Acta Univ. Sapientiae, Mathematica, **15**, 2 (2023) 259–271

DOI: 10.2478/ausm-2023-0014

Oscillatory behavior of second-order nonlinear noncanonical neutral differential equations

Said R. Grace
Department of Engineering Mathematics
Faculty of Engineering
Cairo University
Orman, Giza 12221, Egypt
email: saidgrace@yahoo.com

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA
email: John-Graef@utc.edu

Tongxing Li
School of Control Science and Engineering
Shandong University
Jinan, Shandong 250061, P. R. China
email: litongx2007@163.com

Ercan Tunç
Department of Mathematics
Faculty of Arts and Sciences
Tokat Gaziosmanpasa University
60240, Tokat, Turkey
email: ercantunc72@yahoo.com

Abstract. This paper discusses the oscillatory behavior of solutions to a class of second-order nonlinear noncanonical neutral differential equations. Sufficient conditions for all solutions to be oscillatory are given. Examples are provided to illustrate all the main results obtained.

1 Introduction

In this paper, we examine the oscillatory behavior of solutions of the second-order nonlinear noncanonical neutral differential equation

$$\left[a(t)\left(x(t)+p(t)x(\tau(t))\right)'\right]' + q(t)x^{\alpha}(\sigma(t)) = 0, \tag{1}$$

2010 Mathematics Subject Classification: 34C10, 34K11, 34K40
Key words and phrases: oscillation, neutral differential equations, second-order, noncanonical equations

where $t \geq t_0 > 0$ and α is the ratio of odd positive integers with $0 < \alpha \leq 1$. In the remainder of the paper we assume that:

(i) $a : [t_0, \infty) \to (0, \infty)$ and $q : [t_0, \infty) \to [0, \infty)$ are continuous functions, and $q(t) > 0$ for large t;

(ii) $p : [t_0, \infty) \to \mathbb{R}$ is a continuous function with $0 \leq p(t) \leq d < 1$;

(iii) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\tau(t) \leq t$, σ is nondecreasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

We let

$$I(t) = \int_t^\infty \frac{1}{a(s)} ds, \quad t \geq t_0,$$

and assume that

$$I(t_0) < \infty, \tag{2}$$

i.e., the equation is in noncanonical form. By a *solution* of equation (1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ such that $x(t) + p(t)x(\tau(t)) \in C^1([t_x, \infty), \mathbb{R})$, $a(t)(x(t) + p(t)x(\tau(t)))' \in C^1([t_x, \infty), \mathbb{R})$, and x satisfies (1) on $[t_x, \infty)$. We consider only those solutions of (1) that exist on some half-line $[t_x, \infty)$ and satisfy

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \quad \text{for any} \quad T_1 \geq t_x;$$

in addition, we tacitly assume that (1) possesses such solutions. Such a solution $x(t)$ of (1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Neutral differential equations are functional differential equations in which the highest-order derivative of the unknown function appears in the equation with the argument t (present state) as well as one or more delay or advanced arguments. Equations of this type arise in many areas of applied mathematics and have important applications in the natural sciences and technology. Readers interested in the application of equations of this kind can refer to the monograph by Hale [16] among the most cited sources. For instance, they arise in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits) (see [6, 25]); in the study of vibrating masses attached to an elastic bar, and as the Euler equation in some variational problems containing a delay (see [16, pp. 4–7]).

Oscillation and asymptotic behavior of solutions of second order functional differential equations with linear, sublinear and superlinear neutral term has been a very active area of research in recent years and interest in the subject can be seen from the many articles in the literature. For some typical results, the reader can refer to the papers [2, 3, 5, 7, 8, 17, 18, 19, 20, 21, 23, 24, 26, 28, 29] for equations with linear neutral terms, the papers [1, 9, 22, 27] for equations with sublinear neutral terms, and the papers [4, 10] for the equations with superlinear neutral terms. Motivated by the papers mentioned above and the results in [11], our aim here is to establish some new sufficient conditions under which every solution of (1) is oscillatory. We note that the results presented in this paper extend the results in [11] in some special cases and are not covered by existing results in the literature. Since our equation considered here is fairly simple, it would be possible to extend our results to more general equations (e.g., to the equations in [4, 9, 21, 22, 24, 26, 27, 28, 29]) and to other types that include equation (1) as a special case. For these reasons, it is our hope that the present paper will stimulate additional interest in research on second and higher even-order functional differential equations with linear, sublinear and superlinear neutral terms.

