ell B}$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, z \in \mathbb{E}_1^0. \tag{5}$$

The known family of functions starlike with respect to symmetric points were introduced by Sakaguchi. Subsequently, we make use the same idea along with (6) or (7) and define the family $\mathcal{S}_{SP}^{\ell B}$ of Sakaguchi functions associated with the lemniscate of the Bernoulli as:

$$\left|\left(\frac{zf'(z)}{f(z) - f(-z)}\right)^2 - 1\right| < 1, z \in \mathbb{E}_1^0. \tag{6}$$

Thus w maps $\mathbb{E}_1^0$ onto the the right half of the lemniscate of Bernoulli defined by the inequality $\operatorname{Re}(\psi) > 0 : |\psi^2 - 1| < 1$. It is obvious that $f \in \mathcal{S}_{SP}^{\ell B}$, iff

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \sqrt{1+z},\ z \in \mathbb{E}_1^0. \tag{7}$$

Let $f \in \mathcal{A}$. Then the family $\mathcal{S}_{SP}^{\ell B}$ is defined by

$$\left|\left(\frac{2zf'(z)}{f(z) + \overline{f(\overline{z})}}\right)^2 - 1\right| < 1,\ z \in \mathbb{E}_1^0, \tag{8}$$

where $\operatorname{Re}\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} > 0$. Thus a mapping $f \in \mathcal{S}_{SCP}^{\ell B}$, if $\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}}$ lies to the right half of the lemniscate of Bernoulli as defined by (4). It is evident that $f \in \mathcal{S}_{SCP}^{\ell B}$, if it satisfies

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} \prec \sqrt{1+z},\ z \in \mathbb{E}_1^0, \tag{9}$$

where Re $\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} > 0$. Sokol and Stankiewicz [15] and other [1, 15] introduced the same structure of other related families of these functions.

The coefficient bounds problem plays a significant role in dealing with the geometrical aspects of complex mappings. Hankel matrices or catalecticant matrices are square matrices, where ascending skew-diagonals from left to right are constants. These matrices are obtained for a sequence of outputs, when a realization of a hidden Markov model or a state-space model is required. Some decomposition of such matrices provide a mean of computing those matrices which define these realizations. This matrix is also obtained when signals are assumed useful for separation of non-stationary signals along with time-frequency representation. Certain techniques used in polynomial distributions are leading to the Hankel matrix and it results in obtaining weight parameters of their approximations.

The qth Hankel determinant $H_d(q, j)$ is studied in [9] and it can be defined as:

$$H_d(q,j) = \begin{vmatrix} \eta_j & \eta_{j+1} & \cdot & \cdot & \eta_{j+q-1} \\ \eta_{j+1} & \eta_{j+2} & \cdot & \cdot & \eta_{j+q-2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \eta_{j+q-1} & \eta_{j+q-2} & \cdot & \cdot & \eta_{j+2q-2} \end{vmatrix},$$

where $q \geq 1$, $\eta_j : j = 2, 3, ...$ are the complex coefficients of an analytic function $f \in \mathcal{A}$. This determinant is also significant in the study of singularities, see [3]. This is particularly significant when analyzing integral coefficient in a power series, for detail, again we refer [3]. We also find its applications in the study of meromorphic functions. Fekete-Szegö problem $H_d(2, 1) = \eta_3 - \eta_2^2$ is a particular form of the generalized functional $\eta_3 - \tau\eta_2^2$ for some τ real or complex. For τ real and $f \in \mathcal{S}$, the family of injective, one-to-one or univalent functions, Fekete and Szegö provided sharp estimates for $|\eta_3 - \tau\eta_2^2|$. As seen, it is just a combination of the first two coefficients that describe the known Gronwall's area problems. In addition, we know that the functional $\eta_2\eta_4 - \eta_3^2$ is equivalent to $H_d(2, 2)$. For a few subclasses of holomorphic functions, this determinant $H_d(2, 2)$ has been lately investigated and many authors have looked into the bounds of the functional $\eta_2\eta_4 - \eta_3^2$, see [3, 6]. Babalola [2] also investigated $H_d(3, 1)$ for few other classes involving analytic mappings. Using a well-known

Toeplitz determinants, we find the upper bounds of $H_d(3,1)$ for functions connected to the lemniscate of Bernoulli $\Gamma_{\ell\beta}$.

