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Parametric uniform numerical method for singularly perturbed differential equations having both small and large delay

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Abstract. In this paper, singularly perturbed differential equations having both small and large delay are considered. The considered problem contains large delay parameter on the reaction term and small delay parameter on the convection term. The solution of the problem exhibits interior layer due to the delay parameter and strong right boundary layer due to the small perturbation parameter ε. The resulting singularly perturbed problem is solved using exponential fitted operator method. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, one model problem is considered for numerical experimentation.

1 Introduction

A differential equation is said to be a singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions occurring with different arguments, and also, the highest derivative term is multiplied by a small parameter. Such types of delayed differential equations play a very important role in the mathematical models of science and engineering, such as, the human pupil light reflex with mixed delay type [11],

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variational problems in control theory with small state problem [6], models of HIV infection [1], and signal transition [3].

Any system involving a feedback control usually involves time delay. The delay occurs because a finite time is required to sense the information and then react to it. Finding the solution of singularly perturbed delay differential equations, whose applications were mentioned above, is a challenging problem. In response to these, in recent years, there has been a growing interest in numerical methods on singularly perturbed delay differential equations. The authors of [14], [15], [4] have developed various numerical schemes on uniform meshes for singularly perturbed second order differential equations having small delay on the convection term. The authors of [16],[5], [9] and [8] have have presented second order differential equations with large delay.

In this paper, we consider a new governing problem having both small delay on the convection term and large delay. Additionally, in recent years the in-depth correlative physical analysis of the problem under consideration have been done by the authors [10]–[18]. As far as the researchers' knowledge numerical solution of singularly perturbed boundary value problem containing both small delay and large delay is first being considered. Thus, the purpose of this study is to develop stable, convergent and accurate numerical methods for solving singularly perturbed differential-difference equations having both small and large delay.

Throughout our analysis C is a generic positive constant that is independent of the parameter ε and the number of mesh points is $2N$. We assume that $\bar{\Omega} = [0,2]$, $\Omega = (0,2)$, $\Omega_1 = (0,1)$, $\Omega_2 = (1,2)$, $\Omega* = \Omega_1 \cup \Omega_2$, $\overline{\Omega}^{2N}$ is denoted by $\{0,1,2,...,2N\}$, Ω_1^{2N} is denoted by $\{1,2,...,N\}$ and Ω_2^{2N} is denoted by $\{N+1,N+2,...,2N-1\}$. K_1 and K_2 are the linear operators associated to the domain Ω_1 and Ω_2, respectively.

2 Statement of the problem

Consider the following singularly perturbed problem

$$\begin{aligned} Ly(x) &= -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) + d(x)y'(x-\delta) \\ &= f(x), x \in \Omega, \end{aligned} \tag{1}$$

$$y(x) = \phi(x), x \in [-1,0], y(2) = l, l \in R. \tag{2}$$

where δ is small, that is $\delta = O(\varepsilon)$, $0 < \varepsilon << 1$, $\phi(x)$ is sufficiently smooth on $[-1,0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions

$a(x), b(x)$, $c(x)$ and $d(x)$ satisfy $a(x) \geq a_1 > a > 0, b(x) > b \geq 0, c(x) \leq \gamma < 0, d(x) \geq \zeta \geq 0$, and $2(a+\zeta)+5b+5\gamma > \eta > 0, a(a_1-a)+2\gamma > 0$. The above assumptions ensure that $y \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega*)$.
The boundary value problem 1-2 exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$.
Expand $y'(x-\delta)$ around x using the Taylor's expansion and discard higher order terms. Then, Eqs. (1)-(2) can be approximated by

$$Ky(x) = -c_{\varepsilon,\delta}(x)y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad (3)$$

where $c_\varepsilon = \varepsilon + \delta d(x)$ and $p(x) = a(x) + d(x)$,

$$y(x) = \phi(x), x \in [-1, 0], y(2) = l. \quad (4)$$

As we observed from Eqs. (3) and (4), the values of $y(x-1)$ are known for the domain Ω_1 and unknown for the domain Ω_2 due to the large delay at $x = 1$. So, it impossible to treat the problem throughout the domain $(\bar{\Omega})$. Thus, we have to treat the problem at Ω_1 and Ω_2 separately.
Eqs. (3)–(4) are equivalent to