In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1) since if $x(t)$ is a solution of (1), then $-x(t)$ is also a solution.

2 Main results

For the reader's convenience, we adopt the notation:

$$z(t) := x(t)+p(t)x(\tau(t)), \quad \text{and} \quad \pi(t) := 1-p(\sigma(t))\frac{I(\tau(\sigma(t)))}{I(\sigma(t))} \quad \text{for} \quad t \geq t_1 \geq t_0.$$

Note that $\pi(t) \geq 0$ for $t \geq t_1 \geq t_0$. We first present the following three oscillation criteria for equation (1) in the case where σ is a delay argument, i.e., for the case $\sigma(t) \leq t$.

Theorem 1 *Let* (2) *hold. If*

$$\int_{t_0}^{\infty} I(s)q(s)ds = \infty \tag{3}$$

and

$$\limsup_{t\to\infty}\left(I(t)\int_{t_0}^{t} q(s)\pi^{\alpha}(s)ds + I^{-\alpha}(\sigma(t))\int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))ds\right) > \begin{cases}1, & \text{if } \alpha=1,\\ 0, & \text{if } 0<\alpha<1,\end{cases} \tag{4}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, we see from (1) that

$$\left(a(t)z'(t)\right)' = -q(t)x^{\alpha}(\sigma(t)) \leq 0 \quad \text{for } t \geq t_1, \tag{5}$$

and so, $a(t)z'(t)$ is nonincreasing and eventually of one sign. That is, there exists $t_2 \geq t_1$ such that, for $t \geq t_2$, either

(I) $z(t) > 0, \quad z'(t) > 0, \quad (a(t)z'(t))' \leq 0,$ or

(II) $z(t) > 0, \quad z'(t) < 0, \quad (a(t)z'(t))' \leq 0.$

We first consider case (I). From the definition of z, we see that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - dz(\tau(t)) \geq (1-d)z(t). \tag{6}$$

Using (6) in (1) yields

$$\left(a(t)z'(t)\right)' + (1-d)^{\alpha}q(t)z^{\alpha}(\sigma(t)) \leq 0 \tag{7}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Since $z(t) > 0$ and $z'(t) > 0$, for some $c > 0$, $z(t) \geq c > 0$ for $t \geq t_3$ and so inequality (7) takes the form

$$\left(a(t)z'(t)\right)' + c^{\alpha}(1-d)^{\alpha}q(t) \leq 0 \quad \text{for } t \geq t_3. \tag{8}$$

Integrating inequality (8) twice yields

$$z(t) \leq z(t_3) + \int_{t_3}^{t} \frac{a(t_3)z'(t_3)}{a(s)}ds - c^{\alpha}(1-d)^{\alpha}\int_{t_3}^{t}\frac{1}{a(u)}\int_{t_3}^{u} q(s)ds du \to -\infty$$

as $t \to \infty$ due to (2) and (3). This contradicts the fact that $z(t)$ is positive.

Next, assume that case (II) holds. Then there exists a constant $\kappa \geq 0$ such that

$$\lim_{t\to\infty} z(t) = \kappa < \infty.$$

We claim that $\kappa = 0$. If $\kappa > 0$, then there exist $t_3 \geq t_2$ and $\lambda > 1$ with $\lambda d < 1$ such that

$$\kappa < z(t) < \kappa\lambda \tag{9}$$

for $t \geq t_3$. Now,

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - dz(\tau(t)) \geq \left(\frac{1-\lambda d}{\lambda}\right) z(t). \tag{10}$$