2 Preliminaries

The following lemmas are used in our major results. In the subsequent lemma as seen in [8] on page 162, Section 4, we find bounds on $\vartheta_2 - \tau\vartheta_1^2$.

Lemma 1 *Let $\hbar(z) = 1 + \vartheta_1 z + \vartheta_2 z^2 + ... \in \mathcal{Q}$ be represented by (2). Then we have*

$$|\vartheta_2 - \tau\vartheta_1^2| \leq \begin{cases} -4\tau + 2, & \tau < 0, \\ 2, & 0 \leq \tau \leq 1, \\ 4\tau - 2, & \tau > 1. \end{cases}$$

For $\tau < 0$ or $\tau > 1$, we have the equality iff $\hbar(z) = \frac{1+z}{1-z}$ and $0 < \tau < 1$, we have the equality iff $\hbar(z) = \frac{1+z^2}{1-z^2}$ or its any rotation. If $\tau = 0$, the equality holds iff

$$\hbar(z) = (\frac{1}{2} + \frac{\eta}{2})\frac{1+z}{1-z} + (\frac{1}{2} - \frac{\eta}{2})\frac{1-z}{1+z}, \ 0 \leq \eta \leq 1$$

or its any rotation. However, the previous upper bound can be improved for

$$\left|\vartheta_2 - \tau\vartheta_1^2\right| + \tau|\vartheta_1|^2 \leq 2, \ 0 < \tau \leq \frac{1}{2}$$

and

$$\left|\vartheta_2 - \tau\vartheta_1^2\right| + (1+\tau)|\vartheta_1|^2 \leq 2, \frac{1}{2} < \tau \leq 1.$$

The following lemma also deals with the coefficients bounds for the functions in class $\mathcal{Q}$, when $\tau \in \mathbb{C}$.

Lemma 2 *If $\hbar(z) = 1 + \vartheta_1 z + \vartheta_2 z^2 + ... \in \mathcal{Q}$, then for $\tau \in \mathbb{C}$, we have*

$$|\vartheta_2 - \tau\vartheta_1^2| \leq 2\max\{1, |2\tau - 1|\}$$

This inequality is sharp. The equality is concerned with the function

$$\hbar_1(z) = \frac{1+z}{1-z} \quad or \quad \hbar_2(z) = \frac{1+z^2}{1-z^2}.$$

For the reference of aforementioned lemma, see [4]. The subsequently given lemma also addresses the estimation of the coefficients under specific constraints.

Lemma 3 *If $\hbar \in \mathcal{Q}$, then for some $x : |x| \leq 1$, we have*

$$2\vartheta_2 = x(4-\vartheta_1^2) + \vartheta_1^2$$

and also for some $z : |z| \leq 1$, we obtain

$$4\vartheta_3 = (\vartheta_1^2-4)\vartheta_1 x^2 + \vartheta_1^3 - 2(\vartheta_1^2-4)\vartheta_1 x - 2(\vartheta_1^2-4)(1-|x|^2)z.$$

For reference, see [13].

3 Discussions

In this section, we study some Hankel determinant related problems. The theorem below describes bounds estimates of the Fekete-Szegö functional $\eta_3 - \tau\eta_2^2$.

Theorem 1 *Let $f \in \mathcal{S}_{SCP}^{\ell B}$ be represented by (8) or equivalently, we have (9). Then the bounds on the Fekete-Szegö functional $\eta_3 - \tau\eta_2^2$ can be written as:*

$$|\eta_3 - \tau\eta_2^2| \leq \begin{cases} -\frac{1}{8}(2\tau+1), & \tau < -\frac{5}{2} \\ \frac{1}{2}, & -\frac{5}{2} \leq \tau \leq \frac{3}{2} \\ \frac{1}{8}(2\tau+1), & \tau > \frac{3}{2} \end{cases}.$$

Moreover, we can see that

$$\left|\eta_3 - \tau\eta_2^2\right| + (2\tau+5)\,|\eta_2|^2 \leq \frac{1}{2},\ -\frac{5}{2} < \tau \leq -\frac{1}{2}$$

and

$$\left|\eta_3 - \tau\eta_2^2\right| + (3-\tau)\,|\eta_2|^2 \leq \frac{1}{2},\quad -\frac{1}{2} < \tau \leq \frac{3}{2}$$

These above results are sharp.