$$Ky(x) = R(x), \quad (5)$$

where

$$Ky(x) = \begin{cases} K_1y(x) = -c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x), x \in \Omega_1, \\ K_2y(x) = -c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1), x \in \Omega_2. \end{cases} \quad (6)$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), x \in \Omega_1, \\ f(x), x \in \Omega_2. \end{cases} \quad (7)$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), x \in [-1, 0], \\ y(1^-) = y(1^+), y'(1^-) = y'(1^+), \\ y(2) = l. \end{cases} \quad (8)$$

where $y(1^-)$ and $y(1^+)$ denote the left and right limits of y at $x = 1$, respectively.

3 Properties of continuous solution

Lemma 1 *(Maximum Principle) Let* $\psi(x)$ *be any function in* X *such that* $\psi(0) \geq 0, \psi(2) \geq 0, L_1\psi(x) \geq 0, \forall x \in \Omega_1, L_2\psi(x) \geq 0, \forall x \in \Omega_2$ *and* $\psi'(1^+) - \psi'(1^-) = [\psi'](1) \leq 0$. *Then* $\psi(x) \geq 0, \forall x \in \bar{\Omega}$.

Proof. For the proof, we refer to [16] □

Lemma 2 *(Stability Result) The solution y(x) of the problem* (3)-(4) *satisfies the bound*

$$|y(x)| \leq C \max\{|y(0)|, |y(2)|, \sup_{x \in \Omega*} |Ly(x)|\}, \quad x \in \overline{\Omega}.$$

Proof. For the proof, we refer to [16] □

Lemma 3 *Let* $y(x)$ *be the solution of* (3)-(4). *Then we have the following bounds*

$$||y^{(k)}(x)||_{\Omega^*} \leq Cc_\varepsilon^{-k}, \quad k = 1, 2, 3.$$

Proof. For the proof, we refer to [16] □

Lemma 4 *The bound for derivative of the solution* $y(x)$ *of Eqs.* 1-3 *when* $x \in \Omega_1$ *is given by*

$$|y^{(k)}(x)| \leq C\left(1 + c_\varepsilon^{-k} \exp\left(\frac{-p(1 - x_i)}{c_\varepsilon}\right)\right), \quad 0 \leq k \leq 4, i = 1, 2, 3, ..., N - 1.$$

Proof. For the proof, we refer to [2]. □

4 Numerical scheme formulation

The linear ordinary differential equation in Eq. (1) cannot, in general, be solved analytically because of the dependence of $a(x)$, $b(x)$ and $c(x)$ on the spatial coordinate x. We divide the interval [0, 2] into $2N$ equal parts with constant mesh length h. If we consider the interval $x \in (0, 1)$, the domain [0,1] is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < ... < x_N = 1$ be the points such that $x_i = ih, \;\; i = 1, 2, 3, ..., N$.

We apply an exponentially fitted operator finite difference method (FOFDM). From Eq. (6) and Eq. (7), we have

$$\begin{cases} -c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) = R(x), \quad x \in \Omega_1, \\ y(0) = \phi(0), \quad y(1) = \theta, \end{cases} \tag{9}$$

where $R(x) = f(x) - c(x)\phi(x-1)$.

To find the numerical solution of Eq. (9), we use the theory used in the asymptotic method for solving singularly perturbed BVPs. In the considered case, the boundary layer is in the right side of the domain, i.e. near $x = 1$. From the theory of singular perturbations given in [12] we get an approximation (up to first order) of the asymptotic solution in the form

$$y(x) = y_0(x) + \frac{p(1)}{p(x)}(\theta - y_0(1)) \exp\left(-\int_x^1 \left(\frac{p(x)}{c_\varepsilon} - \frac{b(x)}{p(x)} \right) dx \right) + O(c_\varepsilon),$$