Using (10) in (1) yields

$$\left(a(t)z'(t)\right)' + \left(\frac{1-\lambda d}{\lambda}\right)^{\alpha} q(t)z^{\alpha}(\sigma(t)) \leq 0 \tag{11}$$

for $t \geq t_4$ for some $t_4 \geq t_3$. Integrating (11) from t_4 to t gives

$$-a(t)z'(t) \geq \left(\frac{1-\lambda d}{\lambda}\right)^{\alpha} \kappa^{\alpha} \int_{t_4}^{t} q(s)ds. \tag{12}$$

Integrating again and using (3) gives

$$\begin{aligned} z(t_4) &\geq \left(\frac{1-d\lambda}{\lambda}\right)^{\alpha} \kappa^{\alpha} \int_{t_4}^{\infty} \frac{1}{a(u)} \int_{t_4}^{u} q(s)dsdu \\ &= \left(\frac{1-\lambda d}{\lambda}\right)^{\alpha} \kappa^{\alpha} \int_{t_4}^{\infty} I(s)q(s)ds = \infty, \end{aligned}$$

which is a contradiction, and so $\lim_{t\to\infty} z(t) = 0$.

It follows from case (II) that

$$z(t) \geq -\int_t^{\infty} \frac{a(s)z'(s)}{a(s)} ds \geq -\left(\int_t^{\infty} \frac{1}{a(s)} ds\right) a(t)z'(t) = -I(t)a(t)z'(t),$$

and so

$$z(t) + I(t)a(t)z'(t) \geq 0, \tag{13}$$

which implies that

$$\left(\frac{z(t)}{I(t)}\right)' = \frac{I(t)z'(t) + \frac{z(t)}{a(t)}}{I^2(t)} \geq 0,$$

i.e., $z(t)/I(t)$ is eventually nondecreasing, say for $t \geq t_5$ for some $t_5 \geq t_4$. From this and the definition of z, we observe that

$$x(t) \geq z(t) - p(t)z(\tau(t)) \geq \left(1 - p(t)\frac{I(\tau(t))}{I(t)}\right) z(t). \tag{14}$$

Using (14) in (1) gives

$$\big(a(t)z'(t)\big)' + q(t)\pi^\alpha(t)z^\alpha(\sigma(t)) \le 0, \tag{15}$$

which can be written in the equivalent form

$$(a(t)z'(t)I(t) + z(t))' + I(t)q(t)\pi^\alpha(t)z^\alpha(\sigma(t)) \le 0. \tag{16}$$

Integrating (16) from t to u, letting $u \to \infty$, and using (13) yields

$$a(t)z'(t)I(t) + z(t) \ge \int_t^\infty I(s)q(s)\pi^\alpha(s)z^\alpha(\sigma(s))ds. \tag{17}$$

An integration of (15) from t_5 to t and multiplication by $I(t)$ gives

$$-a(t)z'(t)I(t) \ge I(t)\int_{t_5}^t q(s)\pi^\alpha(s)z^\alpha(\sigma(s))ds. \tag{18}$$

It follows from (17) and (18) that

$$z(t) \ge I(t)\int_{t_5}^t q(s)\pi^\alpha(s)z^\alpha(\sigma(s))ds + \int_t^\infty I(s)q(s)\pi^\alpha(s)z^\alpha(\sigma(s))ds. \tag{19}$$

Since z is decreasing, we see that if $s \le t$, we have $\sigma(s) \le \sigma(t) \le t$, and so $z(\sigma(s)) \ge z(\sigma(t)) \ge z(t)$ for $t \ge t_5$. Thus,

$$\int_{t_5}^t q(s)\pi^\alpha(s)z^\alpha(\sigma(s))ds \ge \left(\int_{t_5}^t q(s)\pi^\alpha(s)ds\right) z^\alpha(t). \tag{20}$$

Also, for $s > t$, we have $\sigma(s) \ge \sigma(t)$ and $\sigma(t) \le t$. Since $z(t)/I(t)$ is nondecreasing and $z(t)$ is decreasing,

$$\frac{z^\alpha(\sigma(s))}{I^\alpha(\sigma(s))} \ge \frac{z^\alpha(\sigma(t))}{I^\alpha(\sigma(t))} \ge \frac{z^\alpha(t)}{I^\alpha(\sigma(t))}.$$

