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Fixed point in $\mathcal{M}_\nu^b-$metric space and applications

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Abstract. The aim is to utilize a new metric called an $\mathcal{M}_\nu^b-$metric which is an improvement and generalization of $\mathcal{M}_\nu-$metric to revisit the celebrated Banach and Sehgal contractions in $\mathcal{M}_\nu^b-$metric space. We demonstrate that the collection of open balls forms a basis on $\mathcal{M}_\nu^b$-metric space. Further, we give some examples for the verification of established results. Towards the end, we solve a non-linear matrix equation and an equation of rotation of a hanging cable to substantiate the utility of these extensions.

1 Introduction and preliminaries

Distance is one of the earliest perceptions appreciated by humans. Initially, the idea of distance appeared during the period of Euclid. In 1906, Maurice Rene Frećhet [7] introduced the general and more axiomatic form of a distance and named it "L-space". Felix Hausdorff [9] reviewed it as a metric space. Subsequently, numerous refined, generalized, and extended versions of the metric

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structure appeared in the literature. For details, about the generalizations of the metric notion, one may refer to Kirk and Shahzad [12]. In most of these improvements, extensions and generalizations of Banach's result [4] have been announced.

The aim of the present work is to utilize a novel notion of distance called an $\mathcal{M}_v^b-$metric [10], which is an improvement and generalization of the $\mathcal{M}_v-$metric [3], to revisit the acclaimed Banach contraction principle [4] and Sehgal [20] besides validating it with suitable examples. We also compare some of the existing structures, $\mathcal{M}-$metric [1], $\mathcal{M}_v-$metric [3], usual metric [4], $\mathfrak{b}-$metric [5], rectangular metric [6], generalized $\nu-$metric [6], rectangular $\mathfrak{b}-$metric [8], generalized partial metric $\mathfrak{p}_v^b$ [11], $\mathcal{M}_\mathfrak{b}-$metric [13], partial metric [14], generalized $\mathfrak{d}_v^b-$metric [15], rectangular $\mathcal{M}-$metric [17], rectangular partial metric [19], partial $\mathfrak{b}-$metric [21] to demonstrate the superiority of $\mathcal{M}_v^b-$metric over existing notions of distances. Besides, we demonstrate that the collection of open balls forms a basis on $\mathcal{M}_v^b-$metric space. Towards the end, we solve a non-linear matrix equation and an equation of rotation of a hanging cable to substantiate the utility of these extensions. These fixed point results promote further examinations and applications in metric fixed point theory.

2 Preliminaries

In the following, we denote:

$$\mathfrak{m}_{\nu_{u,w}} = \min\{\mathfrak{m}_\nu(u,u),\ \mathfrak{m}_\nu(w,w)\} \text{ and } \mathcal{M}_{\nu_{u,w}} = \max\{\mathfrak{m}_\nu(u,u),\ \mathfrak{m}_\nu(w,w)\}.$$

In 2017, Mitrović and Radenović [15] announced a generalized $\mathfrak{d}_v^b-$metric.

Definition 1 *A generalized $\mathfrak{d}_v^b-$ metric on a nonempty set $\mathcal{M}$ with $s \geq 1$, is a map $\mathfrak{d}_v^b : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying:*

($\mathfrak{d}_v^b$(i)) $\mathfrak{d}_v^b(u, \mathfrak{w}) = 0$ *if and only if* $u = \mathfrak{w}$,

($\mathfrak{d}_v^b$(ii)) $\mathfrak{d}_v^b(u, \mathfrak{w}) \geq 0$,

($\mathfrak{d}_v^b$(iii)) $\mathfrak{d}_v^b(u, \mathfrak{w}) = \mathfrak{d}_v^b(\mathfrak{w}, u)$,

($\mathfrak{d}_v^b$(iv)) $(\mathfrak{d}_v^b(u, \mathfrak{w}) \leq s[(\mathfrak{d}_v^b(u, \mathfrak{z}_1) + (\mathfrak{d}_v^b(\mathfrak{z}_1, \mathfrak{z}_2) + \cdots + (\mathfrak{d}_v^b(\mathfrak{z}_v, \mathfrak{w})]$,

$u,\ \mathfrak{z}_1,\ \mathfrak{z}_2,\ \ldots,\ \mathfrak{z}_\mathfrak{v},\ \mathfrak{w} \in \mathcal{M}$ *and are distinct. A pair $(\mathcal{M}, \mathfrak{d}_v^b)$ is called a generalized $\mathfrak{d}_v^b-$metric space.*

Remark 1 *A generalized d_v^b-metric [15] reduces to a $v-$generalized metric [6] on taking $\mathfrak{s} = 1$, a rectangular metric [6] on taking $v = 2$ and $\mathfrak{s} = 1$, a rectangular b$-$metric [8] on taking $v = 2$, b$-$metric [5] on taking $v = 1$ and a usual metric [4] on taking $v = \mathfrak{s} = 1$.*

In 2018, Karahan and Isik [11] introduced the notion of a generalized partial metric space p_v^b.

Definition 2 *A generalized p_v^b-partial metric on a nonempty set $\mathcal{M}$ with $s \geq 1$, is a map $p_v^b : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying:*

$(p_v^b i)$ $p_v^b(\mathfrak{u}, \mathfrak{u}) = p_v^b(\mathfrak{w}, \mathfrak{w}) = p_v^b(\mathfrak{u}, \mathfrak{w})$ *if and only if* $\mathfrak{u} = \mathfrak{w}$,

$(p_v^b ii)$ $p_v^b(\mathfrak{u}, \mathfrak{u}) \leq p_v^b(\mathfrak{u}, \mathfrak{w})$,

$(p_v^b iii)$ $p_v^b(\mathfrak{u}, \mathfrak{w}) = p_v^b(\mathfrak{w}, \mathfrak{u})$,

$(p_v^b iv)$ $p_v^b(\mathfrak{u}, \mathfrak{w}) \leq s[p_v^b(\mathfrak{u}, \mathfrak{z}_1) + p_v^b(\mathfrak{z}_1, \mathfrak{z}_2) + \cdots + p_v^b(\mathfrak{z}_v, \mathfrak{w})] - \Sigma_{i=1}^{v} p_v^b(\mathfrak{z}_i, \mathfrak{z}_i)$,

$\mathfrak{u}, \mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_v, \mathfrak{w} \in \mathcal{M}$ *and are distinct. A pair $(\mathcal{M}, p_v^b)$ is a generalized p_v^b-partial metric space.*

Remark 2 *A generalized p_v^b-partial metric reduces to a rectangular partial metric [19] on taking $v = 2$ and $\mathfrak{s} = 1$, a rectangular partial b$-$metric [11] on taking $v = 2$, a partial b$-$metric [21] on taking $v = 1$ and a partial metric [14] on taking $v = \mathfrak{s} = 1$.*

In 2019, Asim et al. [3] announced M_v-metric.