By using the Taylor series around $x = 1$ for $p(x)$ and $b(x)$ and simplifying we obtain

$$y(x) = y_0(x) + (\theta - y_0(1)) \exp\left(-\frac{p^2(1) - c_\varepsilon b(1)}{c_\varepsilon p(1)}(1-x) \right) + O(c_\varepsilon), \tag{10}$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $c_\varepsilon = 0$) of Eq. (9) which is given by

$$p(x)y'(x) + b(x)y(x) = R(x), \quad y_0 = \phi(0). \tag{11}$$

By considering a small enough h, the discretized form of Eq. (10) becomes

$$y(ih) = y_0(ih) + (\theta - y_0(1)) \exp\left(-\frac{p^2(1) - c_\varepsilon b(1)}{p(1)}(1/c_\varepsilon - i\rho) \right), \tag{12}$$

where $\rho = \frac{h}{c_\varepsilon}$, $h = \frac{1}{N}$. Similarly, we write

$$y_{i\pm1} = y_0((i \pm 1)h) + (\theta - y_0(1)) \exp\left(-\frac{p^2(1) - c_\varepsilon b(1)}{p(1)}(1/c_\varepsilon - (i \pm 1)\rho) \right).$$

Using Taylors series approximation for $y_0((i+1)h)$ and $y_0((i-1)h)$ up to first order, we obtain

$$\begin{cases} y_{i+1} = y_0(ih) + (\theta - y_0(1)) \exp\left(-\frac{p^2(1)-c_\varepsilon b(1)}{p(1)}(1/c_\varepsilon - (i+1)\rho) \right), \\ y_{i-1} = y_0(ih) + (\theta - y_0(1)) \exp\left(-\frac{p^2(1)-c_\varepsilon b(1)}{p(1)}(1/c_\varepsilon - (i-1)\rho) \right). \end{cases} \tag{13}$$

To handle the artificial viscosity caused by the perturbation parameter, the term containing the perturbation parameter is multiplied by the exponentially fitting factor $\sigma(\rho)$) as

$$-c_\varepsilon \sigma(\rho) y''(x) + p(x) y'(x) + b(x) y(x) = R(x), \tag{14}$$

with boundary conditions $y(0) = y_0 = \phi(0)$ and $y(1) = \theta$, where $y(1)$ is evaluated by Runge-Kutta fourth order formula from the reduced solution of Eq. (10).

Next, on a uniform mesh points $\overline{\Omega}^N = \{x_i\}_{i=0}^N$ and with $h = x_{i+1} - x_i$, using the difference approximations

$$\begin{cases} D^0 y(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + \tau_1, \\ D^+ D^- y(x_i) = \frac{y(x_{i-1}) + 2y(x_i) + y(x_{i+1})}{h^2} + \tau_2, \end{cases} \tag{15}$$

where $\tau_1 = \frac{-h^2}{6} y^{(3)}(x_i)$ and $\tau_2 = \frac{-h^2}{12} y^{(4)}(x_i)$. When we apply a central difference formula on Eq. (14), it takes the form

$$-c_\varepsilon \sigma(\rho) \Big(D^+ D^- y(x_i) \Big) + p(x_i) \Big(D^0 y(x_i) \Big) + b(x_i) y(x_i) = R(x_i). \tag{16}$$

Let Y_i be an approximate solution of $y(x)$ and R_i is an approximation of $R(x)$ at grid point x_i, then we write the numerical scheme for Eq. (16) in difference operator form as

$$K^N Y_i = R_i, \tag{17}$$

with boundary conditions $Y_0 = \phi(0)$ and $Y(1) = \theta$, where

$$K^N Y_i = -c_\varepsilon \sigma(\rho) \left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2} \right) + p(x_i) \left(\frac{Y_{i+1} - Y_{i-1}}{2h} \right) + b(x_i) Y_i = R_i. \tag{18}$$

The multiplication of Eq. (18) by h, then tending to zero with h and truncating the term $(R_i - b(x_i)Y_i)h$, results in

$$\frac{-\sigma(\rho)}{\rho} \Big(Y_{i+1} - 2Y_i + Y_{i-1} \Big) + \frac{p(x_i)}{2} \Big(Y_{i+1} - Y_{i-1} \Big) = 0. \tag{19}$$

By substituting the results of Eq.(12) and Eq.(13) into Eq. (19) and simplification, the exponential fitting factor is obtained as

$$\sigma(\rho) = \frac{\rho p(1)}{2} \coth \left(\frac{\rho p(1)}{2} \right). \tag{20}$$

Assume that $\overline{\Omega}^{2N}$ denotes the partition of [0,2] into $2N$ subintervals such that $0 = x_0, x_1, x_2, ..., x_N = 1$ and $x_{N+1}, x_{N+2}, ..., x_{2N} = 2$ with $x_i = ih,\ h = \frac{2}{2N} = \frac{1}{N},\ i = 0, 1, 2, ..., 2N$.