Thus,

$$\begin{aligned}\int_t^\infty I(s)q(s)\pi^\alpha(s)z^\alpha(\sigma(s))ds &\ge \int_t^\infty I(s)q(s)\pi^\alpha(s)I^\alpha(\sigma(s))\frac{z^\alpha(\sigma(s))}{I^\alpha(\sigma(s))}ds \\ &\ge \left(\frac{1}{I^\alpha(\sigma(t))}\int_t^\infty I(s)q(s)\pi^\alpha(s)I^\alpha(\sigma(s))ds\right) z^\alpha(t).\end{aligned} \tag{21}$$

Using (20) and (21) in (19) yields

$$z(t) \geq I(t)\left(\int_{t_5}^{t} q(s)\pi^{\alpha}(s)ds\right) z^{\alpha}(t) + \left(\frac{1}{I^{\alpha}(\sigma(t))}\int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))ds\right) z^{\alpha}(t).$$

Now taking the lim sup as $t \to \infty$ of the resulting inequality, we obtain a contradiction to (4). This completes the proof of the theorem. □

Theorem 2 *Let* (2) *and* (3) *hold. If*

$$\limsup_{t\to\infty}\left(I(\sigma(t))\int_{t_0}^{\sigma(t)} q(s)\pi^{\alpha}(s)ds + \int_{\sigma(t)}^{t} I(s)q(s)\pi^{\alpha}(s)ds + I^{-\alpha}(\sigma(t))\int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))ds\right) > \begin{cases} 1, & \textit{if } \alpha = 1, \\ 0, & \textit{if } 0 < \alpha < 1, \end{cases} \tag{22}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 1, we again arrive at (19), which can be written as below

$$z(\sigma(t)) \geq I(\sigma(t))\int_{t_5}^{\sigma(t)} q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds + \int_{\sigma(t)}^{t} I(s)q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds + \int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds.$$

The remainder of the proof is similar to that of Theorem 1 and hence is omitted. The proof of this theorem is complete. □

Theorem 3 *Let* (2) *and* (3) *hold. If*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} I(s)q(s)\pi^{\alpha}(s)ds > \begin{cases} 1, & \textit{if } \alpha = 1, \\ 0, & \textit{if } 0 < \alpha < 1, \end{cases} \tag{23}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) with $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 1, we again arrive at (13) and (16). Setting

$$y(t) := z(t) + I(t)a(t)z'(t),$$

we have $0 \le y(t) \le z(t)$ since $z'(t) < 0$ in case II. It then follows from (16) that

$$y'(t) + I(t)q(t)\pi^{\alpha}(t)y^{\alpha}(\sigma(t)) \le 0.$$

Integrating this inequality from $\sigma(t)$ to t, we obtain

$$y(\sigma(t)) \ge \int_{\sigma(t)}^{t} I(s)q(s)\pi^{\alpha}(s)y^{\alpha}(\sigma(s))ds \ge y^{\alpha}(\sigma(t)) \int_{\sigma(t)}^{t} I(s)q(s)\pi^{\alpha}(s)ds,$$

so,

$$\frac{y(\sigma(t))}{y^{\alpha}(\sigma(t))} \ge \int_{\sigma(t)}^{t} I(s)q(s)\pi^{\alpha}(s)ds.$$

Now take the $\limsup_{t\to\infty}$ on both sides of the above inequality. Recalling the fact that $z(t) \to 0$, which implies that $y(t) \to 0$, we obtain a contradiction to condition (23), and this proves the theorem. □

Next, we present the following two oscillation criteria for equation (1) in the case where σ is a advanced argument, i.e., for the case $\sigma(t) \ge t$.