Definition 3 *An M_v-metric on a nonempty set $\mathcal{M}$ is a map $m_v : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying:*

$(m_v i)$ $m_v(\mathfrak{u}, \mathfrak{u}) = m_v(\mathfrak{w}, \mathfrak{w}) = m_v(\mathfrak{u}, \mathfrak{w})$ *if and only if* $\mathfrak{u} = \mathfrak{w}$,

$(m_v ii)$ $m_{v_{\mathfrak{u},\mathfrak{w}}} \leq m_v(\mathfrak{u}, \mathfrak{w})$,

$(m_v iii)$ $m_v(\mathfrak{u}, \mathfrak{w}) = m_v(\mathfrak{w}, \mathfrak{u})$,

$(m_v iv)$ $(m_v(\mathfrak{u}, \mathfrak{w}) - m_{v_{\mathfrak{u},\mathfrak{w}}}) \leq (m_v(\mathfrak{u}, \mathfrak{z}_1) - m_{v_{\mathfrak{u},\mathfrak{z}_1}}) + (m_v(\mathfrak{z}_1, \mathfrak{z}_2) - m_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}) + \cdots + (m_v(\mathfrak{z}_v, \mathfrak{w}) - m_{v_{\mathfrak{z}_v,\mathfrak{w}}})$,

$\mathfrak{u}, \mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_v, \mathfrak{w} \in \mathcal{M}$ *and are distinct. A pair $(\mathcal{M}, m_v)$ is an M_v-metric space.*

Remark 3 *If $v = 1$, M_v is an $M-$metric [1] and if $v = 2$, it is a rectangular metric [17].*

Example 1 [3] *Let $\mathcal{M} = \mathbb{R}$. Define $m_v : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ by $m_v(\mathfrak{u}, \mathfrak{w}) = \frac{|\mathfrak{u}|+|\mathfrak{w}|}{2}$, $\mathfrak{u}, \mathfrak{w} \in \mathcal{M}$, then m_v is an M_v-metric.*

3 Main results

Joshi et al. [10] used the following notations

$$\mathfrak{m}_{v_{\mathfrak{u},\mathfrak{w}}}^b = \min\{\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u}),\ \mathfrak{m}_v^b(\mathfrak{w},\mathfrak{w})\} \text{ and } M_{v_{\mathfrak{u},\mathfrak{w}}}^b = \max\{\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u}),\ \mathfrak{m}_v^b(\mathfrak{w},\mathfrak{w})\},$$

and introduced M_v^b-metric space.

Definition 4 *An M_v^b−metric on a non-empty set $\mathcal{M}$ with $s \geq 1$, is a map $\mathfrak{m}_v^b : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying:*

$(\mathfrak{m}_v^b i)$ $\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u}) = \mathfrak{m}_v^b(\mathfrak{w},\mathfrak{w}) = \mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w})$ *if and only if* $\mathfrak{u} = \mathfrak{w}$,

$(\mathfrak{m}_v^b ii)$ $\mathfrak{m}_{v_{\mathfrak{u},w}}^b \leq \mathfrak{m}_v^b(\mathfrak{u},w)$,

$(\mathfrak{m}_v^b iii)$ $\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}) = \mathfrak{m}_v^b(\mathfrak{w},\mathfrak{u})$,

$(\mathfrak{m}_v^b iv)$ $(\mathfrak{m}_v^b(\mathfrak{u},w) - \mathfrak{m}_{v_{\mathfrak{u},w}}^b) \leq s[(\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{z}_1) - \mathfrak{m}_{v_{\mathfrak{u},\mathfrak{z}_1}}^b) + (\mathfrak{m}_v^b(\mathfrak{z}_1,\mathfrak{z}_2) - \mathfrak{m}_{v_{\mathfrak{z}_1,\mathfrak{z}_2}}^b) + \cdots + (\mathfrak{m}_v^b(\mathfrak{z}_v,w) - \mathfrak{m}_{v_{\mathfrak{z}_v,w}}^b)] - \Sigma_{i=1}^{v}\mathfrak{m}_v^b(\mathfrak{z}_i,\mathfrak{z}_i)$,

$\mathfrak{u},\ \mathfrak{z}_1,\ \mathfrak{z}_2,\ \ldots,\ \mathfrak{z}_v,\ w \in \mathcal{M}$ *and are distinct. A pair $(\mathcal{M}, \mathfrak{m}_v^b)$ is called an M_v^b−metric space.*

Remark 4 *If $s = 1$, $(\mathcal{M}, \mathfrak{m}_v^b)$ is an improvement and extension of M_v−metric space [3]. In particular, if $v = \mathfrak{s} = 1$, $(\mathcal{M}, \mathfrak{m}_v^b)$ is an M_b−metric space [13].*

Example 2 *Let $\mathcal{M} = \mathbb{R}^+$ and $\mathfrak{m}_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ be defined as: $\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}) = \frac{1+|\mathfrak{u}-\mathfrak{w}|^\alpha}{|\mathfrak{u}-\mathfrak{w}|^\alpha} + \max\{\mathfrak{u},\ \mathfrak{w}\}^\alpha$, $\alpha > 1$. By routine calculations, one may verify that $(\mathcal{M}, \mathfrak{m}_v^b)$ is an M_v^b−metric space with $s \geq 2^{\alpha-1}$. But $(\mathcal{M}, \mathfrak{m}_v^b)$ is not an M_v−metric space. Since, for $\mathfrak{u} = 1$, $\mathfrak{w} = n$ and $\mathfrak{z}_1 = 2$, $\mathfrak{z}_2 = 3$, $\ldots$, $\mathfrak{z}_v = n-1$, we obtain*

$$\mathfrak{m}_v^b(1,n) - \mathfrak{m}_{v_{1,n}}^b = \frac{|1-n|^\alpha}{1+|1-n|^\alpha} + \max\{1,n\}^\alpha - 1^\alpha = \frac{|1-n|^\alpha}{1+|1-n|^\alpha} + n^\alpha - 1^\alpha,$$

$$\mathfrak{m}_v^b(1,2) - \mathfrak{m}_{v_{1,2}}^b = \frac{|1-2|^\alpha}{1+|1-2|^\alpha} + \max\{1,2\}^\alpha - 1^\alpha = \frac{1}{2} + 2^\alpha - 1^\alpha,$$

$$\mathfrak{m}_v^b(2,3) - \mathfrak{m}_{v_{2,3}}^b = \frac{|2-3|^\alpha}{1+|2-3|^\alpha} + \max\{2,3\}^\alpha - 2^\alpha = \frac{1}{2} + 3^\alpha - 2^\alpha,$$

$$\vdots$$

$$\mathfrak{m}_v^b(n-2,n-1) - \mathfrak{m}_{v_{n-2,n-1}}^b = \frac{|n-2-n+1|^\alpha}{1+|n-2-n+1|^\alpha} + \max\{n-2,n-1\}^\alpha - (n-2)^\alpha$$
$$= \frac{1}{2} + (n-1)^\alpha - (n-2)^\alpha.$$

Therefore, $\mathfrak{m}_v^b(1,n) - \mathfrak{m}_{v_{1,n}}^b > \mathfrak{m}_v^b(1,2) - \mathfrak{m}_{v_{1,2}}^b + \mathfrak{m}_v^b(2,3) - \mathfrak{m}_{v_{2,3}}^b + \cdots + \mathfrak{m}_v^b(n-2,n-1) - \mathfrak{m}_{v_{n-2,n-1}}^b$.

To discuss the topology corresponding to M_ν^b−metric, Joshi et al. [10] defined the open ball centered at $\mathfrak{u}$ and radius $\varepsilon \in (0,\infty)$ as
$\mathcal{U}_{M_\nu^b}(\mathfrak{u},\varepsilon) = \{\mathfrak{w} \in \mathcal{M} : \mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}) < \mathfrak{m}_{\nu_{\mathfrak{u},\mathfrak{w}}}^b + \frac{\varepsilon}{\mathfrak{s}}\}$.
Similarly, the closed ball [10] centered at $\mathfrak{u}$ and radius $\varepsilon \in (0,\infty)$ is defined as
$\mathcal{U}_{M_\nu^b}[\mathfrak{u},\varepsilon] = \{\mathfrak{w} \in \mathcal{M} : \mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}) \le \mathfrak{m}_{\nu_{\mathfrak{u},\mathfrak{w}}}^b + \frac{\varepsilon}{\mathfrak{s}}\}$.