Case (1): Consider Eqs. (6) and (7) on the domain Ω_1 which is given by

$$-c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) = f(x) - c(x)\phi(x-1), \tag{21}$$

Hence, the required finite difference scheme becomes

$$\left(\frac{-c_\varepsilon\sigma(\rho)}{h^2} - \frac{p(x_i)}{2h}\right)Y_{i-1} + \left(\frac{2c_\varepsilon\sigma(\rho)}{h^2} + b(x_i)\right)Y_i + \left(\frac{-c_\varepsilon\sigma(\rho)}{h^2} + \frac{p(x_i)}{2h}\right)Y_{i+1} = f_i - c_i\phi(x_i - N), \tag{22}$$

for $i = 0, 1, 2, ..., N$.

The numerical scheme in Eq. (22) can be written in three term recurrence relation as

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} = H_i, \quad i = 1, 2, ..., N, \tag{23}$$

where $E_i = \frac{-c_\varepsilon\sigma(\rho)}{h^2} - \frac{p_i}{2h}$, $F_i = \frac{2c_\varepsilon\sigma(\rho)}{h^2} + b_i$, $G_i = \frac{-c_\varepsilon\sigma(\rho)}{h^2} + \frac{p_i}{2h}$, $H_i = f_i - c_i\phi(x_i - N)$.

Case (2): Consider Eqs. (6) and (7) on the domain Ω_2 using exponentially fitted finite difference method, which is given by

$$-c_\varepsilon\sigma(\rho)\left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2}\right) + p_i\left(\frac{Y_{i+1} - Y_{i-1}}{2h}\right) + b_iY_i + c_iY(x_i - 1) + \tau_1 = f_i. \tag{24}$$

Similarly, this equation can be written as

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} + C_i = H_i, \quad i = N+1, N+2, ..., 2N-1, \tag{25}$$

where $E_i = \frac{-c_\varepsilon\sigma(\rho)}{h^2} - \frac{p_i}{2h}$, $F_i = \frac{2c_\varepsilon\sigma(\rho)}{h^2} + b_i$, $G_i = \frac{-c_\varepsilon\sigma(\rho)}{h^2} + \frac{p_i}{2h}$, $C_i = c_iy(x_i - 1)$ and $H_i = f_i$.

Therefore, on the whole domain $\overline{\Omega} = [0, 2]$, the basic schemes to solve Eqs. (1)-(3) are the schemes given in Eqs. (23) and (25) together with the local truncation error of τ_1 .

Uniform convergence analysis

The discrete scheme corresponding to the original Eqs. (6)-(7) is as follows For $i = 1, 2, 3, ..., N$

$$K_1^N Y_i = f_i - c_i\phi_{i-N}. \tag{26}$$

For $i = N+1, N+2, ..., 2N-1$

$$K_2^N Y_i = f_i, \tag{27}$$

subject to the boundary conditions:

$$Y_i = \phi_i, \quad i = -N, -N+1, ..., 0 \tag{28}$$

$$Y_{2N} = l, \tag{29}$$

where

$$\begin{cases} K_1^N Y_i = -c_\varepsilon D^+ D^- Y_i + p(x_i) D^0 Y_i + b(x_i) Y_i \\ K_2^N Y_i = -c_\varepsilon D^+ D^- Y_i + p(x_i) D^0 Y_i + b(x_i) Y_i + c(x_i) Y_{i-N} \end{cases} \tag{30}$$

Lemma 5 *(Discrete Maximum Principle) Assume that the mesh function* $\psi(x_i)$ *satisfies* $\psi(x_0) \geq 0$ *and* $\psi(x_{2N}) \geq 0$ *. Then* $K_1^N \psi(x_i) \geq 0, \quad \forall x_i \in \Omega_1^{2N}$, $K_2^N \psi(x_i) \geq 0$, $\forall x_i \in \Omega_2^{2N}$ *and* $\psi'(1^+) - \psi'(1^-) = [\psi'](1) \leq 0$*. Then* $\psi(x_i) \geq 0$, $\forall x_i \in \overline{\Omega}^{2N}$.