Proof. For the mapping $f \in \mathcal{S}_{SCP}^{\ell B}$, from the definition which is equivalent to (8), we see that $\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}} \prec \frac{1}{2}\phi(z)$, when $\phi(z) = \sqrt{1+z}$. Assuming a functional $\hbar$ such that

$$\hbar(z) = \frac{1+\vartheta(z)}{1-\vartheta(z)} = 1 + \vartheta_1 z + \vartheta_2 z^2 + \ldots$$

Obviously $\vartheta(z) = \frac{\hbar(z)-1}{\hbar(z)+1}$. Thus, $\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = \phi\left(\vartheta(z)\right)$ or $\phi\left(\vartheta(z)\right) = \left(\frac{2\hbar(z)}{\hbar(z)+1}\right)^{\frac{1}{2}}$. Now we see that

$$\left(\frac{2\hbar(z)}{\hbar(z)+1}\right)^{\frac{1}{2}} = 1+\frac{1}{4}\vartheta_1 z+\left(\frac{1}{4}\vartheta_2 - \frac{5}{32}\vartheta_1^2\right)z^2+\left(\frac{1}{4}\vartheta_3 - \frac{5}{16}\vartheta_1\vartheta_2 + \frac{13}{128}\vartheta_1^3\right)z^3+\ldots.$$

Similarly, we can write

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = 1+\eta_2 z+\eta_3 z^2+\eta_4 z^3+\dots$$

Therefore, we conclude that

$$\eta_2 = \frac{1}{4}\vartheta_1, \tag{10}$$

$$\eta_3 = \frac{1}{4}\vartheta_1 - \frac{5}{32}\vartheta_1^2 \tag{11}$$

and also we see that

$$\eta_4 = \frac{1}{4}\vartheta_3 - \frac{5}{16}\vartheta_1\vartheta_2 + \frac{13}{128}\vartheta_1^3. \tag{12}$$

This implies that

$$\left|\eta_3 - \tau\eta_2^2\right| = \frac{1}{4}\left|\vartheta_2 - \frac{1}{8}(2\tau+5)\vartheta_1^2\right|.$$

Applying Lemma 1, we obtain the required result. The equality follows from the functions $\hbar_j(z)$, $j = 1, 2, 3, 4$, such that

$$\frac{z\hbar'(z)}{\hbar(z)} = \begin{cases} \sqrt{1+z} \text{ if } \tau < \frac{-5}{2} \text{ or } \tau > \frac{3}{2}, \\ \sqrt{1+z^2} \text{ if } \frac{-5}{2} < \tau < \frac{3}{2} \\ \sqrt{1+\phi(z)} \text{ if } \tau = \frac{-5}{2}, \\ \sqrt{1-\phi(z)}, \text{ if } \tau = \frac{3}{2}. \end{cases}$$

where $\phi(z) = \frac{z(z+\eta)}{1+\eta}$ with $0 \le \eta \le 1$. □

The subsequent theorem describes $|\eta_3 - \tau\eta_2^2|$, when τ is a complex number.

Theorem 2 *Let $f \in \mathcal{S}^{\ell B}_{SCP}$ and τ be a complex number. Then*

$$|\eta_3 - \tau\eta_2^2| \le \frac{1}{2}\max\left\{1; \frac{1}{4}|2\tau+1|\right\}.$$

Proof. From (10) and (12), we observe that

$$|\eta_3 - \tau\eta_2^2| \le \frac{1}{4}\left|\vartheta_2 - \frac{1}{8}(\tau+5)\vartheta_1^2\right|.$$

Thus application of Lemma 2 leads to the desired result. This result is sharp and equality holds for the functions

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = \sqrt{1+z}$$

or

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = \sqrt{1+z^2}.$$

□

Remark 1 *For* $\tau = 1$, $\mathrm{Hd}_2(1) = \alpha_3 - \alpha_2^2$ *and for* $f \in \mathcal{S}^*_{\ell\beta}$, $|\alpha_3 - \alpha_2^2| \leq \frac{1}{2}$.

In context of the lemniscate of Bernoulli and in the view of above Lemma 3, we state that:

Theorem 3 *Let* $f \in \mathcal{S}^{\ell B}_{SCP}$. *Then* $|\eta_2\eta_4 - \eta_3^2| \leq \frac{1}{4}$.