Theorem 4 *Let* (2) *and* (3) *hold. If*

$$\limsup_{t\to\infty} \left(\frac{I^{\alpha}(\sigma(t))}{I^{\alpha-1}(t)} \int_{t_0}^{t} q(s)\pi^{\alpha}(s)ds + \frac{1}{I^{\alpha}(t)} \int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))ds \right) > \begin{cases} 1, & \text{if } \alpha = 1, \\ 0, & \text{if } 0 < \alpha < 1, \end{cases} \tag{24}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for $t \ge t_1$ for some $t_1 \ge t_0$. Proceeding similarly to the proof of Theorem 1, we again arrive at (19). Using the fact that z is decreasing and $z(t)/I(t)$ is nondecreasing, we see that if $s \le t$, we have $s \le \sigma(s) \le \sigma(t)$ and $\sigma(t) \ge t$, so

$$z(\sigma(s)) \ge z(\sigma(t)).$$

Now,

$$\int_{t_5}^{t} q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds \ge \left(\int_{t_5}^{t} q(s)\pi^{\alpha}(s)ds \right) I^{\alpha}(\sigma(t)) \frac{z^{\alpha}(\sigma(t))}{I^{\alpha}(\sigma(t))}$$

$$\geq \left(\int_{t_5}^{t} q(s)\pi^{\alpha}(s)ds\right)\left(\frac{I^{\alpha}(\sigma(t))}{I^{\alpha}(t)}\right)z^{\alpha}(t). \quad (25)$$

Also, if $s > t$, we have $\sigma(s) \geq \sigma(t)$, and so

$$\frac{z^{\alpha}(\sigma(s))}{I^{\alpha}(\sigma(s))} \geq \frac{z^{\alpha}(\sigma(t))}{I^{\alpha}(\sigma(t))} \geq \frac{z^{\alpha}(t)}{I^{\alpha}(t)}.$$

Thus,

$$\int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds \geq \int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))\frac{z^{\alpha}(t)}{I^{\alpha}(t)}ds$$
$$\geq \left(\frac{1}{I^{\alpha}(t)}\int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))ds\right)z^{\alpha}(t). \quad (26)$$

Using (25) and (26) in (19) yields

$$z^{1-\alpha}(t) \geq \frac{I^{\alpha}(\sigma(t))}{I^{\alpha-1}(t)}\int_{t_5}^{t} q(s)\pi^{\alpha}(s)ds + \frac{1}{I^{\alpha}(t)}\int_{t}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))ds.$$

The rest of the proof is similar to that of Theorem 1 and is omitted. □

Theorem 5 *Let* (2) *and* (3) *hold. If*

$$\limsup_{t\to\infty}\left(I(\sigma(t))\int_{t_0}^{t} q(s)\pi^{\alpha}(s)ds + I^{1-\alpha}(\sigma(t))\int_{t}^{\sigma(t)} I^{\alpha}(\sigma(s))q(s)\pi^{\alpha}(s)ds\right.$$
$$\left.+I^{-\alpha}(\sigma(t))\int_{\sigma(t)}^{\infty} I(s)q(s)\pi^{\alpha}(s)I^{\alpha}(\sigma(s))ds\right) > \begin{cases}1, & \text{if } \alpha = 1,\\ 0, & \text{if } 0 < \alpha < 1,\end{cases} \quad (27)$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) with $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 1, we again arrive at (19), which can be written as

$$z(\sigma(t)) \geq I(\sigma(t))\int_{t_5}^{t} q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds + I(\sigma(t))\int_{t}^{\sigma(t)} q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds$$
$$+\int_{\sigma(t)}^{\infty} I(s)q(s)\pi^{\alpha}(s)z^{\alpha}(\sigma(s))ds.$$

As in the proof of Theorem 1, using the fact that z is decreasing and $z(t)/I(t)$ is nondecreasing, we arrive at the desired conclusion. This completes the proof of the theorem. □

Next, we us illustrate all the results obtained here on linear and nonlinear second-order neutral differential equations with delay and advanced arguments.