Proof. Let us define

$$s(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, x_i \in [0,1] \cap \overline{\Omega}^{2N} \\ \frac{3}{8} + \frac{x_i}{4}, x_i \in [1,2] \cap \overline{\Omega}^{2N} \end{cases}$$

Note that $s(x_i) > 0, \forall x_i \in \overline{\Omega}^{2N}$, $K^N s(x_i) > 0, \forall x_i \in \Omega_1^{2N} \cup \Omega_2^{2N}$ and $[s'](x_N) < 0$. Let use the notation $\mu = \max\left(\frac{-\psi(x_i)}{s(x_i)} : x_i \in \overline{\Omega}^{2N}\right)$. Then there exists $x_i \in \overline{\Omega}^{2N}$ such that $\psi(x_k) + \mu s(x_k) = 0$ and $\psi(x_k) + \mu s(x_k) \geq 0$, $\forall x_i \in \overline{\Omega}^{2N}$. Therefore, the function $\psi + \mu s$ attains its minimum at $x = x_k$. Suppose the theorem does not hold true, then $\mu > 0$.

Case (i): $x_k = x_0$

$$0 < (\psi + \mu s)(x_0) = 0, \text{ it is a contradiction.}$$

Case (ii): $x_k \in \Omega_1^{2N}$

$$\begin{aligned} 0 < K_1^N(\psi + \mu s)(x_k) &= -c_\varepsilon(\psi + \mu s)''(x_k) + p(x_k)(\psi + \mu s)'(x_k) \\ &\quad + b(x_k)(\psi + \mu s)(x_k) \leq 0, \end{aligned}$$

it is a contradiction.

Case (iii): $x_k = x_N$

$$0 < [(\psi + \mu s)'](x_N) = [\psi'](x_N) + [s'](x_N) < 0, \text{ it is a contradiction.}$$

Case (iv): $x_k \in \Omega_2^{2N}$

$$0 < K_2^N(\psi + \mu s)(x_k) = -c_\varepsilon(\psi + \mu s)''(x_k) + p(x_k)(\psi + \mu s)'(x_k) \\ +b(x_k)(\psi + \mu s)(x_k) + c(x_k)(\psi + \mu s)(x_k - 1) \leq 0,$$

it is a contradiction.
Case (v): $x_k = x_{2N}$

$$0 < (\psi + \mu s)x_{2N} \leq 0, \text{ it is a contradiction.} \tag{31}$$

Hence, the proof of the lemma is finished.
Remark: The above problem (3) has a solution (see [16]), and further the solution is unique due to the above maximum principle. □

Lemma 6 *Let* $\psi(x)$ *be any mesh function. Then, for* $0 < i < 2N$

$$|\psi(x_i)| \leq C \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |K^N \psi(x_i)|\}$$

Proof. Consider the barrier functions

$$\theta^\pm(x_i) = CMs(x) \pm \psi(x_i), \quad \forall x_i \in \overline{\Omega}^{2N} \tag{32}$$

where $M = \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |L^N \psi(x_i)|\}$.
From Eq. (32) it is clear that $\theta^\pm(x_0) \geq 0$ and $\theta^\pm(x_{2N}) \geq 0$

$$K_1^N \theta^\pm(x_i) \geq 0, \quad \forall x_i \in \Omega_1^{2N}$$
$$K_2^N \theta^\pm(x_i) \geq 0, \quad \forall x_i \in \Omega_2^{2N}$$
$$[\theta^{\pm\prime}](x_N) \leq 0$$

Using Lemma 4, $\theta^\pm(x_i) \geq 0, \quad \forall x_i \in \overline{\Omega}^{2N}$. □

We proved above that the discrete operator K^N satisfies the maximum principle. Next, we analyze the uniform convergence of the method.