Lemma 1 *The collection of all open balls in an M_ν^b−metric space $(\mathcal{M},\mathfrak{m}_\nu^b)$, $\mathcal{U}_{\mathfrak{m}_\nu^b}(\mathfrak{u},\mathfrak{r}) = \{\mathfrak{w} \in \mathcal{M} : \mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}) < \mathfrak{m}_{\nu_{\mathfrak{u},\mathfrak{w}}}^b + \frac{\varepsilon}{\mathfrak{s}}\}$, forms a basis on $\mathcal{M}$.*

Proof. Let $\mathfrak{w}_0 \in \mathcal{U}_{\mathfrak{m}_\nu^b}(\mathfrak{u},\mathfrak{r})$, then $\mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}_o) < \mathfrak{m}_{\nu_{\mathfrak{u},\mathfrak{w}_o}}^b + \frac{\mathfrak{r}}{\mathfrak{s}}$. Choose, $\frac{\varepsilon}{\mathfrak{s}} = \mathfrak{m}_{\nu_{\mathfrak{u},\mathfrak{w}_o}}^b + \frac{\mathfrak{r}}{\mathfrak{s}} - \mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}_o) > 0$.
Again, let $\mathfrak{w}_1 \in \mathcal{U}_{\mathfrak{m}_\nu^b}(\mathfrak{w}_o,\varepsilon)$, so $\mathfrak{m}_\nu^b(\mathfrak{w}_1,\mathfrak{w}_o) < \mathfrak{m}_{\nu_{\mathfrak{w}_1,\mathfrak{w}_o}}^b + \frac{\varepsilon}{\mathfrak{s}}$ and choose $\frac{\varepsilon_1}{\mathfrak{s}} = \mathfrak{m}_{\nu_{\mathfrak{w}_1,\mathfrak{w}_o}}^b + \frac{\varepsilon}{\mathfrak{s}} - \mathfrak{m}_\nu^b(\mathfrak{w}_1,\mathfrak{w}_o) > 0$.
In same way, let $\mathfrak{w}_\nu \in \mathcal{U}_{\mathfrak{m}_\nu^b}(\mathfrak{w}_{\nu-1},\varepsilon_\nu)$, so $\mathfrak{m}_\nu^b(\mathfrak{w}_\nu,\mathfrak{w}_{\nu-1}) < \mathfrak{m}_{\nu_{\mathfrak{w}_\nu,\mathfrak{w}_{\nu-1}}}^b + \frac{\varepsilon_{\nu-1}}{\mathfrak{s}}$, choose $\frac{\varepsilon_\nu}{\mathfrak{s}} = \mathfrak{m}_{\nu_{\mathfrak{w}_\nu,\mathfrak{w}_{\nu-1}}}^b + \frac{\varepsilon_{\nu-1}}{\mathfrak{s}} - \mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}_1) > 0$.
Now, for $\mathfrak{u},\mathfrak{w}_o,\mathfrak{w}_1,\ldots,\mathfrak{w}_\nu$,

$$\begin{aligned}
\mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}_\nu) - \mathfrak{m}_{\nu_{\mathfrak{u},\mathfrak{w}_\nu}} &\le \mathfrak{s}[(\mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}_0) - \mathfrak{m}_{1_{\mathfrak{u},\mathfrak{w}_0}}) + (\mathfrak{m}_\nu^b(\mathfrak{w}_0,\mathfrak{w}_1) \\
&\quad - \mathfrak{m}_{\nu_{\mathfrak{w}_0,\mathfrak{w}_1}}) + \cdots + (\mathfrak{m}_\nu^b(\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu) - \mathfrak{m}_{\nu_{\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu}})] \\
&\quad - \mathfrak{m}_\nu^b(\mathfrak{w}_1,\mathfrak{w}_1) - \mathfrak{m}_\nu^b(\mathfrak{w}_2,\mathfrak{w}_2) - \cdots - \mathfrak{m}_\nu^b(\mathfrak{w}_{\nu-1},\mathfrak{w}_{\nu-1}) \\
&\le \mathfrak{s}[(\mathfrak{m}_\nu^b(\mathfrak{u},\mathfrak{w}_0) - \mathfrak{m}_{1_{\mathfrak{u},\mathfrak{w}_0}}) + (\mathfrak{m}_\nu^b(\mathfrak{w}_0,\mathfrak{w}_1) - \mathfrak{m}_{\nu_{\mathfrak{w}_0,\mathfrak{w}_1}}) \\
&\quad + \cdots + (\mathfrak{m}_\nu^b(\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu) - \mathfrak{m}_{\nu_{\mathfrak{w}_{\nu-1},\mathfrak{w}_\nu}})] \\
&= \mathfrak{s}\left[\left(\frac{\mathfrak{r}}{\mathfrak{s}} - \frac{\varepsilon}{\mathfrak{s}}\right) + \left(\frac{\varepsilon}{\mathfrak{s}} - \frac{\varepsilon_1}{\mathfrak{s}}\right) + \cdots + \left(\frac{\varepsilon_{\nu-1}}{\mathfrak{s}} - \frac{\varepsilon_\nu}{\mathfrak{s}}\right)\right] \\
&= \mathfrak{r} - \varepsilon_\nu.
\end{aligned}$$

Hence, $\mathcal{U}_{\mathfrak{m}_\nu^b}(\mathfrak{w}_o,\varepsilon) \subseteq \mathcal{U}_{\mathfrak{m}_\nu^b}(\mathfrak{u},\mathfrak{r})$. □

Joshi et al. [10] discussed the convergence of the sequence and introduced definitions related to it.

Definition 5 (i) *A sequence $\{\mathfrak{u}_n\}$ in $(\mathcal{M},\mathfrak{m}_\nu^b)$ is $\mathfrak{m}_\nu^b$−convergent to $\mathfrak{u} \in \mathcal{M}$ if and only if* $\lim_{n\longrightarrow\infty} \mathfrak{m}_\nu^b(\mathfrak{u}_n,\mathfrak{u}) - \mathfrak{m}_{\nu_{\mathfrak{u}_n,\mathfrak{u}}} = 0$.
In other words, a sequence $\{\mathfrak{u}_n\}$ in a topological space $(\mathcal{M},\tau_\nu^b)$ converges to a point $\mathfrak{u}$ in $\mathcal{M}$ if for each open ball $\mathcal{U}_{M_\nu^b}(\mathfrak{u},\varepsilon)$ containing $\mathfrak{u}$, there exists a number k such that for each $n > k$, $\mathfrak{u}_n \in \mathcal{U}_{M_\nu^b}(\mathfrak{u},\varepsilon)$.

(ii) *A sequence $\{\mathfrak{u}_n\}$ in $(\mathcal{M},\mathfrak{m}_\nu^b)$ is an $\mathfrak{m}_\nu^b$−Cauchy if and only if* $\lim_{n,m\longrightarrow\infty}$ $(\mathfrak{m}_\nu^b(\mathfrak{u}_n,\mathfrak{u}_m) - \mathfrak{m}_{\nu_{\mathfrak{u}_n,\mathfrak{u}_m}}^b)$ *and* $\lim_{n,m\longrightarrow\infty}(M_{\nu_{\mathfrak{u}_n,\mathfrak{u}_m}}^b - \mathfrak{m}_{\nu_{\mathfrak{u}_n,\mathfrak{u}_m}}^b)$ *exist and are finite.*

(iii) *An M_v^b-metric space is an m_v^b-complete if every m_v^b-Cauchy sequence $\{u_n\}$ converges to a point $u \in \mathcal{M}$ such that* $\lim_{n,m\longrightarrow\infty}(m_v^b(u_n,u) - m^b_{v_{u_n,u}}) = 0$ *and* $\lim_{n,m\longrightarrow\infty}(M^b_{v_{u_n,u}} - m^b_{v_{u_n,u}}) = 0$.