Example 1 *Consider the equation*

$$\left(t^2\left(x(t)+\frac{t+1}{4t+6}x\left(\frac{t}{2}\right)\right)'\right)'+q_0x\left(\frac{t}{4}\right)=0,\quad t\geq 1. \tag{28}$$

Here we have $a(t)=t^2$, $p(t)=(t+1)/(4t+6)$, $\tau(t)=t/2$, $\sigma(t)=t/4$, $\alpha=1$, *and* $q(t)=q_0>0$ *is a constant. Then*

$$I(t)=\frac{1}{t},\ I(\sigma(t))=\frac{4}{t},\ I(\tau(\sigma(t)))=\frac{8}{t},\ p(\sigma(t))=\frac{t+4}{4t+24},\ \textit{and}\ \pi(t)=\frac{t+8}{2t+12}.$$

Now, if we apply Theorems 1–3 to equation (28), we see that (28) is oscillatory by Theorem 1 if $q_0>1$, *the same equation is oscillatory by Theorem 2 if* $q_0>\frac{1}{1+2\ln 2}$, *and the same equation is oscillatory by Theorem 3 if* $q_0>\frac{1}{\ln 2}$.

Example 2 *Consider the equation*

$$\left(\frac{t^3}{2}\left(x(t)+\frac{1}{8}x\left(\frac{t}{2}\right)\right)'\right)'+tx^{1/3}(2t)=0,\quad t\geq 1. \tag{29}$$

Here we have $a(t)=t^3/2$, $p(t)=1/8$, $\tau(t)=t/2$, $\sigma(t)=2t$, $\alpha=1/3$, *and* $q(t)=t$. *Then*

$$I(t)=\frac{1}{t^2},\ I(\sigma(t))=\frac{1}{4t^2},\ I(\tau(\sigma(t)))=\frac{1}{t^2},\ p(\sigma(t))=1/8,\ \textit{and}\ \pi(t)=1/2.$$

Since $I(t_0)=1$ *and*

$$\int_{t_0}^{\infty}I(s)q(s)ds=\int_1^{\infty}\frac{1}{s}ds=\infty,$$

conditions (2) and (3) hold. Also, we can easily see that conditions (24) and (27) are satisfied. Hence, By Theorem 4 and Theorem 5, equation (29) is oscillatory.

In conclusion, as some suggestions for future research, it would be of interest to obtain results similar to those we obtained here, but for different ranges on the value of the coefficient $p(t)$. The asymptotic behavior of solutions of neutral equations changes significantly, for example, if p takes on negative values. In this regard we refer the reader to the paper of Graef, Grammatikopoulos, and Spikes [12] as well as [13, 14, 15].

References

[1] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, Oscillation of second-order differential equations with a sublinear neutral term, *Carpathian J. Math.*, **30** (2014), 1–6.

[2] R. P. Agarwal, C. Zhang, and T. Li, Some remarks on oscillation of second order neutral differential equations, *Appl. Math. Comput.*, **274** (2016), 178–181.

[3] M. Bohner, S. R. Grace, and I. Jadlovská, Oscillation criteria for second-order neutral delay differential equations, *Electron. J. Qual. Theory Diff. Equ.*, **2017** (2017), No. 60, 1–12.

[4] M. Bohner, B. Sudha, K. Tangavelu, and E. Thandapani, Oscillation criteria for second-order differential equations with superlinear neutral term, *Nonlinear Stud.*, **26** (2019), 425–434.

[5] M. Bohner, S. R. Grace and I. Jadlovská, Sharp oscillation criteria for second-order neutral delay differential equations, *Math. Meth. Appl. Sci.*, **43** (2020), 10041–10053.

[6] R. K. Brayton, and R. A. Willoughby, On the numerical integration of a symmetric system of difference-differential equations of neutral type, *J. Math. Anal. Appl.*, **18** (1967), 182–189.

[7] J. G. Dong, Oscillation behavior of second order nonlinear neutral differential equations with deviating arguments, *Comput. Math. Appl.*, **59** (2010), 3710–3717.

[8] S. R. Grace, J. Džurina, I. Jadlovská and T. Li, An improved approach for studying oscillation of second-order neutral delay differential equations, *J. Inequalities Appl.*, **2018** (2018), 1–13.