Proof. Keeping in view the values for η_2, η_3 and η_4 as given in (10), (11) and (12) respectively, we calculate $\eta_2\eta_4 - \eta_3^2$ as:

$$\begin{aligned}\eta_2\eta_4 - \eta_3^2 &= \frac{1}{16}\left(\vartheta_1\vartheta_3 - \frac{5}{4}\vartheta_1^2\vartheta_2 + \frac{13}{32}\vartheta_1^4\right) - \left(\frac{1}{4}\vartheta_2 - \frac{5}{32}\vartheta_1^2\right)^2 \\ &= \frac{1}{16}\vartheta_1\vartheta_3 + \frac{1}{1024}\vartheta_1^4 - \frac{1}{16}\vartheta_2^2.\end{aligned}$$

By taking $C = \left|\eta_2\eta_4 - \eta_3^2\right|$, $(4-\vartheta_1^2) = c$ and then assuming that $t = \vartheta_1 \in (0, 2]$ and using the value of ϑ_2 and ϑ_3 in term of t, from Lemma 3, we write

$$C = \frac{1}{1024}\left|16t\left\{t^3 + 2ctx - ctx^2 + 2c(1-|x|^2)z - 16\{t^2+xc\}^2 + t_1^4\right\}\right|.$$

After some simplification, we apply triangular inequality and replace $|x|$ by ρ to obtain

$$C = \frac{1}{1024}\left[t^4 + \{16t^2 + 16(4-t^2)\}(4-t^2)\rho^2 + 32t(4-t^2)(1-\rho^2)\right] = F(t, \rho).$$

On differentiating partially with ρ, we see that $\frac{\partial F(t,\rho)}{\partial \rho}$ is positive which means that the multivariable function $F(t, \rho)$ is increasing on the compact set $[0, 1]$. Thus the greatest value occurs at $\rho = 1$. Therefore, we take $\max F(t, \rho) = G(t)$.

Considering G, we calculate G' and G'' and find that $G' > 0$ along with $G''(z) < 0$ for $t = 0$. Thus the max $G(t)$ occurs at $t = 0$. Therefore, we can write

$$|\eta_2\eta_4 - \eta_3^2| \leq \frac{1}{4}.$$

This is a sharp result and equality holds for the functions $\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}} = \frac{1}{2}\sqrt{1+z^2}$ or $\frac{1}{2}\sqrt{1+z}$. □

In context of the lemniscate of Bernoulli, we determine the value of the modulus of $\eta_2\eta_3 - \eta_4$:

Theorem 4 *For* $f \in \mathcal{S}_{SCP}^{\ell B}$, *we have* $|\eta_2\eta_3 - \eta_4| \leq \frac{1}{2}$.

Proof. For $f \in \mathcal{S}_{SCP}^{\ell B}$, we can write □

$$\eta_2 = \frac{1}{4}\vartheta_1,\ \eta_3 = \frac{1}{4}\vartheta_2 - \frac{5}{32}\vartheta_1^2 \text{ and } \eta_4 = \frac{1}{4}\vartheta_3 - \frac{5}{16}\vartheta_1\vartheta_2 + \frac{1}{4}\vartheta_1^3,$$

which leads to

$$\begin{aligned}
\eta_2\eta_4 - \eta_3^2 &= \frac{1}{16}\left(\vartheta_1\vartheta_2 - \frac{5}{8}\vartheta_1^3\right) - \left(\frac{1}{4}\vartheta_2 - \frac{5}{16}\vartheta_1\vartheta_2 + \frac{13}{128}\vartheta_1^3\right) \\
&= \frac{3}{8}\vartheta_1\vartheta_2 + \frac{1}{4}\vartheta_3 - \frac{9}{64}\vartheta_1^3 \\
&= \frac{1}{64}(24\vartheta_1\vartheta_2 + 16\vartheta_3 - 9\vartheta_1^3).
\end{aligned}$$

Therefore, in view of Lemma 3, we note that

$$\left|\eta_2\eta_4 - \eta_3^2\right| = \frac{1}{64}\left|\vartheta_1^3 + 2c\vartheta_1 x - c\vartheta_1 x^2 + 2c(1-|x|^2)z - 12\vartheta_1\{\vartheta_1^2 + xc\} + 9\vartheta_1^3\right|$$

where $4 - \vartheta_1^2 = c$. Applying triangle inequality, replacing $|x|$ with ρ, $|z|$ by 1 and assuming that $t > 0$, such that $\vartheta_1 = t \in [0, 2]$, we can write

$$\left|\eta_2\eta_4 - \eta_3^2\right| \leq \frac{1}{64}\left\{t^3 + 4(4-t^2)t\rho + 4(4-t^2)t\rho^2 + 8(4-t^2)(1-\rho^2)\right\}.$$