Theorem 1 *Let* $y(x_i)$ *and* Y_i *be the exact solution of Eqs.* (1)-(3) *and numerical solutions of Eq.* (17) *respectively. Then, for a sufficiently large* N*, the following parameter uniform error estimate holds*

$$|L^N(y(x_i) - Y_i)| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\left(1 + c_\varepsilon^{-3} \exp\left(-\frac{p(1 - x_i)}{c_\varepsilon}\right)\right). \tag{33}$$

Proof. Let us consider the local truncation error defined as

$$\begin{aligned} L^N(y(x_i)-Y_i) &= -c_\varepsilon \sigma(\rho)(y''(x_i)-D^+D^-y(x_i))+p(x_i)(y'(x_i)-D^0y(x_i)),\\ &= -c_\varepsilon\left[\frac{\rho p(1)}{2}\coth\left(\frac{\rho p(1)}{2}\right)-1\right]D^+D^-y(x_i)\\ &+c_\varepsilon(y''(x_i)-D^+D^-y(x_i))+p(x_i)(y'(x_i)-D^0y(x_i)), \end{aligned} \tag{34}$$

where $\sigma(\rho)=p(1)\frac{\rho}{2}\coth(p(1)\frac{\rho}{2})$, and $\rho=\frac{N^{-1}}{c_\varepsilon}$.
since $|z\coth(z)-1|\le z^2$ holds if $z\neq 0$ and also $|z\coth(z)-1|\le z$ if $z>0$ values, Now, for $z>0$, C_1 and C_2 are constants, and we have $|z\coth(z)-1|\le C_1z^2, \quad z\le 1$. Similarly, for $z\longrightarrow\infty$, since $\lim_{z\longrightarrow\infty}\coth(z)=1$, $|z\coth(z)-1|\le C_1 z$ is given.
In general, for all $z>0$, we write

$$C_1\frac{z^2}{z+1}\le z\coth(z)-1\le C_2\frac{z^2}{z+1} \tag{35}$$

implying that

$$c_\varepsilon[p(1)\frac{\rho}{2}\coth(p(1)\frac{\rho}{2})-1]\le c_\varepsilon\left(\frac{(N^{-1}/c_\varepsilon)^2}{(N^{-1}/c_\varepsilon)+1}\right)=\frac{N^{-2}}{N^{-1}+c_\varepsilon}. \tag{36}$$

Using Taylor series expansion, we can rewrite $y(x_{i-1})$ and $y(x_{i+1})$ in terms of the values and derivatives of $y(x_i)$ as

$$\begin{cases} y(x_{i-1})=y(x_i)-hy'(x_i)+\frac{h^2}{2!}y''(x_i)-\frac{h^3}{3!}y^{(3)}(x_i)+\frac{h^4}{4!}y^{(4)}(x_i)+O(h^5),\\ y(x_{i+1})=y(x_i)+hy'(x_i)+\frac{h^2}{2!}y''(x_i)+\frac{h^3}{3!}y^{(3)}(x_i)+\frac{h^4}{4!}y^{(4)}(x_i)+O(h^5). \end{cases}$$

We obtain the bound for the second order derivatives as

$$\begin{cases} |D^+D^-y(x_i)|\le C|y''(x_i)|,\\ |y''(x_i)-D^+D^-y(x_i)|\le CN^{-2}|y^{(4)}(x_i)|. \end{cases} \tag{37}$$

Similarly, for the first derivative term

$$|y'(x_i)-D^0y(x_i)|\le CN^{-2}|y^{(3)}(x_i)|, \tag{38}$$

where $|y^{(k)}(x_i)|=\sup_{x_i\in(x_0,x_N)}|y^{(k)}(x_i)|, \quad k=2,3,4$.
Using the bounds in Eq.(37) and Eq.(38), we obtain

$$\begin{aligned} |L^N(y(x_i)-Y_i)| &\le C\frac{N^{-2}}{N^{-1}+c_\varepsilon}|y''(x_i)|+c_\varepsilon CN^{-2}|y^{(4)}(x_i)|+CN^{-2}|y^{(3)}(x_i)|,\\ &\le C\frac{N^{-2}}{N^{-1}+c_\varepsilon}|y''(x_i)|+CN^{-2}[c_\varepsilon|y^{(4)}(x_i)|+|y^{(3)}(x_i)|]. \end{aligned}$$