[9] S. R. Grace and J. R. Graef, Oscillatory behavior of second order nonlinear differential equations with a sublinear neutral term, *Math. Model. Anal.*, **23** (2018), 217–226.

[10] S. R. Grace, J. R. Graef, T. Li and E. Tunç, Oscillatory behavior of second-order nonlinear differential equations with mixed neutral terms, *Tatra Mt. Math. Publ.*, **79** (2021), 119–134.

[11] S. R. Grace, J. R. Graef and E. Tunç, Oscillation of second-order nonlinear noncanonical dynamic equations with deviating arguments, *Acta Math. Univ. Comenianae*, **91** (2022), 113–120.

[12] J. R. Graef, M. K. Grammatikopoulos, and P. W. Spikes, Asymptotic behavior of nonoscillatory solutions of neutral delay differential equations of arbitrary order, *Nonlinear Anal.*, **21** (1993), 23–42.

[13] J. R. Graef, M. K. Grammatikopoulos, and P. W. Spikes, On the asymptotic behavior of solutions of a second order nonlinear neutral delay differential equation, *J. Math. Anal. Appl.*, **156** (1991), 23–39.

[14] J. R. Graef and P. W. Spikes, Some asymptotic properties of solutions of a neutral delay equation with an oscillatory coefficient, *Canadian Math. Bull.*, **36** (1993), 263–272.

[15] J. R. Graef and P. W. Spikes, Asymptotic and oscillatory behavior of solutions of nonlinear neutral delay equations of arbitrary order, *Appl. Math. Comput.*, **65** (1994), 209–221.

[16] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[17] I. Jadlovská, New criteria for sharp oscillation of second-order neutral delay differential equations, *Mathematics*, **9** (2021), 1–23.

[18] T. Li, Y. V. Rogovchenko, and C. Zhang, Oscillation of second-order neutral differential equations, *Funkcial. Ekvac.*, **56** (2013), 111–120.

[19] T. Li, E. Thandapani, J. R. Graef and E. Tunç, Oscillation of second-order Emden-Fowler neutral differential equations, *Nonlinear Stud.*, **20** (2013), 1–8.

[20] T. Li and Y. V. Rogovchenko, Oscillation of second-order neutral differential equations, *Math. Nachr.*, **288** (2015), 1150–1162.

[21] T. Li and Y. V. Rogovchenko, Oscillation criteria for second-order super-linear Emden–Fowler neutral differential equations, *Monatsh. Math.*, **184** (2017), 489–500.

[22] N. Prabaharan, C. P. Dharuman, J. R. Graef, and E. Thandapani, New oscillation criteria for second order quasi-linear differential equations with sub-linear neutral term, *Appl. Math. E-Notes*, **19** (2019), 563–574.

[23] S. H. Saker, R. P. Agarwal, and D. O'Regan, Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales, *Appl. Anal.*, **86** (2007), 1–17.

[24] A. K. Sethi and A. K. Tripathy, On oscillatory second-order differential equations with variable delays, *Palestine J. Math.*, **10** (2021), 487–501.

[25] M. Slemrod and E. F. Infante, Asymptotic stability criteria for linear systems of difference-differential equations of neutral type and their discrete analogues, *J. Math. Anal. Appl.*, **38** (1972), 399–415.

[26] Y. Sui and Z. Han, Oscillation of second order neutral dynamic equations with deviating arguments on time scales, *Adv. Differ. Equ.*, **2018** (2018), 1–10.

[27] S. Tamilvanan, E. Thandapani, and J. Džurina, Oscillation of second order nonlinear differential equation with sub-linear neutral term, *Differ. Equ. Appl.*, **9** (2017), 29–35.

[28] A. K. Tripathy and S. S. Santra, Necessary and sufficient conditions for oscillation of second-order differential equations with nonpositive neutral coefficients, *Math. Bohem.*, **146** (2021), 185–197.

[29] R. Xu and F. Meng, Some new oscillation criteria for second order quasi-linear neutral delay differential equations, *Appl. Math. Comput.*, **182** (2006), 797–803.

Received: June 12, 2022