We shall use the following lemma to revisit the Banach contraction principle [4] in M_v^b-metric space $(\mathcal{M}, m_v^b)$.

Lemma 2 [10] *Let $(\mathcal{M}, m_v^b)$ be an M_v^b-metric space and $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ be a self map on $\mathcal{M}$. If there exists $\eta \in [0, \frac{1}{\mathfrak{s}})$, satisfying:*

$$m_v^b(\mathcal{A}u, \mathcal{A}\mathfrak{w}) \leq \eta m_v^b(u, \mathfrak{w}). \tag{1}$$

Consider the sequence $\{u_n\}$ defined as $u_{n+1} = \mathcal{A}u_n$. If $u_n \longrightarrow u$ as $n \longrightarrow \infty$, then $\mathcal{A}u_n \longrightarrow \mathcal{A}u$ as $n \longrightarrow \infty$.

Theorem 1 *Let $(\mathcal{M}, m_v^b)$ be an M_v^b-complete metric space. Suppose a self map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ satisfies*

$$m_v^b(\mathcal{A}u, \mathcal{A}\mathfrak{w}) \leq \eta m_v^b(u, \mathfrak{w}), \quad \eta \in [0, \frac{1}{\mathfrak{s}}) \ \ \textit{and} \ \ u,\ \mathfrak{w} \in \mathcal{M}. \tag{2}$$

Then, $\mathcal{A}$ has a unique fixed point $u \in \mathcal{M}$ such that $m_v^b(u, u) = 0$..

Proof. Starting from the given element $u_0 \in \mathcal{M}$, form the sequence $\{u_n\}$, where $u_n = \mathcal{A}u_{n-1},\ n \in \mathbb{N}$. If $m_v^b(u_n, u_{n+1}) = 0,\ n \geqslant 0$, then $\mathcal{A}u_n = u_{n+1} = u_n$ and $m_v^b(u_n, u_n) = 0$ and this completes the proof.
Further, take $m_v^b(u_n, u_{n+1}) > 0,\ \ n \geqslant 0$. For $u = u_n,\ \mathfrak{w} = u_{n+1},$ utilizing condition (2),

$$\begin{aligned} m_v^b(u_{n+1}, u_{n+2}) &= m_v^b(\mathcal{A}u_n, \mathcal{A}u_{n+1}) \\ &\leq \eta m_v^b(u_n, u_{n+1}) \\ &\leq \eta^n m_v^b(u_0, u_1) \longrightarrow 0, \text{ as, } n \longrightarrow \infty. \end{aligned}$$

Also,

$$\begin{aligned} m_v^b(u_{n+1}, u_{n+1}) &= m_v^b(\mathcal{A}u_n, \mathcal{A}u_n) \\ &\leq \eta m_v^b(u_n, u_n) \\ &\leq \eta^n m_v^b(u_0, u_0) \longrightarrow 0, \text{ as, } n \longrightarrow \infty. \end{aligned}$$

First, we show that $u_n \neq u_m$, for $n \neq m$. Suppose $u_n = u_m$, for $n > m$, then $\mathcal{A}u_n = u_{n+1} = \mathcal{A}u_m = u_{m+1}$. Now, by using inequality (2), for $u = u_n$ and

$\mathfrak{w} = u_{n+1}$,
$m_v^b(u_m, u_{m+1}) = m_v^b(\mathcal{A}u_{n-1}, \mathcal{A}u_n) \leq \eta m_v^b(u_{n-1}, u_n) \leq \eta^2 m_v^b(u_{n-2}, u_{n-1}) \leq \cdots \leq \eta^{n-m} m_v^b(u_m, u_{m+1}) < m_v^b(u_m, u_{m+1})$, a contradiction. Thus, $u_n \neq u_m$, for $n \neq m$.
Now, we show that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{M}, m_v^b)$. We discuss two cases:

Case(i) First, let l be odd, that is, $l = 2m+1$, for $n, m \in \mathbb{N}$. Now, by using $(m_v^b iv)$ for $n \leq v \leq n+l$,

$$\begin{aligned}
m_v^b(u_n, u_{n+l}) &= m_v^b(u_n, u_{n+2m+1}) \\
&\leq \mathfrak{s}\big[m_v^b(u_n, u_{n+1}) + m_v^b(u_{n+1}, u_{n+2}) + \cdots + m_v^b(u_{n+v-1}, u_{n+v}) \\
&\quad + m_v^b(u_{n+v}, u_{n+2m+1})\big] - m_v^b(u_{n+1}, u_{n+1}) \\
&\quad - m_v^b(u_{n+2}, u_{n+2}) - \cdots - m_v^b(u_{n+v}, u_{n+v}) \\
&\leq \mathfrak{s}\big(\eta^{n-1} + \eta^n + \ldots \eta^{n+v-2}\big) m_v^b(u_0, u_1) \\
&\quad - \big(\eta^n + \eta^{n+1} + \cdots + \eta^{n+v-1}\big) m_v^b(u_0, u_1) + \mathfrak{s} m_v^b(u_{n+v}, u_{n+2m+1}) \\
&= \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^v)}{1-\eta}\right) m_v^b(u_0, u_1) - \frac{\eta^n(1-\eta^v)}{1-\eta} m_v^b(u_0, u_1) \\
&\quad + \mathfrak{s} m_v^b(u_{n+v}, u_{n+2m+1}) \\
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^v)}{1-\eta}\right) m_v^b(u_0, u_1) - \frac{\eta^n(1-\eta^v)}{1-\eta} m_v^b(u_0, u_1) \\
&\quad + \mathfrak{s}^2[m_v^b(u_{n+v}, u_{n+v+1}) + m_v^b(u_{n+v+1}, u_{n+v+2}) \\
&\quad + \cdots + m_v^b(u_{n+2v-1}, u_{n+2v}) + m_v^b(u_{n+2v}, u_{n+2m+1})] \\
&\quad - \mathfrak{s}[m_v^b(u_{n+v+1}, u_{n+v+1}) + m_v^b(u_{n+v+2}, u_{n+v+2}) + \ldots + m_v^b(u_{n+2v}, u_{n+2v})] \\
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^v)}{1-\eta}\right) m_v^b(u_0, u_1) - \frac{\eta^n(1-\eta^v)}{1-\eta} m_v^b(u_0, u_1) \\
&\quad + \mathfrak{s}^2(\eta^{n+v-1} + \eta^{n+v} + \cdots + \eta^{n+2v-2}) m_v^b(u_0, u_1) \\
&\quad + \mathfrak{s}^2 m_v^b(u_{n+2v}, u_{n+2m+1}) - \mathfrak{s}(\eta^{n+v} + \eta^{n+v+1} + \cdots + \eta^{n+2v-1}) m_v^b(u_0, u_1) \\
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^v)}{1-\eta}\right) m_v^b(u_0, u_1) - \frac{\eta^n(1-\eta^v)}{1-\eta} m_v^b(u_0, u_1) \\
&\quad + \mathfrak{s}^2\left(\frac{\eta^{n+v-1}(1-\eta^v)}{1-\eta}\right) m_v^b(u_0, u_1) - \mathfrak{s}\frac{\eta^{n+v}(1-\eta^v)}{1-\eta} m_v^b(u_0, u_1) \\
&\quad + \cdots + \mathfrak{s}^{\frac{2m}{v}-1} m_v^b(u_{n+2m-v}, u_{n+2m+1})
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^{\nu})}{1-\eta}\right)m_v^b(u_0,u_1) - \frac{\eta^{n}(1-\eta^{\nu})}{1-\eta}m_v^b(u_0,u_1)\\
&\quad + \mathfrak{s}^2\left(\frac{\eta^{n+\nu-1}(1-\eta^{\nu})}{1-\eta})m_v^b(u_0,u_1\right) - \mathfrak{s}\frac{\eta^{n+\nu}(1-\eta^{\nu})}{1-\eta}m_v^b(u_0,u_1)\\
&\quad + \cdots + \mathfrak{s}^{\frac{2m}{\nu}}[m_v^b(u_{n+2m-\nu},u_{n+2m-\nu+1}) + m_v^b(u_{n+2m-\nu+1},u_{n+2m-\nu+2})\\
&\quad + \cdots + m_v^b(u_{n+2m},u_{n+2m+1})] - \mathfrak{s}^{\frac{2m}{\nu}-1}[m_v^b(u_{n+2m-\nu+1},u_{n+2m-\nu+1})\\
&\quad + \cdots + m_v^b(u_{n+2m},u_{n+2m})]\\
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^{\nu})}{1-\eta}\right)m_v^b(u_0,u_1) - \frac{\eta^{n}(1-\eta^{\nu})}{1-\eta}m_v^b(u_0,u_1)\\
&\quad + \mathfrak{s}^2\left(\frac{\eta^{n+\nu-1}(1-\eta^{\nu})}{1-\eta}\right)m_v^b(u_0,u_1) - \mathfrak{s}\frac{\eta^{n+\nu}(1-\eta^{\nu})}{1-\eta}m_v^b(u_0,u_1)\\
&\quad + \cdots + \mathfrak{s}^{\frac{2m}{\nu}}\left(\frac{\eta^{n+2m-\nu-1}(1-\eta^{\nu})}{1-\eta}\right)m_v^b(u_0,u_1)\\
&\quad - \mathfrak{s}^{\frac{2m}{\nu}-1}\frac{\eta^{n+2m-\nu}(1-\eta^{\nu})}{1-\eta}m_v^b(u_0,u_1) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty,
\end{aligned}
$$