Now, using the bounds for the derivatives of the solution in lemma (3)and the assumption $c_\varepsilon \leq N^{-1}$, Eq. (3), we have

$$\begin{aligned}|L^N(y(x_i)-Y_i)| &\leq \frac{CN^{-2}}{N^{-1}+c_\varepsilon}\left(1+c_\varepsilon^{-2}\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)\right)\\ &+CN^{-2}\left[c_\varepsilon\left(1+c_\varepsilon^{-4}\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)\right)\right.\\ &\left.+\left(1+c_\varepsilon^{-3}\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)\right)\right]\\ &\leq \frac{CN^{-2}}{N^{-1}+c_\varepsilon}\left(1+c_\varepsilon^{-2}\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)\right)\\ &+CN^{-2}\left[\left(c_\varepsilon\right.\right.\\ &\left.\left.+c_\varepsilon^{-3}\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)\right)+\left(1+c_\varepsilon^{-3}\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)\right)\right],\end{aligned}$$

which simplifies to

$$|L^N(y(x_i)-Y_i)| \leq \frac{CN^{-2}}{N^{-1}+c_\varepsilon}\left(1+c_\varepsilon^{-3}\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)\right), \quad since \quad c_\varepsilon^{-3} \geq c_\varepsilon^{-2}. \tag{39}$$

□

Lemma 7 *For a fixed mesh and for* $c_\varepsilon \to 0$, *the following holds:*

$$\lim_{c_\varepsilon\to 0}\max_{1\leq j\leq N-1}\frac{\exp\left(\frac{-px_j}{c_\varepsilon}\right)}{c_\varepsilon^m}=0, \quad m=1,2,3,....$$

$$\lim_{c_\varepsilon\to 0}\max_{1\leq j\leq N-1}\frac{\exp\left(\frac{-p(1-x_j)}{c_\varepsilon}\right)}{c_\varepsilon^m}=0, \quad m=1,2,3,....$$

Proof. We refer to [2] □

Theorem 2 *Let* $y(x_i)$ *and* Y_i *be the exact solution of Eqs.* (1)-(2) *and numerical solutions of Eq.* (17) *respectively. Then, the following error bound holds*

$$\sup_{0<c_\varepsilon<<1}|(y(x_i)-Y_i)\| \leq \frac{CN^{-2}}{N^{-1}+c_\varepsilon} \leq CN^{-1}. \tag{40}$$

Proof. By substituting the results of lemma 4 in to Theorem 4 and applying the discrete maximum principle, we obtain the required bound. □

For the case $c_\varepsilon > N^{-1}$ the scheme secures second order convergence and we expect to lose an order of convergence for $c_\varepsilon \leq N^{-1}$, and in fact it turns out that the scheme guarantees second order uniformly convergent.

5 Numerical examples and results

In this section, one example is given to illustrate the numerical method discussed above. The exact solutions of the test problem is not known. Therefore, we use the double mesh principle to estimate the error and compute the experimental rate of convergence to the computed solution. For this we put

$$E_\varepsilon^N = \max_{0\leq i\leq 2N} |Y_i^N - Y_{2i}^{2N}|, \tag{41}$$

where Y_i^N and Y_{2i}^{2N} are the i^{th} and $2i^{th}$ components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_{\varepsilon} E_\varepsilon^N, \text{and} R^N = \log_2\left(\frac{E^N}{E^{2N}}\right). \tag{42}$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{ 10^{-4}, 10^{-8}, ..., 10^{-20}\}$.

Example 1 *Consider the model singularly perturbed boundary value problem:*

$$-\varepsilon y''(x) + 10y'(x) - y(x-1) + y'(x-\varepsilon) = x \quad x \in (0,1) \cup (1,2),$$

subject to the boundary conditions

$$y(x) = 1, \;\; x \in [-1, 0], \quad y(2) = 2.$$

6 Discussion and conclusion

This study introduces exponential fitted operator method for singularly perturbed differential equations having both small and large delay. The numerical scheme is developed on uniform mesh using fitted operator in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, one model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Table 1). Further, behavior of the numerical solution (Figure 1), point-wise absolute error (Figure 2) and the ε -uniform convergence of the method is shown by the log-log plot (Figure 3). The method is shown to be ε-uniformly convergent with order of convergence $O(h)$. The proposed method gives an accurate, stable and ε-uniform numerical result.