Let us consider that

$$F(t,\rho) = \frac{1}{64}\left\{t^3 + 4(4-t^2)t\rho + 4(4-t^2)t\rho^2 + 8(4-t^2)(1-\rho^2)\right\}. \tag{13}$$

We further suppose the upper bounds exist in the interior of $[0,2]\times[0,1]$. Differentiating 13 partially with ρ, we see that

$$\frac{\partial}{\partial\rho}\left(F(t,\rho)\right)=\frac{1}{64}\left\{4t(4-t^2)+8\rho(t-2)(4-t^2)\right\}.$$

For $0<\rho<1$ and fixed $t\in[0,2]$, we see that $\frac{\partial F(\omega,\rho)}{\partial\rho}<0$.This shows that $F(t,\rho)$ is decreasing which contradicts to our supposition. Hence, $\max F(t,\rho)=F(t,0)=G(t)$ and

$$G(t)=\frac{1}{64}[t^3-8t^2+32],\, G'(t)=\frac{1}{64}[3t^2-16t],$$

which shows that $G''(t)=\frac{1}{64}[6t-16]<0$ for $t=0$. Therefore, at $t=0$ a maximum is achieved. Hence, we obtain the required proof.

4 Conclusion

The Fekete-Szegö inequality denoted as F-S inequality is one of the inequalities involving certain coefficients related to the Bieberbach conjecture and associated with this inequality is the Hankel determinant, which is used in the investigations of the singularities and determination of integral coefficients. In this investigation, we studied F-S inequalities for certain mappings f as defined by (8) for which the image domain is related with the lemniscate of Bernoulli.

References

[1] R. M. Ali, N. E. Cho, V. Ravichandran and S. S. Kumar, Differential Subordination for Functions Associated with the Lemniscate of Bernoulli, *Taiwan. J. Math.*, **16** (3) (2012), 1017–1026.

[2] K. O. Babalola, On $H_d(3,1)$ Hankel Determinant for Some Classes of Univalent Functions, *Inequal. Th. Appl.* **6** (2007), 1–7.

[3] D. Bansal, Upper Bound of Second Hankel Determinant for a New Class of Analytic Functions, *Appl. Math. Lett.*, **26** (1) (2013), 103–107.

[4] I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, *J. Math. Anal. Appl.* **435**, 2016, 369–379.

[5] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math., Inst. Math. Polish Acad. Sci.*, **23** (1970), 159–177.

[6] A. Janteng, S. A. Halim and M. Darus, Hankel Determinant for Starlike and Convex Functions, *Int. J. Math. Anal.*, **1** (3) (2007), 619–625.

[7] S. Kumar, V. Ravichandran, A subclass starlike functions associated with a rational; function, *South East Asian Bull. Math.*, **40** (2016), 199–212.

[8] W. Ma, and D. Minda, A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis; Z. Li, F. Ren, L. Yang and S. Zhang, S., Eds.; International Press Incorporated: Cambridge, MA, USA, 1992; pp. 157–169.

[9] J. W. Noonan and D. K. Thomas, On the Second Hankel Determinant of Areally Mean p-Valent Functions, *Trans. Amer. Math. Soc.*, **223** (2) (1976), 337–346.

[10] R. K. Raina, J. Sokol, On coefficient estimates for a certain class of starlike functions, *Hacet. J. Math. Stat.*, *44* (2015), 1427–1433.

[11] M. S. Robertson, Certain classes of starlike functions, *Mich. Math. J.*, **32** (1985), 135–140.

[12] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Am. Math. Soc.*, **118** (1993), 189–196.

[13] G. Singh and G. Singh, On the second Hankel determinant for a new subclass of analytic functions, *J. Math. Sci. Appl.* **2**, 2014, 1–3.

[14] J. Sokol, Coefficient Estimates in a Class of Strongly Starlike Functions, *Kyung-pook Math. J.*, **49** (2) (2009), 349–353.

[15] J. Sokol and J. Stankiewicz, Radius of Convexity of Some Subclasses of Strongly Starlike Functions, *Zeszty Nauk. politech. Rzeszowskiej Mat.* **19**(1996), 101–105.

[16] R. Vein, and P. Dale, Determinants and their Applications in Mathematical Physics, *Appl. Math. Sci.*, 134 (1999), Springer, New York.

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