that is, $\lim_{n,m\longrightarrow\infty} m_v^b(u_n,u_{n+2m+1}) = 0$.

Case (ii) Now, let l is even, that is, $l = 2m$ for $n, m \in \mathbb{N}$.
Now, by using $(m_v^b iv)$ for $n \leq \nu \leq n + l$,

$$
\begin{aligned}
m_v w^b(u_n,u_{n+l}) &= m_v^b(u_n,u_{n+2m})\\
&\leq \mathfrak{s}[m_v^b(u_n,u_{n+1}) + m_v^b(u_{n+1},u_{n+2}) + \cdots + m_v^b(u_{n+\nu-1},u_{n+\nu})\\
&\quad + m_v^b(u_{n+\nu},u_{n+2m})] - m_v^b(u_{n+1},u_{n+1}) - m_v^b(u_{n+2},u_{n+2})\\
&\quad - \cdots - m_v^b(u_{n+\nu},u_{n+\nu})\\
&\leq \mathfrak{s}(\eta^{n-1} + \eta^{n} + \ldots \eta^{n+\nu-2})m_v^b(u_0,u_1)\\
&\quad - (\eta^{n} + \eta^{n+1} + \cdots + \eta^{n+\nu-1})m_v^b(u_0,u_1) + \mathfrak{s}m_v^b(u_{n+\nu},u_{n+2m})\\
&= \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^{\nu})}{1-\eta}\right)m_v^b(u_0,u_1) - \frac{\eta^{n}(1-\eta^{\nu})}{1-\eta}m_v^b(u_0,u_1)\\
&\quad + \mathfrak{s}m_v^b(u_{n+\nu},u_{n+2m})\\
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^{\nu})}{1-\eta}\right)m_v^b(u_0,u_1) - \frac{\eta^{n}(1-\eta^{\nu})}{1-\eta}m_v^b(u_0,u_1)\\
&\quad + \mathfrak{s}^2[m_v^b(u_{n+\nu},u_{n+\nu+1}) + m_v^b(u_{n+\nu+1},u_{n+\nu+2})\\
&\quad + \cdots + m_v^b(u_{n+2\nu-1},u_{n+2\nu}) + m_v^b(u_{n+2\nu},u_{n+2m+1})]
\end{aligned}
$$

$$
\begin{aligned}
&-\mathfrak{s}[\mathfrak{m}_\nu^b(u_{n+\nu+1},u_{n+\nu+1})+\mathfrak{m}_\nu^b(u_{n+\nu+2},u_{n+\nu+2})+\cdots+\mathfrak{m}_\nu^b(u_{n+2\nu},u_{n+2\nu})]\\
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^\nu)}{1-\eta}\right)\mathfrak{m}_\nu^b(u_0,u_1)-\frac{\eta^{n}(1-\eta^\nu)}{1-\eta}\mathfrak{m}_\nu^b(u_0,u_1)+\mathfrak{s}^2(\eta^{n+\nu-1}+\eta^{n+\nu}\\
&+\cdots+\eta^{n+2\nu-2})\mathfrak{m}_\nu^b(u_0,u_1)+\mathfrak{s}^2\mathfrak{m}_\nu^b(u_{n+2\nu},u_{n+2m})\\
&-\mathfrak{s}(\eta^{n+\nu}+\eta^{n+\nu+1}+\cdots+\eta^{n+2\nu-1})\mathfrak{m}_\nu^b(u_0,u_1)\\
&\leq \mathfrak{s}\left(\frac{\eta^{n-1}(1-\eta^\nu)}{1-\eta}\right)\mathfrak{m}_\nu^b(u_0,u_1)-\frac{\eta^{n}(1-\eta^\nu)}{1-\eta}\mathfrak{m}_\nu^b(u_0,u_1)\\
&+\mathfrak{s}^2\left(\frac{\eta^{n+\nu-1}(1-\eta^\nu)}{1-\eta}\right)\mathfrak{m}_\nu^b(u_0,u_1)-\mathfrak{s}\frac{\eta^{n+\nu}(1-\eta^\nu)}{1-\eta}\mathfrak{m}_\nu^b(u_0,u_1)\\
&+\cdots+\mathfrak{s}^{\frac{2m}{\nu}}\left(\frac{\eta^{n+2m-\nu-2}(1-\eta^\nu)}{1-\eta}\right)\mathfrak{m}_\nu^b(u_0,u_1)\\
&-\mathfrak{s}^{\frac{2m}{\nu}-1}\frac{\eta^{n+2m-\nu-1}(1-\eta^\nu)}{1-\eta}\mathfrak{m}_\nu^b(u_0,u_1)\longrightarrow 0,\quad \text{as},\quad n\longrightarrow\infty,
\end{aligned}
$$

that is, $\lim_{n,m\longrightarrow\infty}\mathfrak{m}_\nu^b(u_n,u_{n+2m})=0.$

So, $\lim_{n,m\longrightarrow\infty}(\mathfrak{m}_\nu^b(u_n,u_m)-\mathfrak{m}^b_{\nu_{u_n,u_m}})=0.$
Let $\mathfrak{M}_\nu^b(u_n,u_m)=\mathfrak{m}_\nu^b(u_n,u_n)$. Now,
$\mathfrak{M}_\nu^b(u_n,u_m)-\mathfrak{m}_\nu^b(u_n,u_m)\leq \mathfrak{M}_\nu^b(u_n,u_m)=\mathfrak{m}_\nu^b(u_n,u_n)\leq\eta^{n-1}\mathfrak{m}_\nu^b(u_0,u_0)\longrightarrow 0$, as $n\longrightarrow\infty$.
So, $\lim_{n,m\longrightarrow\infty}\mathfrak{M}_\nu^b(u_n,u_m)-\mathfrak{m}_\nu^b(u_n,u_m)=0.$
Consequently, the sequence $\{u_n\}$ is $\mathfrak{m}_\nu^b-$Cauchy in $\mathcal{M}$. Since, $\mathcal{M}$ is $\mathfrak{m}_\nu^b-$complete, there exists $u\in\mathcal{U}$ so that $u_n\longrightarrow u$. Now, we assert that $\mathcal{A}u=u$.