Table 1: Maximum absolute errors for Example 5 at number of mesh points $2N$.

ε	N=32	N=64	N=128	N=256	N=512
10^{-4}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-8}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-12}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-16}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-20}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
E^N	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
R^N	0.9849	0.9925	0.9962	0.9982	

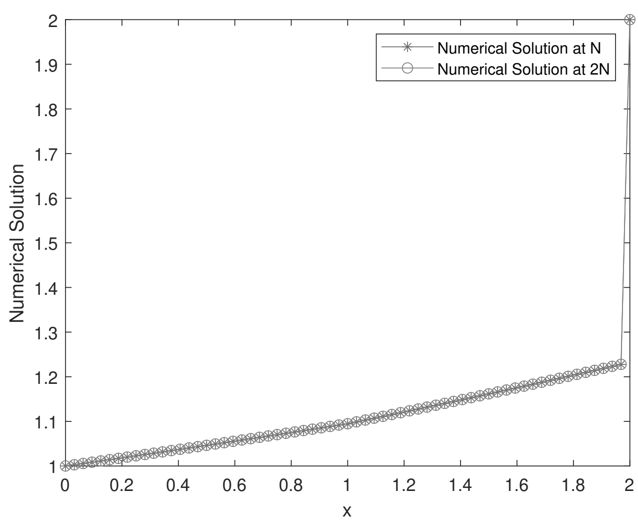

Figure 1: The behavior of the Numerical Solution for Example 5 at $\varepsilon = 10^{-12}$ and $N = 32$.

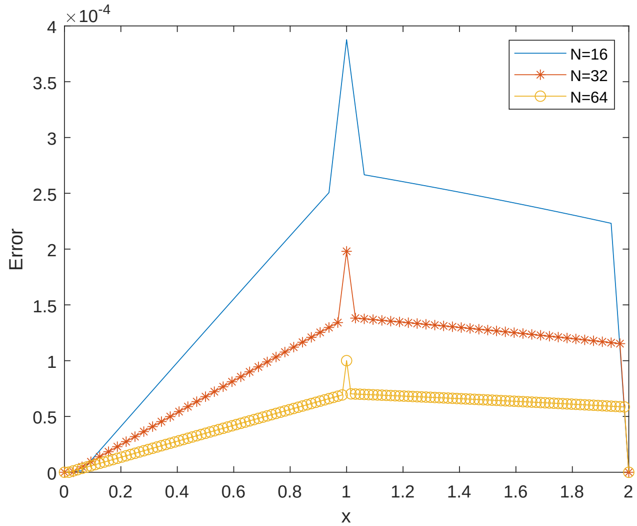

Figure 2: Point wise absolute error of Example 5 at $\varepsilon = 10^{-12}$ with different mesh point N.

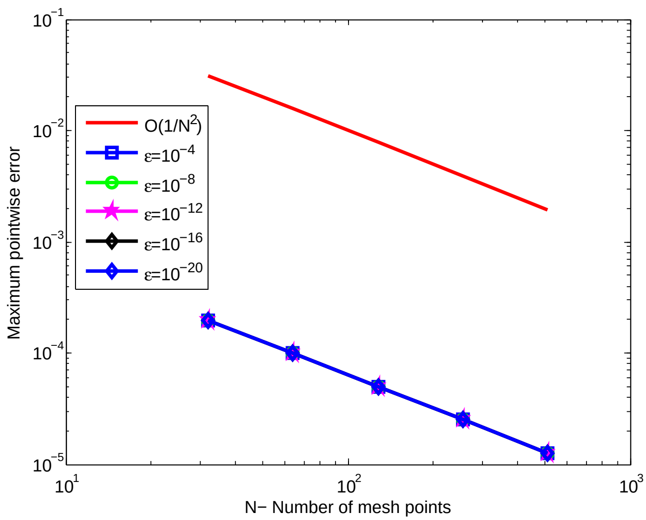

Figure 3: ε-uniform convergence with fitted operator in log-log scale for Example 5.

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