$$
\begin{aligned}
&\lim_{n\longrightarrow\infty}(\mathfrak{m}_\nu^b(u_n,u)-\mathfrak{m}^b_{\nu_{u_n,u}})=0\\
&\Rightarrow\lim_{n\longrightarrow\infty}(\mathfrak{m}_\nu^b(u_{n+1},u)-\mathfrak{m}^b_{\nu_{u_{n+1},u}})=0\\
&\Rightarrow\lim_{n\longrightarrow\infty}(\mathfrak{m}_\nu^b(\mathcal{A}u_n,u)-\mathfrak{m}^b_{\nu_{\mathcal{A}u_n,u}})=0\\
&\Rightarrow \mathfrak{m}_\nu^b(\mathcal{A}u,u)-\mathfrak{m}^b_{\nu_{\mathcal{A}u,u}}=0,\quad\text{(using Lemma 2)},
\end{aligned}
$$

that is, $\mathfrak{m}_\nu^b(\mathcal{A}u,u)=\min\{\mathfrak{m}_\nu^b(\mathcal{A}u,\mathcal{A}u),\ \mathfrak{m}_\nu^b(u,u)\}$
$\Rightarrow\mathfrak{m}_\nu^b(\mathcal{A}u,u)=\mathfrak{m}_\nu^b(\mathcal{A}u,\mathcal{A}u)$ or $\mathfrak{m}_\nu^b(\mathcal{A}u,u)=\mathfrak{m}_\nu^b(u,u)$.
Hence, $\mathcal{A}u=u$, that is, u is a fixed point of $\mathcal{A}$.
To conclude the theorem, suppose u and $\mathfrak{w}$ are two different fixed points of $\mathcal{A}$, so
$\mathfrak{m}_\nu^b(u,\mathfrak{w})=\mathfrak{m}_\nu^b(\mathcal{A}u,\mathcal{A}\mathfrak{w})\leq\eta\mathfrak{m}_\nu^b(u,\mathfrak{w})\Rightarrow\mathfrak{m}_\nu^b(u,\mathfrak{w})=0$. Hence, $u=\mathfrak{w}$.
Next, we assert that if u is a fixed point, then $\mathfrak{m}_\nu^b(u,u)=0$.

$\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u}) = \mathfrak{m}_v^b(\mathcal{M}\mathfrak{u},\mathcal{M}\mathfrak{u}) \leq \eta \mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u}) < \mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u})$, a contradiction.
Hence, $\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u}) = 0$. □

Example 3 *Consider $\mathcal{M} = [0, 10]$. Let an M_v^b-metric $\mathfrak{m}_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined as $\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}) = (\frac{\mathfrak{u}+\mathfrak{w}}{2})^2$, $\mathfrak{s} = 3$, $\mathfrak{u},\mathfrak{w} \in \mathcal{M}$. Then, $(\mathcal{M},\mathfrak{m}_v^b)$ is a complete M_v^b-metric space. Define a self map $\mathcal{A}$ on $\mathcal{M}$ by $\mathcal{A}\mathfrak{u} = \frac{2}{15}\mathfrak{u}$, $\mathfrak{u} \in \mathcal{M}$. Observe that, for all $\mathfrak{u},\mathfrak{w} \in \mathcal{M}$, we obtain*

$$\mathfrak{m}_v^b(\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{w}) = \left(\frac{\mathcal{A}\mathfrak{u}+\mathcal{A}\mathfrak{w}}{2}\right)^2 = \left(\frac{\frac{2}{15}\mathfrak{u}+\frac{2}{15}\mathfrak{w}}{2}\right)^2 = \frac{4}{225}\left(\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)^2 \leq \frac{4}{225}\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}).$$

Consequently, all the postulates of Theorem 1 are verified and $\mathcal{A}$ has a unique fixed point at $0 \in \mathcal{M}$. Clearly, $\mathfrak{m}_v^b(0,0) = 0$.

The contractive condition used in the next result is the generalization of the Sehgal contraction [20] in M_v^b-metric space, which uses four possible combinations of distances $\big(\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w});\ \mathfrak{m}_v^b(\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{w});\ \mathfrak{m}_v^b(\mathfrak{u},\mathcal{A}\mathfrak{w});\ \mathfrak{m}_v^b(\mathfrak{w},\mathcal{A}\mathfrak{u})\big)$ in a linear way. On the other hand, Banach [4] utilized only the first two distances.

Theorem 2 *Let $(\mathcal{M},\mathfrak{m}_v^b)$ be an M_v^b-complete metric space. Suppose a self map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ satisfies*

$$\begin{gathered}\mathfrak{m}_v^b(\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{w}) \leq \eta \max\{\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{v}),\ \mathfrak{m}_v^b(\mathfrak{u},\mathcal{A}\mathfrak{u}),\ \mathfrak{m}_v^b(\mathfrak{w},\mathcal{A}\mathfrak{w})\},\\ \eta \in \left[0, \frac{1}{\mathfrak{s}}\right) \ \text{and}\ \ \mathfrak{u},\mathfrak{w} \in \mathcal{M}.\end{gathered} \tag{3}$$

Then, $\mathcal{A}$ has a unique fixed point $\sqcap$ such that $\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{u}) = 0$.

Proof. Let the sequence $\{\mathfrak{u}_n\}$ be defined as in the proof of Theorem 1, $\mathfrak{u}_n \neq \mathfrak{u}_{n+1}$, $\mathfrak{u}_0 \in \mathcal{M}$, $n \in \mathbb{N}$. Now,

$$\begin{aligned}\mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1}) &= \mathfrak{m}_v^b(\mathcal{A}\mathfrak{u}_{n-1},\mathcal{A}\mathfrak{u}_n)\\ &\leq \eta \max\{\mathfrak{m}_v^b(\mathfrak{u}_{n-1},\mathfrak{u}_n),\ \mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1})\}.\end{aligned}$$

We discuss two cases:

(i) If $\mathfrak{m}_v^b(\mathfrak{u}_{n-1},\mathfrak{u}_n) \leq \mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1})$, then $\mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1}) \leq \eta\mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1}) < \mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1})$,
a contradiction.

(ii) If $\mathfrak{m}_v^b(\mathfrak{u}_{n-1},\mathfrak{u}_n) \geq \mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1})$, then $\mathfrak{m}_v^b(\mathfrak{u}_n,\mathfrak{u}_{n+1}) \leq \eta\mathfrak{m}_v^b(\mathfrak{u}_{n-1},\mathfrak{u}_n)$.

Hence, the sequence $\{u_n\}$ verifies the postulates of Theorem 1. So, following similar steps as in Theorem 2, we may conclude that $\mathcal{A}$ has a unique fixed point $u \in \mathcal{M}$ and $m_v^b(u,u) = 0$. $\square$

Example 4 *Let $\mathcal{M} = \mathbb{R}$ and an M_v^b-metric $m_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined as:*
$m_v^b(u, \mathfrak{w}) = \max\{|u|^2, |\mathfrak{w}|^2\} + |u - \mathfrak{w}|^2, \quad u, \mathfrak{w} \in \mathcal{M}$. *$(\mathcal{M}, m_v^b)$ is an M_v^b-metric with $\mathfrak{s} = 3$. Define a self map $\mathcal{A} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}$ by* $\mathcal{A}u = \begin{cases} \frac{u}{9}, & u \in [-9, 9] \\ \frac{3u}{5}, & \text{otherwise} \end{cases}$.
Observe that, for all u, $\mathfrak{w} \in \mathcal{M}$, we obtain
$m_v^b(\mathcal{A}u, \mathcal{A}\mathfrak{w}) = \max\{|\mathcal{A}u|^2, |\mathcal{A}\mathfrak{w}|^2\} + |\mathcal{A}u - \mathcal{A}\mathfrak{w}|^2 \leq \frac{9}{25} \max\{|u|^2, |\mathfrak{w}|^2\} + |u - \mathfrak{w}|^2 = \frac{9}{25} m_v^b(u, \mathfrak{w})$.
Consequently, all the postulates of Theorem 2 are verified and $\mathcal{A}$ has a unique fixed point at $0 \in \mathcal{M}$ and clearly, $m_v^b(0,0) = 0$. It is fascinating to see that a self map $\mathcal{A}$ is not continuous.

Remark 5 *Theorems 1 and 2 are generalizations and extensions of Asadi et al. [1], Asim et al. [2]-[3], Banach [4], Bakhtin [5], Branciari [6], George [8], Karahan and Isik [11], Mlaiki et al. [13], Matthews [14], Özgür [17], Sehgal [20], and so on to M_v^b-metric space. Noticeably, the map under consideration is not even continuous in Theorem 2 (see Example 4).*

4 Applications

Motivated by the fact that the theory of linear systems is the foundation of numerical linear algebra, which performs a significant role in chemistry, physics, computer science, engineering, and economics, we resolve the system of linear equations in an m_b^v-metric space using Theorem 1.

Let $\mathcal{H}_n$ denote the set of all $n \times n$ Hermitian matrices, $\mathcal{P}_n$ the set of all $n \times n$ Hermitian positive definite matrices, $\mathcal{P}_{n_0}$ the set of all $n \times n$ positive semidefinite matrices. In the following, the symbol $\|.\|$ is the spectral norm of a matrix $\mathcal{B} = [b_{ij}]_{n \times n}$, that is, $\|\mathcal{B}\| = \sqrt{\lambda^+(\mathcal{B}^*\mathcal{B})}$, $\lambda^+(\mathcal{B}^*\mathcal{B})$ is the largest eigenvalue of $\mathcal{B}^*\mathcal{B}$, where $\mathcal{B}^*$ is the conjugate transpose of $\mathcal{B}$. Further, $\|.\|_{tr}$ denotes the trace norm of $\mathcal{B}$ and $\|\mathcal{B}\|_{tr} = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n |b_{ij}|^2} = \sqrt{tr(\mathcal{B}^*\mathcal{B})} = \sqrt{\Sigma_{i=1}^n \sigma_i^2(\mathcal{B})}$, $\sigma_i(\mathcal{B})$, $i = 1, 2, \ldots, n$, denotes largest singular values of $\mathcal{B} \in M_n(\mathbb{C})$. Let $\mathcal{M} = \mathcal{P}_n$ and $m_v^b : \mathcal{M} \longrightarrow \mathcal{M}$ be defined as

$$m_v^b(\mathcal{U}, \mathcal{W}) = \max\{|tr(\mathcal{U})|, |tr(\mathcal{W})|\}^2 + |tr(\mathcal{U} - \mathcal{W})|^2, \ \mathcal{U},\ \mathcal{W} \in \mathcal{M} \text{ and } s = 3.$$

Theorem 3 *Let a nonlinear matrix equation be*

$$\mathcal{U} = \Sigma_{i=1}^{n} \mathcal{B}_i^* f(\mathcal{U}) \mathcal{B}_i, \tag{4}$$

where $\mathcal{B}_i \in M_n(\mathbb{C})$ are the arbitrary matrix of order n. Let $f : \mathcal{H}_n(\mathbb{C}) \longrightarrow \mathcal{H}_n(\mathbb{C})$ be a monotone self map, which maps $\mathcal{P}_n(\mathbb{C})$ into $\mathcal{P}_n(\mathbb{C})$.

(i) $\max\{|tr(f\mathcal{U})|, \ |tr(f\mathcal{W})|\} \preccurlyeq \frac{1}{\sqrt{4\eta}} \max\{|tr(\mathcal{U})|, \ |tr(\mathcal{W})|\}$,

(ii) $|tr(f\mathcal{U}) - f\mathcal{W}| \preccurlyeq \frac{1}{\sqrt{4\eta}} |tr(\mathcal{U} - \mathcal{W})|$,

(iii) $tr(\mathcal{W}\mathcal{V}) \leq \|\mathcal{W}\| tr(\mathcal{V}), \quad \mathcal{W} \in M_n(\mathbb{C})$,

(iv) $\Sigma_{i=1}^{n} \mathcal{P}_i^* \mathcal{P} \leq (4\eta^2 I_n)^{\frac{1}{2}}$, *where I_n is the identity matrix of order n and* $\eta \in (0, \frac{1}{s})$.

Then, the matrix equation (4) has one and only solution $\mathcal{U}^* \in \mathcal{M}$. Further, the iteration $\mathcal{U}_n = \Sigma_{i=1}^{n} \mathcal{B}_i^* f(\mathcal{U}) \mathcal{B}_i, \quad \mathcal{U}_0 \in M_n(\mathbb{C})$ such that $\mathcal{U}_0 \leq \Sigma_{i=1}^{n} \mathcal{B}_i^* f(\mathcal{U}) \mathcal{B}_i$, converges to $\mathcal{U}^* \in \mathcal{M}$ satisfying the nonlinear matrix equation (4).

Proof. Let a self map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ be defined as

$$\mathcal{A}(\mathcal{U}) = \Sigma_{i=1}^{n} \mathcal{B}_i^* f(\mathcal{U}) \mathcal{B}_i. \tag{5}$$

Noticeably, a fixed point of $\mathcal{A}$ is a solution of a matrix Equation (4).

$$\begin{aligned}
m_v^b(\mathcal{A}\mathcal{U}, \mathcal{A}\mathcal{W}) &= \max\{|tr(\mathcal{A}\mathcal{U})|, |tr(\mathcal{A}\mathcal{W})|\}^2 + |tr(\mathcal{A}\mathcal{U} - \mathcal{A}\mathcal{W})|^2 \\
&= \max\{|tr(\Sigma_{i=1}^{n} \mathcal{B}_i^* f(\mathcal{U}) \mathcal{B}_i)|, \ |tr(\Sigma_{i=1}^{n} \mathcal{B}_i^* f(\mathcal{W}) \mathcal{B}_i)|\}^2 \\
&\quad + |tr(\Sigma_{i=1}^{n} \mathcal{B}_i^* (f(\mathcal{U}) - f(\mathcal{W}) \mathcal{B}_i))|^2 \\
&= \max\{|tr(\Sigma_{i=1}^{n} \mathcal{B}_i^* \mathcal{B}_i f(\mathcal{U}))|, \ |tr(\Sigma_{i=1}^{n} \mathcal{B}_i^* \mathcal{B}_i f(\mathcal{W}))|\}^2 \\
&\quad + |tr(\Sigma_{i=1}^{n} \mathcal{B}_i^* \mathcal{B}_i f(\mathcal{U}) - f(\mathcal{W}))|^2 \\
&\leq (\|\Sigma_{i=1}^{n} \mathcal{B}_i^* \mathcal{B}_i\|)^2 \big[\max\{|tr(f(\mathcal{U}))|, \ |tr(f(\mathcal{W}))|\}^2 + tr|f(\mathcal{U}) - f(\mathcal{W})|^2 \big] \\
&\leq \|4\eta^2 I\| \left(\frac{1}{\sqrt{4\eta}} \right)^2 [\max\{|tr(\mathcal{U})|, \ |tr(\mathcal{W})|\}^2 + |tr(\mathcal{U} - \mathcal{W})|^2] \\
&= \eta \big[\max\{|tr(\mathcal{U})|, \ |tr(\mathcal{W})|\}^2 + |tr(\mathcal{U} - \mathcal{W})|^2 \big] \\
&= \eta m_v^b(\mathcal{U}, \mathcal{W}).
\end{aligned}$$

We may observe that postulates of Theorem 1 are verified, and $\mathcal{A}$ has only one fixed point $\mathcal{U}^* \in \mathcal{M}$, that is, matrix equation (4) has only one solution $\mathcal{U}^* \in \mathcal{M}$. □

As an application of the main result, we solve the equation of the motion of rotation of a cable. Let $I = [-1,1]$ and $\mathcal{M} = C[I,\mathbb{R}]$ denote the set of all continuous functions on $[0,1]$. Define $m_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ by $m_v^b(u,\mathfrak{w}) = \left(\frac{|u|+|\mathfrak{w}|}{2}\right)^2$.

Theorem 4 *The equation of motion of a rotation of cable is :*

$$\frac{d}{dt}\left[\left(1-t^2\right)\frac{du}{dt}\right] + \eta u = \mathcal{K}(t,u(t)),\ t \in [-1,1],\ \eta \in \left[0,\frac{1}{s}\right), \tag{6}$$

with finite Dirichlet boundary conditions $u(-1)$ *and* $u(1)$, *where* η *is a constant and* $\mathcal{K} : \mathcal{M} \times [-1,1] \longrightarrow \mathbb{R}$, *is a continuous function satisfying*

$$|\mathcal{K}(s,u(s))| + |\mathcal{K}(s,\mathfrak{w}(s))| \le \frac{4\eta}{[\ln 4+1]^2}\max\{m_v^b(u,\mathfrak{w}),\ m_v^b(u,\mathcal{A}u),\ m_v^b(\mathfrak{w},\mathcal{A}\mathfrak{w})\},$$

where $u,\ \mathfrak{w} \in \mathbb{R},\ a \in [0,1)$.
Then, the Dirichlet boundary value problem (6) has a solution in $\mathcal{M}$.

Proof. A Dirichlet boundary value problem (6) is identical to

$$u(t) = \int_{-1}^{1} \mathcal{G}(s,t)\mathcal{K}(s,u(s))ds,\ t \in [-1,1], \tag{7}$$

Here,

$$\mathcal{G}(t,s) = \begin{cases} \ln 2 - \frac{1}{2} - \frac{1}{2}\ln(1-s)(1+t) & ,-1 \le t \le s \le 1 \\ \ln 2 - \frac{1}{2} - \frac{1}{2}\ln(1+s)(1-t) & ,-1 \le s \le t \le 1 \end{cases}, \tag{8}$$

is a continuous Green function on $[-1,1]$. Let $\mathcal{M} = (C[-1,1],\mathbb{R}^+)$ be the set of non negative real-valued continuous function. Define a map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ given by

$$\mathcal{A}u(t) = \textstyle\int_{-1}^{1} G(s,t)\mathcal{K}(s,u(s))ds.$$

Then, u is a solution of (7) if and only if u is a fixed point of $\mathcal{A}$.

Clearly, $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ is well defined, so

$$\begin{aligned}
\mathfrak{m}_v^b(\mathcal{A}\mathfrak{u}(t), \mathcal{A}\mathfrak{w}(t)) &= \left(\frac{|\mathcal{A}\mathfrak{u}(t)| + |\mathcal{A}\mathfrak{w}(t)|}{2}\right)^2 \\
&= \left(\frac{\left|\int_{-1}^{1} \mathcal{G}(s,t)\mathcal{K}(s,\mathfrak{u}(s))ds\right| + \left|\int_{-1}^{1} \mathcal{G}(s,t)\mathcal{K}(s,\mathfrak{w}(s))ds\right|}{2}\right)^2 \\
&\le \left(\frac{\int_{-1}^{1} \mathcal{G}(s,t)\left|\mathcal{K}(s,\mathfrak{u}(s))\right|ds + \int_{-1}^{1} \mathcal{G}(s,t)\left|\mathcal{K}(s,\mathfrak{w}(s))\right|ds}{2}\right)^2 \\
&= \frac{1}{4}\left(\int_{-1}^{1} \mathcal{G}(t,s)(|\mathcal{K}(s,\mathfrak{u}(s))| + |\mathcal{K}(s,\mathfrak{w}(s))|)ds\right)^2 \\
&\le \frac{1}{4}\max(|\mathcal{K}(s,\mathfrak{u}(s))| + |\mathcal{K}(s,\mathfrak{w}(s))|)^2\left(\int_{-1}^{1} \mathcal{G}(t,s)ds\right)^2 \\
&\le \frac{1}{4}\frac{4\eta}{[\ln 4+1]^2}\max\{\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}), \mathfrak{m}_v^b(\mathfrak{u},\mathcal{A}\mathfrak{u}), \mathfrak{m}_v^b(\mathfrak{w},\mathcal{A}\mathfrak{w})\} \\
&\quad \left(\int_{-1}^{1} \mathcal{G}(t,s)ds\right)^2 \\
&= \frac{\eta}{\ln 4+1}\max\{\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}), \mathfrak{m}_v^b(\mathfrak{u},\mathcal{A}\mathfrak{u}), p(\mathfrak{w},\mathcal{A}\mathfrak{w}) \\
&\quad \left(\int_{t}^{1}(\ln 2 - \frac{1}{2} - \frac{1}{2}\ln(1-s)(1+s))ds\right)^2 \\
&\le \frac{\eta}{[\ln 4+1]^2}\max\{\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}), \mathfrak{m}_v^b(\mathfrak{u},\mathcal{A}\mathfrak{u}), \mathfrak{m}_v^b(\mathfrak{w},\mathcal{A}\mathfrak{w})\}[\ln 4+1]^2 \\
&= \eta\max\{\mathfrak{m}_v^b(\mathfrak{u},\mathfrak{w}),\ \mathfrak{m}_v^b(\mathfrak{u},\mathcal{A}\mathfrak{u}),\ \mathfrak{m}_v^b(\mathfrak{w},\mathcal{A}\mathfrak{w})\}.
\end{aligned} \tag{9}$$

Thus, all the postulates of Theorem 2 are verified and $\mathcal{A}$ has a fixed point, which is indeed a solution to the problem (6). □

5 Conclusion

We utilized the M_v^b−metric which is an improvement and generalization of an M_v−metric to create an environment for the survival of a unique fixed point. Further, we demonstrated that the collection of open balls forms a basis on M_v^b−metric space. Examples and applications to solve the system of linear equations and the equation of a motion of rotation of a cable substantiate the utility of these extensions.

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